Two invariants under continuity and the incomparability of fans

by

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Introduction

Two invariants under continuous mappings are defined and investigated in this paper (see main theorems T 18, T 19). The first of them, which I call the *degree of non-local connectedness*, is related to the hereditary unicoherence of metric continua, and the second, which I call *uniform arcwise connectedness*, is related to the arcwise connectedness of curves.

As an application of these invariants a construction of arbitrarily numerous finite families of plane fans is given, none of which is a continuous image of another (incomparability under continuity).

Recall that a space H is said to be *unicoherent* provided that it is connected and, for every decomposition $H = A \cup B$ on closed and connected sets, the intersection $A \cap B$ is connected. A space H is said to be *hereditarily unicoherent* provided it is unicoherent and every connected set in it is unicoherent. For continua this definition is equivalent (see [6], Theorem 1.1, p. 179) to the following:

D1. A continuum is *hereditarily unicoherent* if every two points of it can be joined by exactly one irreducible continuum between them.

In the continuation of this paper hereditarily unicoherent continua will be denoted by H, H_1, H_2, etc., and irreducible continua between points x and y by xy. By virtue of D1 they are uniquely determined.

More generally, if X is an arbitrary set in a space $\mathfrak{X}$, $I(X)$ will denote a continuum in $\mathfrak{X}$ irreducible with respect to containing the set X (an irreducible continuum about X), i.e. a continuum which has no proper subcontinuum containing X.

First we prove five theorems on hereditarily unicoherent continua, beginning with the proof of the uniqueness of $I(X)$ for hereditarily unicoherent spaces.

T1. *If H is a hereditarily unicoherent continuum, then for every subset X of it there exists in H exactly one irreducible continuum $I(X)$ about X.*

13*

Indeed, let $X \subset H$. The existence of the continuum $I(X) \subset H$ follows from the compactness of the family of subcontinua of H containing X (see [4], § 38, V, Theorem 1, p. 27).

Suppose that there exists more than one continuum. Let $X \subset I_1(X) \cap \cap I_2(X)$, whence $I_1(X) \cap I_2(X) \neq 0$. Therefore the union $I_1(X) \cup I_2(X)$ is the subcontinuum of the hereditarily unicoherent continuum H. Consequently $I_1(X) \cap I_2(X)$ is a continuum, then $I_1(X) = I_1(X) \cap I_2(X) = I_2(X)$ by the irreducibility of the continua $I_1(X)$ and $I_2(X)$.

T2. *The operation I is monotonic for hereditarily unicoherent continua: if X_1 and X_2 are subsets of a hereditarily unicoherent continuum H, then $X_1 \subset X_2$ implies $I(X_1) \subset I(X_2)$.*

This is a direct consequence of the hereditary unicoherence of the space H and of the irreducibility of its subcontinua $I(X_1)$ and $I(X_2)$.

T3. *If X_1 and X_2 are subsets of a hereditarily unicoherent continuum, then $I(X_1) \cap I(X_2) \neq 0$ implies $I(X_1) \cup I(X_2) = I(X_1 \cup X_2)$.*

Indeed, it follows from T2 that $I(X_1) \cup I(X_2) \subset I(X_1 \cup X_2)$. Inversely, the union of the two continua $I(X_1) \cup I(X_2)$ being a continuum, we have $I(X_1 \cup X_2) \subset I(X_1) \cup I(X_2)$ by the irreducibility of the continuum $I(X_1 \cup X_2)$.

Let $N(X)$ be a set of points of X at which the space X is not locally connected. Thus the equality $N(X) = 0$ is equivalent to the local connectedness of X.

T4. *The operation N is monotonic for hereditarily unicoherent continua: if H_1 and H_2 are hereditarily unicoherent continua, then $H_1 \subset H_2$ implies $N(H_1) \subset N(H_2)$.*

Proof. Suppose that

(1) $$p \in N(H_1) - N(H_2).$$

Thus there exists a neighbourhood $U_1(p)$ of p such that for every neighbourhood $V_1(p) \subset U_1(p)$ the intersection $V_1(p) \cap H_1$ is not connected. Therefore it follows from (1) that there exists a sequence of points $p_n \in U_1(p) \cap H_1$ such that $p = \lim_{n \to \infty} p_n$ and that the points p_n and p belong to different components of the set $U_1(p) \cap H_1$. Since $pp_n \subset H_1$ for $n = 1, 2, ...$, we have

(2) $$pp_n - U_1(p) \neq 0 \quad \text{for sufficiently great } n.$$

The continuum H_2 being, by (1), locally connected at the point p, there exists in every neighbourhood $U_2(p)$ of p a neighbourhood $V_2(p)$ of this point such that

(3) $$V_2(p) \subset U_2(p)$$

and that the intersection $V_2(p) \cap H_2$ is connected. Therefore $\overline{V_2(p) \cap H_2}$ is a subcontinuum of H_2, whence it is also hereditarily unicoherent.

Let us take a neighbourhood $U_2(p)$ of the point p such that

(4) $$\overline{U_2(p)} \subset U_1(p).$$

The continuum $\overline{V_2(p) \cap H_2}$ contains all the points p_n beginning from a sufficiently great n; therefore, by virtue of the hereditary unicoherence, it contains also the continua pp_n, whence by (3) and (4) we have $pp_n \subset V_1(p)$, contrary to (2).

