Concerning connected and dense subsets of indecomposable continua

by

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In this paper it is shown that, if the continuum hypothesis is true, every compact metric indecomposable continuum M contains a connected point set K, containing no perfect point set, such that each composant of M contains one and only one point of K. The terminology and notation used is similar to that used by R. L. Moore in [2]; S denotes the set of all points; and, in theorems 2 through 6, it is assumed that S is a compact metric indecomposable continuum.

THEOREM 1. *Suppose that S is a complete metric separable continuum such that no countable point set separates S and M is a point set such that every perfect point set intersects M. Then* (1) *M is dense in S,* (2) *if H and K are two closed proper subsets of S such that $S = H+K$, then M intersects $H \cdot K$, and* (3) *M is connected.*

Proof. Every domain contains a perfect point set and, hence, M intersects every domain. Then M is dense in S.

Suppose that H and K are two closed proper subsets of S such that $S = H+K$. Then $H \cdot K$ separates S and is, therefore, uncountable. Since every closed and uncountable point set contains a perfect point set, $H \cdot K$ intersects M.

Suppose that M is the sum of two mutually separated point sets U and V. Then $\overline{U}$ and $\overline{V}$ are closed proper subsets of S and $S = \overline{U}+\overline{V}$. Then M intersects $\overline{U} \cdot \overline{V}$ and, hence, U and V are not mutually separated.

THEOREM 2. *If M is a point set such that every perfect point set intersects M and C is a composant of S, then $M \cdot C$ is dense in C and, therefore, is dense in S.*

Proof. Suppose that P is a point of C and D is a domain containing P but no point of $M \cdot C$. Then the component containing P of D contains a nondegenerate continuum K, containing P, and, therefore, K is a perfect point set which does not intersect M.

THEOREM 3. *If M is a point set such that every perfect point set intersects M and H and K are two closed proper subsets of S such that $S = H+K$, then there exist uncountably many composants C of S such that M intersects $C \cdot H \cdot K$.*

Proof. Suppose that Q, the collection of all composants C of S such that M intersects $C \cdot H \cdot K$, is countable. Then $H \cdot K \cdot (S - Q^*)$ is an uncountable inner limiting set and contains a perfect point set which does not intersect M.

THEOREM 4. *Suppose that M is a point set such that, for infinitely many composants C of S, $M \cdot C$ is dense in S. Then there exists a countable dense subset L of M such that no composant of S contains two points of L.*

Proof. Let H denote a countable dense subset of S and let $P_1, P_2, P_3, \ldots$ denote the points of H. For each positive integer i and each positive integer j, let C_{ij} denote a composant of S such that $M \cdot C_{ij}$ is dense in S and $C_{pq} = C_{ij}$ if and only if $p = i$ and $q = j$, and let X_{ij} denote a point of C_{ij} at a distance less than $1/j$ from P_i. Let L denote the set of all points X_{ij} for all positive integers i and j. Then L is a countable dense subset of M such that no composant of S contains two points of L.

THEOREM 5. *There exists a connected point set W such that each composant of S contains one and only one point of W.*

Theorem 5 can be proven by replacing, in the proof of [3], Theorem 1, M by S and (C) by the set of all closed point sets which separate S.

THEOREM 6. *If the continuum hypothesis is true, there exists a connected point set K such that each composant of S contains one and only one point of K and K contains no perfect point set.*

Proof. Let M denote a point set such that every perfect point set intersects both M and $S-M$. That such a point set M exists was shown by F. Bernstein, [1]. Let L denote a countable dense subset of M such that no composant of S contains two points of L, let G denote the collection of all subsets g of L such that g and $L-g$ are mutually separated, and let Q denote the collection of all composants C of S which do not intersect L. Assume that the continuum hypothesis is true. Let a_G denote a meaning of the word precede according to which G is well-ordered and no set of the collection G is preceded by uncountable many sets of G, and let a_Q denote a meaning of the word precede according to which Q is well-ordered and no set of the collection Q is preceded by uncountable many sets of Q. Let β denote a well-ordered sequence of ordered pairs such that, according to the meanings a_G and a_Q of the word precede, (1) if g is the first element of G and C is the first element of Q which contains a point of $M \cdot \bar{g} \cdot (\overline{L-g})$, then (g, C) is the first term of the sequence β; and (2) if β' is an initial segment of the sequence β, then (g, C) is the first term of β which is not a term of β' if and only if g is the first element of G which is not in any ordered pair which is a term of β' and C is the first element of Q which intersects $M \cdot \bar{g} \cdot (\overline{L-g})$ and is not in any ordered pair which is a term of β'. For each term (g, C) of β, let $P(g, C)$ denote a definite point of $C \cdot M \cdot \bar{g} \cdot (\overline{L-g})$ and let K' denote the set of all points of L together with all points $P(g, C)$ for all terms (g, C) of β. No composant of S contains two points of K'. Suppose that K' is the sum of two mutually separated points set U and V. Then $L \cdot U$ and $L \cdot V$ are mutually separated. Let $(L \cdot U, C)$ denote the appropriate term of β. Then $P(L \cdot U, C)$ is a limit point of both U and V and belongs to either U or V, and, thus, U and V are not mutually separated. Then K' is connected. For each composant of S which does not intersect K', if there are any, let X_C denote a definite point of $C \cdot M$. Let K denote K' if K' intersects every composant of S, and, if K' does not intersect every composant of S, let K denote the set of all points of K' together with all points X_C for all composants C of S which do not intersect K'. Since K' is connected and dense in S, K is connected; each composant of S contains one and only one point of K; and, since K is a subset of M, K contains no perfect point set.

$S-K$ is also connected and dense in S.

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References

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