Note that the hypothesis of theorem T4, that H_2 is hereditarily unicoherent, is essential. Indeed, every non-hereditarily locally connected continuum X_2 contains by the definition a non-locally connected subcontinuum X_1. Since $N(X_2) = 0$ and $N(X_1) \neq 0$, the condition $X_1 \subset X_2$ does not imply the condition $N(X_1) \subset N(X_2)$. Of course such a continuum X_2 is not hereditarily unicoherent.

T5. *If H_1, H_2 and $H_1 \cup H_2$ are hereditarily unicoherent continua, then $N(H_1 \cup H_2) = N(H_1) \cup N(H_2)$.*

Proof. By T4 we have $N(H_1) \subset N(H_1 \cup H_2)$ and $N(H_2) \subset N(H_1 \cup H_2)$; thus $N(H_1) \cup N(H_2) \subset N(H_1 \cup H_2)$.

Inversely, we have

(5) $$N(H_1 \cup H_2) \subset N(H_1) \cup N(H_2) \cup (H_1 - H_2) \cup (H_2 - H_1)$$

(see [4], § 44, I, 3, p. 162 taking $H_1 \cup H_2$ as the space, and putting $A_0 = H$ and $A_1 = H_2$). It follows from the definition of the operation N (see p. 188) that $N(H_1 - H_2) \subset N(H_1)$, since the set $H_1 - H_2$ is open in H_1, and that $N(H_1 \cup H_2) \cap (H_1 - H_2) = N(H_1 \cup H_2) \cap N(H_1 - H_2)$, since the set $H_1 - H_2$ is open in $H_1 \cup H_2$. Thus we have $N(H_1 \cup H_2) \cap (H_1 - H_2) \subset N(H_1 \cup H_2) \cap \cap N(H_1) \subset N(H_1)$. Similarly $N(H_1 \cup H_2) \cap (H_2 - H_1) \subset N(H_2)$, whence $N(H_1 \cup H_2) \subset N(H_1) \cup N(H_2)$ by (5).

Now, define for a hereditarily unicoherent continuum $H \neq 0$

D2. $J(H) = I(N(H))$,

and assume for ordinals $a > 0$,

D3. $J^0(H) = H$ and $J^a(H) = \begin{cases} J(J^\beta(H)) & \text{when } a = \beta + 1, \\ \bigcap_{\beta < a} J^\beta(H) & \text{when } a = \lim_{\beta < a} \beta. \end{cases}$

Definitions D2 and D3 in particular immediately imply the following three theorems:

T6. $J^a(J^\beta(H)) = J^{\beta+a}(H)$.

T7. $\min_{a} \{J^{a+1}(H) = 0\} = \lambda + \min_{a} \{J^{\lambda+a+1}(H) = 0\}$

$\qquad \qquad \qquad \qquad \text{for } \lambda \leqslant \min_{a} \{J^{a+1}(H) = 0\}.$

T8. $\beta < a$ implies $J^a(H) \subset J^\beta(H)$.

Note that by T8 and by theorem from [3], § 19, II, 2, p. 146 there is always an α beginning from which we have constantly $J^{\alpha+1}(H) = J^{\alpha}(H)$, and such an α is always the highest countable.

T9. *If*

$$(6) \qquad \xi_1 < \xi_2 < \ldots < \xi_n < \ldots ,$$

$$(7) \qquad \alpha = \lim_{n \to \infty} \xi_n ,$$

then

$$(8) \qquad \operatorname{Lim}_{n \to \infty} J^{\xi_n}(H) = J^{\alpha}(H) .$$

Proof. By T8 and (6) we have $\operatorname{Lim}_{n \to \infty} J^{\xi_n}(H) = \bigcap_{n=1}^{\infty} J^{\xi_n}(H)$ (see [3], § 25, VI, 8, p. 245). The sequence of the continua $J^{\xi_n}(H)$ being decreasing by T8 and (6), we have $\bigcap_{n=1}^{\infty} J^{\xi_n}(H) = \bigcap_{\beta < \alpha} J^{\beta}(H)$ by virtue of (7); then $\operatorname{Lim}_{n \to \infty} J^{\xi_n}(H) = \bigcap_{\beta < \alpha} J^{\beta}(H)$. This implies (8) by D3.

T10. *The operation J^{α} is monotonic for hereditarily unicoherent continua: if H_1 and H_2 are hereditarily unicoherent continua, then $H_1 \subset H_2$ implies*

$$(9) \qquad J^{\alpha}(H_1) \subset J^{\alpha}(H_2)$$

for every countable ordinal α.

Proof. Apply transfinite induction. If $\alpha = 0$, then (9) is true by D3.

Assume (9) for every $\xi < \alpha$. Firstly, let $\alpha = \beta + 1$. Since $J^{\beta}(H_1) \subset J^{\beta}(H_2)$ implies $N(J^{\beta}(H_1)) \subset N(J^{\beta}(H_2))$ by T4, putting in T2 $N(J^{\beta}(H_1)) = X_1$ and $N(J^{\beta}(H_2)) = X_2$ we have $I(N(J^{\beta}(H_1))) \subset I(N(J^{\beta}(H_2)))$, i.e. $J^{\alpha}(H_1) \subset J^{\alpha}(H_2)$ by D3.

Secondly, let $\alpha = \lim_{n \to \infty} \xi_n$, where ξ_n satisfy (6). Then $J^{\xi_n}(H_1) \subset J^{\xi_n}(H_2)$ for every $n = 1, 2, \ldots$ Thereby $\operatorname{Lim}_{n \to \infty} J^{\xi_n}(H_1) \subset \operatorname{Lim}_{n \to \infty} J^{\xi_n}(H_2)$ (see [3], § 25, VI, 2, p. 245); thus $J^{\alpha}(H_1) \subset J^{\alpha}(H_2)$ by T9.

The degree $\tau(H)$ of the non-local connectedness

Introduce the following notion:

D4. By the *degree of the non-local connectedness* of a hereditarily unicoherent continuum H we understand the value

$$\tau(H) = \begin{cases} \min_{\alpha}\{J^{\alpha+1}(H) = 0\}, \\ \infty \text{ when such ordinals } \alpha \text{ do not exist.} \end{cases}$$

For instance, H being the Cantor fan (see [1], E2, p. 240), we have $\tau(H) = \infty$.

Recall that a locally connected continuum is said to be a *dendrite* provided that it contains no simple closed curve. It is easy to see that this definition is equivalent to the following: a dendrite is a locally connected and hereditarily unicoherent continuum. Thus from D4 follows immediately

T11. $\tau(H) = 0$ *if and only if H is a dendrite.*

Further we have (by substitution) from T7, T6 and D4

T12. $\tau(H) = \lambda + \tau(J^{\lambda}(H))$ *for $\lambda \leqslant \tau(H)$.*

Now we prove

T13. *The function $\tau(H)$ is monotonic for hereditarily unicoherent continua: if H_1 and H_2 are hereditarily unicoherent continua, then $H_1 \subset H_2$ implies $\tau(H_1) \leqslant \tau(H_2)$.*

Proof. Suppose that $\tau(H_2) < \tau(H_1)$. By T12 for $H_1 = H$ and $\tau(H_2) = \lambda$ we have $\tau(H_1) = \tau(H_2) + \tau(J^{\tau(H_2)}(H_1))$. Thus $\tau(J^{\tau(H_2)}(H_1)) > 0$, i.e. $J^{\tau(H_2)}(H_1)$ is not locally connected by T11. Further, by taking in T12 $H_2 = H$ and $\tau(H_2) = \lambda$ we obtain $\tau(J^{\tau(H_2)}(H_2)) = 0$, whence by T11 it follows that $J^{\tau(H_2)}(H_2)$ is a dendrite. However, we conclude from T10 in particular for $\alpha = \tau(H_2)$ that $J^{\tau(H_2)}(H_1) \subset J^{\tau(H_2)}(H_2)$, contrary to the hereditary local connectedness of the dendrite $J^{\tau(H_2)}(H_2)$.

Invariability of $\tau(H)$. It is known (see [2], (3), p. 28) that

T14. *If X is a compact space and f is a continuous mapping, then*

$$N(f(X)) \subset f(N(X)) .$$

Let f be a continuous mapping of a hereditarily unicoherent continuum H onto a hereditarily unicoherent continuum H_1.

T15. *If $X \subset H$, $X_1 \subset H_1$ and*

$$(10) \qquad X_1 \subset f(X) ,$$

then $I(X_1) \subset f(I(X))$.

Indeed, since $X \subset I(X)$, we have $f(X) \subset f(I(X))$, and thus $X_1 \subset f(I(X))$ by (10). It remains to apply the irreducibility of the continuum $I(X_1)$.

T16. $J(f(J^{\alpha}(H))) \subset f(J^{\alpha+1}(H))$.

In fact, by T14 for $J^{\alpha}(H) = X$ we have $N(f(J^{\alpha}(H))) \subset f(N(J^{\alpha}(H)))$. The sets $N(f(J^{\alpha}(H))) = X_1 \subset H_1$ and $N(J^{\alpha}(H)) = X \subset H$ satisfy the hypotheses of T15. Thus it suffices to refer to D2 and D3.

T17. $J^{\alpha}(f(H)) \subset f(J^{\alpha}(H))$.

Proof. Apply transfinite induction. If $\alpha = 0$, then the theorem is true by D3. Assume T17 for every $\xi < \alpha$. Firstly, let $\alpha = \beta + 1$. Thus

$J^\beta(f(H)) \subset f(J^\beta(H))$, whence $J^{\beta+1}(f(H)) \subset J(f(J^\beta(H)))$ by T10 and D3. Therefore we have $J^{\beta+1}(f(H)) \subset f(J^{\beta+1}(H))$ by T16.

Secondly, let $\alpha = \lim_{n\to\infty} \xi_n$, where ξ_n satisfy (6). Then

(11) $\qquad J^{\xi_n}(f(H)) \subset f(J^{\xi_n}(H)) \quad$ for every $\quad n = 1, 2, \ldots,$

whence

(12) $$\operatorname*{Lim}_{n\to\infty} J^{\xi_n}(f(H)) = J^\alpha(f(H))$$

by T9. Since the sequence of the continua $J^{\xi_n}(H)$ is decreasing by T8 and (6), we have $\operatorname*{Lim}_{n\to\infty} f(J^{\xi_n}(H)) = \bigcap_{n=1}^{\infty} f(J^{\xi_n}(H)) = f(\bigcap_{n=1}^{\infty} J^{\xi_n}(H)) = f(\bigcap_{\xi<\alpha} J^\xi(H))$. Therefore

(13) $$\operatorname*{Lim}_{n\to\infty} f(J^{\xi_n}(H)) = f(J^\alpha(H))$$

by D3. It follows by (11) (see [3], § 25, VI, 2, p. 245) that $\operatorname*{Lim}_{n\to\infty} J^{\xi_n}(f(H)) \subset \operatorname*{Lim}_{n\to\infty} f(J^{\xi_n}(H))$; thus it is sufficient to refer to (12) and (13).

T17 immediately implies by definition D4 of $\tau(H)$ the following

T18 (First main theorem). *If H is a hereditarily unicoherent continuum and f is a continuous mapping of H onto a hereditarily unicoherent continuum $f(H)$, then $\tau(f(H)) \leqslant \tau(H)$.*

Remarks. A problem arises whether we can assign to an arbitrary continuum K, or even only to an arbitrary curve, a number $\nu(K)$ which would characterise the non-local connectedness of the continuum K and which would not increase under continuous mappings.

Another problem is whether we can find a number $\nu(K)$ which has the previous properties and, moreover, is such that $\nu(K) = \tau(K)$ for the hereditarily unicoherent K. Hence this would be a generalization of the function $\tau(H)$ defined only for hereditarily unicoherent continua.

It seems that one cannot use the notion of $I(X)$ in the definition of number $\nu(K)$, because $I(X)$ is a multivalued operation for non-hereditarily unicoherent continua. However, one cannot simply omit the operation $I(X)$ in the definition of the operation $J(X)$ (thus—indirectly—in the definition of the number $\tau(X)$); neither can one change it by the operation of the closure. It has been suggested to me, for instance, that we should take

$$N_1^0(X) = X, \quad N_1^\alpha(X) = \begin{cases} N(N_1^\beta(X)) & \text{when} \quad \alpha = \beta+1, \\ \bigcap_{\beta<\alpha} N_1^\beta(X) & \text{when} \quad \alpha = \lim_{\beta<\alpha}\beta, \end{cases}$$

or

$$N_2^0(X) = X, \quad N_2^\alpha(X) = \begin{cases} \overline{N(N_2^\beta(X))} & \text{when} \quad \alpha = \beta+1, \\ \bigcap_{\beta<\alpha} N_2^\beta(X) & \text{when} \quad \alpha = \lim_{\beta<\alpha}\beta, \end{cases}$$

and for $i = 1$ or 2

$$\nu_i(X) = \begin{cases} \min_\alpha \{N_i^{\alpha+1}(X) = 0\}, \\ \infty \text{ when such ordinals } \alpha \text{ do not exist.} \end{cases}$$

Unfortunately there exist continua X (even already in the domain of hereditarily unicoherent continua) and their continuous images $f(X)$ such that $\nu_i(X) < \nu_i(f(X))$ for $i = 1$ and 2. Hence it follows that values $\nu_i(H)$ defined in this manner differ from $\tau(H)$. There exists even a hereditarily unicoherent continuum H' such that $\tau(H') = 2$ and $\nu_1(H') = \nu_2(H') = \infty$.

Uniform arcwise connectedness

Introduce the following notion:

D5. A point set X is said to be *uniformly arcwise connected* if it is arcwise connected and if for every number $\eta > 0$ there is a number k such that every arc $A \subset X$ contains points $a_0, a_1, \ldots, a_k$ such that

(14) $$A = \bigcup_{i=0}^{k-1} a_i a_{i+1},$$

(15) $\qquad \delta(a_i a_{i+1}) < \eta \quad$ for every $\quad i = 0, 1, \ldots, k-1.$

The following corollaries arise immediately from D5:

C1. *If an arcwise connected X contains a sequence of rectifiable arcs $\{A_j\}$ whose lengths $\{l(A_j)\}$ are not bounded in common, and if there exist positive numbers ε and η such that*

(16) $\qquad l(A_j) \geqslant \varepsilon \quad implies \quad \delta(A_j) \geqslant \eta \quad for \quad j = 1, 2, \ldots,$

then X is not uniformly arcwise connected.

Indeed, there exists by hypothesis, for an arbitrarily great number k, an index j such that $l(A_j) > k\varepsilon$. Thus, for every $k+1$ points $a_0, a_1, \ldots, a_k$ of A_j satisfying (14) with $A = A_j$, there exists an index i such that $l(a_i a_{i+1}) \geqslant \varepsilon$, whence the negation of (15) follows by (16).

C2. *If all arcs of an arcwise connected continuum X are rectifiable and if they have the lengths bounded in common, then X is uniformly arcwise connected.*

C3. *If a continuum X is uniformly arcwise connected, then every arcwise connected subcontinuum of X is also uniformly arcwise connected* (the heredity of uniform arcwise connectedness for continua).

D6. A space X is said to be *one-arcwise connected* if X is arcwise connected and if for every two points x and y of X it contains exactly one arc xy joining these points.

T19 (Second main theorem). *If X is a uniformly arcwise connected continuum and if f is a continuous mapping of X onto a one-arcwise connected continuum $f(X)$, then $f(X)$ is also uniformly arcwise connected.*

Proof. Take an arbitrary $\varepsilon > 0$. The mapping f being uniformly continuous, there exists for this ε a number $\eta > 0$ such that

(17) for every two points x_1 and x_2 of X the condition $\varrho(x_1, x_2) < \eta$
   implies the condition $\varrho(f(x_1), f(x_2)) < \varepsilon$.

From the hypothesis that the continuum X is uniformly arcwise connected and from D5 it follows that there is for this η a number k such that every arc $A \subset X$ contains points $a_0, a_1, ..., a_k$ which satisfy (14) and (15).

Take an arbitrary arc $B \subset f(X)$, and let b and b' be end-points of this arc.

Further, let

(18) $a \in f^{-1}(b)$ and $a' \in f^{-1}(b')$

be points of X such that there is in the arc aa' no other point of the inverse image of the set $(b) \cup (b')$, i.e. $aa' \cap f^{-1}(b \cup b') = (a) \cup (a')$.

Put $aa' = A$. Thus it follows from (18) that $B \subset f(A)$. By virtue of the one-arcwise connectedness of $f(X)$

(19) $f(A)$ is a dendrite,

whence we conclude that the set $f(a_i a_{i+1}) \cap B$ is a continuum as an intersection of two subcontinua contained in $f(A)$, and therefore it is an arc as a subcontinuum of the arc B.

Let $k' \leqslant k$ be the number of those arcs $a_i a_{i+1}$ for which

(20) $f(a_i a_{i+1}) \cap B \neq 0$.

Take into consideration the set of the end-points of arcs (20). Since there are k' such arcs, the number of their end-points is finite. Let k'' be that number. We can, of course, set those end-points in a sequence $b_0, b_1, ..., b_j, ..., b_{k''}$ so that

(i) b_0 and b_1 are the end-points of that one of the arcs (20) for which the index i is the smallest,

(ii) if b_{j-1} and b_j are the end-points of a certain arc (20) with $i = i_1$, then b_j and b_{j+1} are the end-points of an arc $f(a_{i_2} a_{i_2+1}) \cap B$ such that $i_1 < i < i_2$ implies $f(a_i a_{i+1}) \cap B = 0$.

The continuum X being by hypothesis uniformly arcwise connected, we have by virtue of (i), (ii) and (14)

$$B = \bigcup_{j=0}^{k''-1} b_j b_{j+1}.$$

Further, from (15) and (17) we have $\delta(f(a_i a_{i+1})) < \varepsilon$; thus $\delta(b_j b_{j+1}) < \varepsilon$ and, consequently, $f(X)$ is uniformly arcwise connected.

Applications
Families of n incomparable plane fans for each $n = 2, 3, ...$

Recall that a space is said to be *connected between the sets A and B* provided that it contains a set $F = \overline{F}$ such that the set $\overline{A} \cap F \cup F \cup F \cap \overline{B}$ is connected.

A space is said to be *irreducibly connected between A and B* provided that it is connected between A and B, and that no proper subset of it, $A \cup F \cup B$, where $F = \overline{F}$, is connected between A and B (see [4], § 43, VIII, p. 156). Consequently, if a set L is irreducibly connected between the sets A and B, then, considered as a space, it is connected and contains these sets which are non-empty and separated ([4], 1, p. 156).

D7. Let $L(A, B)$ be a set irreducibly connected between $L(A, B) \cap A$ and $L(A, B) \cap B$. If A and B are empty or non-separated, put $L(A, B) = 0$.

If X, $A \subset X$ and $B \subset X$ are continua, $L(A, B)$ is an irreducible continuum about the set $L(A, B) \cap A \cup L(A, B) \cap B$ (see [4], § 43, IX, 2, p. 159):

(21) $L(A, B) = I(L(A, B) \cap A \cup L(A, B) \cap B)$.

T20. *If H_1 and H_2 are subcontinua of a hereditarily unicoherent continuum H, then there is in H only one continuum $L(H_1, H_2)$.*

The existence of $L(H_1, H_2)$ follows from [4], § 43, IX, 1, p. 158, and its uniqueness is a direct consequence of T1 and (21).

T21. *If X_1 and X_2 are subsets of a hereditarily unicoherent continuum, then*

$$I(X_1) \cup L(I(X_1), I(X_2)) \cup I(X_2) = I(X_1 \cup X_2).$$

Indeed, if $I(X_1) \cap I(X_2) \neq 0$, this equality follows from T3. In the other case we have $L(I(X_1), I(X_2)) \cap I(X_1) \neq 0 \neq L(I(X_1), I(X_2)) \cap I(X_2)$ by (21); hence $I(X_1) \cup L(I(X_1), I(X_2)) \cup I(X_2)$ is a continuum containing X_1 and X_2 by the definition of the operation I. Thus $I(X_1 \cup X_2) \subset I(X_1) \cup L(I(X_1), I(X_2)) \cup I(X_2)$ by the irreducibility of $I(X_1 \cup X_2)$. Inversely, $I(X_1) \cup I(X_2) \subset I(X_1 \cup X_2)$ by T2, whence $L(I(X_1), I(X_2)) \subset I(X_1 \cup X_2)$ by T20. Thus $I(X_1) \cup L(I(X_1), I(X_2)) \cup I(X_2) \subset I(X_1 \cup X_2)$.

T22. *Let H_1 and H_2 be subcontinua of a hereditarily unicoherent continuum H. Let $a \in H_1$, $b \in H_2$ and let $K \subset H$ be an irreducible continuum between a and b. Then $L(H_1, H_2) \subset K$.*

In fact, the continuum K is connected between H_1 and H_2 (see [4], § 41, IV, 7, p. 91). Thus there exists in K a continuum C irreducibly connected between $C \cap H_1$ and $C \cap H_2$ (see [4], § 43, IX, 1, p. 158). Since the continuum $K \subset H$ is hereditarily unicoherent, it follows that $C = L(H_1, H_2)$ by T20.

T23. *If $\{A_n\}$ and $\{B_n\}$ are decreasing sequences of subcontinua of a hereditarily unicoherent continuum and if*

$$(22) \qquad A = \operatorname*{Lim}_{n \to \infty} A_n \quad and \quad B = \operatorname*{Lim}_{n \to \infty} B_n \,,$$

then

$$(23) \qquad L(A, B) = \operatorname*{Lim}_{n \to \infty} L(A_n, B_n) \,.$$

Proof. The continuum $L(A, B)$ is irreducible between every pair of points $a \in L(A, B) \cap A$, $b \in L(A, B) \cap B$ as a continuum irreducibly connected between these sets (see [4], § 43, IX, 2, p. 159). Similarly, the continuum $L(A_n, B_n)$ is irreducible between every pair of points

$$(24) \qquad a_n \in L(A_n, B_n) \cap A_n \,, \quad b_n \in L(A_n, B_n) \cap B_n \,.$$

Let

$$(25) \qquad a = \lim_{n \to \infty} a_n \quad and \quad b = \lim_{n \to \infty} b_n \,.$$

Since by (22)

$$(26) \qquad A = \bigcap_{n=1}^{\infty} A_n \quad and \quad B = \bigcap_{n=1}^{\infty} B_n$$

(see [3], § 25, VI, 8, p. 245), the sets $A_n = H_1$, $B_n = H_2$ and $L(A, B) = K$ satisfy the hypotheses of T22 for $n = 1, 2, \ldots$ Thus

$$(27) \qquad L(A_n, B_n) \subset L(A, B) \quad for \quad n = 1, 2, \ldots$$

By the irreducibility of $L(A, B)$ between a and b and of $L(A_n, B_n)$ between a_n and b_n equality (23) follows from (25) and (27).

Now recall the definitions and some properties of dendroids (see [1], p. 239 and [5], p. 301) and fans.

D8. A *dendroid* is an arcwise connected and hereditarily unicoherent continuum.

This definition is equivalent by D1 to the following

D9. A *dendroid* is an arcwise connected continuum every two distinct points of which can be joined by exactly one irreducible continuum between them.

The following two propositions are contained in paper [1], p. 239 and 240:

T24. *Every dendroid is a curve* (i.e. a continuum of dimension 1).

T25. *Every subcontinuum of a dendroid is a dendroid.*

Adopt the following definition:

D10. A continuum X is said to be *hereditarily arcwise connected* provided that every subcontinuum of X (thus also the whole X) is arcwise connected.

T26. *In order that a continuum be a dendroid it is necessary and sufficient that it be one-arcwise connected and hereditarily arcwise connected.*

Proof. If a continuum K is a dendroid, then it is one-arcwise connected by virtue of D9, and it is hereditarily arcwise connected by virtue of D8 and T25.

Inversely, suppose that a hereditarily arcwise connected continuum K is not a dendroid. Thus it contains by D9 two different irreducible continua N_1 and N_2 between the same points a and b. In consequence of the hereditary arcwise connectedness of K these continua are arcwise connected; therefore by virtue of their irreducibility they are arcs. Thus points a and b can be joined in K by two different arcs N_1 and N_2. Hence K is not one-arcwise connected.

From T26 and T19 follows the invariability of uniform arcwise connectedness under continuous mappings of dendroids onto dendroids:

C4. *If f is a continuous mapping of a uniformly arcwise connected dendroid Δ onto a dendroid $f(\Delta)$, then $f(\Delta)$ is also uniformly arcwise connected.*

T27. *If Δ_1 and Δ_2 are subcontinua of a dendroid Δ, then $L(\Delta_1, \Delta_2)$ is an arc.*

Indeed, the continuum $L(\Delta_1, \Delta_2)$ is irreducible between every pair of points $a \in L(\Delta_1, \Delta_2) \cap \Delta_1$, $b \in L(\Delta_1, \Delta_2) \cap \Delta_2$ (see loco cit. [4], § 43, IX, 2, p. 159). There is in Δ only one irreducible continuum between a and b, namely the arc ab by D8. Thus $L(\Delta_1, \Delta_2) = ab$.

T28. *If Δ_1 and Δ_2 are subcontinua of a dendroid Δ, then*

$$(28) \qquad \Delta_1 \cap \Delta_2 \neq 0$$

implies

$$(29) \qquad J^a(\Delta_1 \cup \Delta_2) = J^a(\Delta_1) \cup L\big(J^a(\Delta_1), J^a(\Delta_2)\big) \cup J^a(\Delta_2)$$

for every countable ordinal a.

Proof. Note that $\Delta_1 \cup \Delta_2$ is a continuum by (28), whence it is a dendroid by T25. Thus, by D8, $J^a(\Delta_1 \cup \Delta_2)$ exists (see D3).

Apply transfinite induction. If $a = 0$, then $J^a(\Delta_1) = \Delta_1$, $J^a(\Delta_2) = \Delta_2$ and $J^a(\Delta_1 \cup \Delta_2) = \Delta_1 \cup \Delta_2$ by D3, whence $L\big(J^a(\Delta_1), J^a(\Delta_2)\big) = 0$ by (28) and D7; thus (29) is true.

Assume (29) for every $\xi < \alpha$. Firstly, let $\alpha = \beta + 1$. Since $N\big(L\big(J^\beta(\varDelta_1),$ $J^\beta(\varDelta_2)\big)\big) = 0$ by T27, we have $N\big(J^\beta(\varDelta_1 \cup \varDelta_2)\big) = N\big(J^\beta(\varDelta_1)\big) \cup N\big(J^\beta(\varDelta_2)\big)$ by T5. Thus $I\big(N\big(J^\beta(\varDelta_1 \cup \varDelta_2)\big)\big) = I\big(N\big(J^\beta(\varDelta_1)\big)\big) \cup L\big(I\big(N\big(J^\beta(\varDelta_1)\big)\big), I\big(N\big(J^\beta(\varDelta_2)\big)\big)\big) \cup$ $\cup\, I\big(N\big(J^\beta(\varDelta_2)\big)\big)$ by T21. Therefore $J^{\beta+1}(\varDelta_1 \cup \varDelta_2) = J^{\beta+1}(\varDelta_1) \cup L\big(J^{\beta+1}(\varDelta_1),$ $J^{\beta+1}(\varDelta_2)\big) \cup J^{\beta+1}(\varDelta_2)$ by D2 and D3.

Secondly, let $\alpha = \lim\limits_{n \to \infty} \xi_n$, where ξ_n satisfy (6). Then $J^{\xi_n}(\varDelta_1 \cup \varDelta_2)$ $= J^{\xi_n}(\varDelta_1) \cup L\big(J^{\xi_n}(\varDelta_1), J^{\xi_n}(\varDelta_2)\big) \cup J^{\xi_n}(\varDelta_2)$ for every $n = 1, 2, \ldots$, whence $\operatorname*{Lim}\limits_{n \to \infty} J^{\xi_n}(\varDelta_1 \cup \varDelta_2) = \operatorname*{Lim}\limits_{n \to \infty} J^{\xi_n}(\varDelta_1) \cup \operatorname*{Lim}\limits_{n \to \infty} L\big(J^{\xi_n}(\varDelta_1), J^{\xi_n}(\varDelta_2)\big) \cup \operatorname*{Lim}\limits_{n \to \infty} J^{\xi_n}(\varDelta_2)$. Thus, the sequences $\{J^{\xi_n}(\varDelta_1)\}$ and $\{J^{\xi_n}(\varDelta_2)\}$ being decreasing by T8, (29) follows from T9 and T23.

T29. *If $\varDelta_1$ and $\varDelta_2$ are subcontinua of a dendroid $\varDelta$ and if $\varDelta_1 \cap \varDelta_2 \neq 0$, then*

$$(30) \qquad \tau(\varDelta_1 \cup \varDelta_2) = \max\{\tau(\varDelta_1), \tau(\varDelta_2)\}.$$

Proof. Note that we can prove in the same way as at the beginning of the proof of T28 that $J^\alpha(\varDelta_1 \cup \varDelta_2)$ exists, whence $\tau(\varDelta_1 \cup \varDelta_2)$ is determined.

If $\tau(\varDelta_1) = \infty$ or $\tau(\varDelta_2) = \infty$, then (30) holds by T13.

If $\tau(\varDelta_1) \neq \infty \neq \tau(\varDelta_2)$, let

$$(31) \qquad \tau(\varDelta_1) \leqslant \tau(\varDelta_2).$$

By D4 we have $J^\alpha(\varDelta_1) = 0$ for $\alpha > \tau(\varDelta_1)$; thus $J^\alpha(\varDelta_1 \cup \varDelta_2) = J^\alpha(\varDelta_2)$ by T28 and D7. Hence it follows by D4 that $\tau(\varDelta_1 \cup \varDelta_2) = \tau(\varDelta_2)$ and (30) holds by (31).

Further, recall the definitions of an end-point and a ramification point in the classical sence in arcwise connected continua (see [1], p. 230 and [5], p. 301).
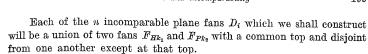
A point p of an arcwise connected continuum X is called an *end-point of X in the classical sense* if p is an end-point of every arc contained in X and containing p. The set of all end-points of X in this sense will be denoted by $E(X)$.

A point p of an arcwise connectet continuum X is called a *ramification point of X in the classical sense* if p is a common end-point of at least three arcs disjoint from one another beyond p and contained in X.

Henceforward the words "in the classical sense" will be omitted.

D11. A *fan* signifies a dendroid which has only one ramification point. Call this point *top*.

For some examples of fans, in particular the harmonic fan and the Cantor fan, see [1], p. 240.

Each of the n incomparable plane fans D_i which we shall construct will be a union of two fans F_{Hk_1} and F_{Pk_2} with a common top and disjoint from one another except at that top.

Call, for shortness, a sequence of points k-*harmonic* if it arises from a $(k-1)$-harmonic sequence by the inscription of the whole harmonic sequence suitably diminished and tending to the second of every two consecutive points of it (meaning here by harmonic sequence, and also by 1-harmonic sequence the sequence $\{2^{-n}\}_{n=1,2,\ldots}$). It is clear how one ought to understand, according to this, such names as a k-harmonic sequence of segments and others. Of course, the greatest order of the non-empty derivative of a k-harmonic sequence is equal to k.

Now F_{Hk_1} will consist of a k_1-harmonic sequence of straight segments with length 1, and it will have a finite degree of the non-local connectedness $\tau(F_{Hk_1}) = k_1$, i.e. a degree equal to the greatest order of non-empty derivative of the set of the end-points of F_{Hk_1}. Since in the fan F_{Hk_1} defined in this way all the arcs have the length at most 2, this fan will be uniformly arcwise connected by virtue of C2.

F_{Pk_2} will consist of the fan F_{Hk_2} and of polygonal lines whose lengths are finite but infinitely increasing and which have only one end-point in common with F_{Hk_2}. These polygonal lines will be subjoined to all those end-points of the fan F_{Hk_2} in which it is locally connected, and only to those end-points. Thus by C1 the fan F_{Pk_2} will not be uniformly arcwise connected.

Construction. Generally let $\overline{ab}$ denote a straight segment with end-points a and b.

Suppose we are given in the Euclidean plane E^2 a system of polar coordinates ϱ, φ with the pole at the point O.

For every $k = 1, 2, \ldots$ arrange in a sequence all systems (i.e. the arbitrarily ordered sets) consisting of k natural numbers. Let

$$(32) \qquad k; s = n_1, n_2, \ldots, n_k$$

be an s-th term of this sequence for any fixed k. Put

$$(33) \qquad \varphi_{k;s} = 2^{-n_1} + 2^{-(n_1+n_2)} + \ldots + 2^{-(n_1+n_2+\ldots n_k)},$$

$$(34) \qquad p_{k;s} = (1, \varphi_{k;s}).$$

Let the set A_0 consist of only one point, $(1, 0)$, i.e.

$$(35) \qquad A_0 = \{(1, 0)\},$$

and let for $j = 1, 2, \ldots, k$ the set A_j consist of A_{j-1} and of all points of the sequence $p_{j;s}$ where $s = 1, 2, \ldots$ Denote by B_j the set of those points, i.e.

$$(36) \qquad B_j = \{p_{j;s}\colon s = 1, 2, \ldots\}.$$

Thus by (33) and (34)

$$(37) \qquad B_j = \bigcup_{n_1=1}^{\infty} \bigcup_{n_2=1}^{\infty} ... \bigcup_{n_j=1}^{\infty} \{(1, \varphi) \colon \varphi = 2^{-n_1} + 2^{-(n_1+n_2)} + ... + 2^{-(n_1+n_2+...+n_j)}\}$$

and

$$(38) \qquad A_j = A_{j-1} \cup B_j .$$

It is easy to see that $A_{j-1} \cap B_j = 0$; therefore

$$(39) \qquad B_j = A_j - A_{j-1} .$$

Hence the $k+1$ sets A_j form a finite sequence

$$A_0 \subset A_1 \subset A_2 \subset ... \subset A_k$$

such that the derivative of the set A_j (in the sense of the theory of sets) is exactly the set A_{j-1} for $j = 1, 2, ..., k$:

$$(40) \qquad A'_j = A_{j-1} \quad \text{for} \quad j = 1, 2, ..., k .$$

Thus the last non-empty derivative of the set A_k has the order k and is equal to the set A_0:

$$(41) \qquad A_k^{(k)} = A_0 .$$

A_0 consisting of only one point by (35) and B_j being countable by (36) it is quite obvious that

$$(42) \qquad A_k \text{ is countable for every natural } k.$$

Join the origin O with every point $p \in A_k$ by a straight segment and put

$$(43) \qquad F_{Hk} = \bigcup_{p \in A_k} \overline{Op} .$$

Hence F_{Hk} is a fan with the point O as its top and with

$$(44) \qquad E(F_{Hk}) = A_k .$$

Consequently, the straight segment $\overline{Op_{j-1;s}}$ without the point O, i.e. the segment whose end-point $p_{j-1;s}$, as belonging to $A_{j-1} - A_{j-2}$ by (39) and (36), is by (40) a limit point of the set A_j, consist of points of the non-local connectedness of the fan F_{Hj}.

It follows that $(F_{Hj} - F_{Hj-1}) \cup (O)$ is exactly the set of points of the local connectedness of the fan F_{Hj}, and by virtue of D2 that

$$(45) \qquad J(F_{Hj}) = F_{Hj-1} \quad \text{for} \quad j = 1, 2, ..., k,$$

where F_{H0} denotes, according to (35), the straight segment $\overline{O(1, 0)}$. Hence $J^j(F_{Hk}) = F_{Hk-j}$ and, in particular, for $j = k$ we have $J^k(F_{Hk}) = F_{H0} = O(1, 0)$; thus $J^{k+1}(F_{Hk}) = 0$. It follows by D4 that

$$(46) \qquad \tau(F_{Hk}) = k .$$

Since every straight segment $\overline{Op}$ in (43) has length equal to 1, all the arcs contained in F_{Hk} have length at most 2; therefore by virtue of C2

$$(47) \qquad F \text{ is uniformly arcwise connected.}$$

Now we define F_{Pk}. Take the points of the set $A_k - A_{k-1}$ which is, by (39) and (40), a set of isolate points of the set A_k. Assign to every point $p_{k;s}$ of this set a system of n_k+1 points $p_{k;s,i}$ of the circumference $\varrho = 1$ with arguments

$$\varphi_{k;s,i} = \varphi_{k;s} - \frac{i}{n_k} \cdot 2^{-(n_1+n_2+...+n_k+2)},$$

where $i = 0, 1, ..., n_k$ and n_k is the last term of system (32). These points will be successive odd vertices of a polygonal line beginning from the point $p_{k;s,0} = p_{k;s}$ as its first vertex. Let the points $q_{k;s,i} = (2^{-1}, \varphi_{k;s,i})$, where $i = 1, 2, ..., n_k$, be the successive even vertices of this polygonal line. In this manner to every point $p_{k;s} \in A_k - A_{k-1}$ is assigned a polygonal line

$$(48) \qquad P_{k;s} = \bigcup_{i=1}^{n_k} \overline{p_{k;s,i-1} q_{k;s,i}} \cup \bigcup_{i=1}^{n_k} \overline{p_{k;s,i} q_{k;s,i}}$$

starting from that point and consisting of $2n_k$ straight segments with total length greater than n_k.

Put

$$(49) \qquad F_{Pk} = F_{Hk} \cup \bigcup_{s=1}^{\infty} P_{k;s} .$$

F_{Pk} defined in this way is a fan with the top O. $E(F_{Pk})$ consist of the points $p_{k;s,n_k}$ by (48) and (49), and of the end-points of F_{Hk-1}, i.e. of points of the set A_{k-1} by (45) and (44). Thus

$$E(F_{Pk}) = E_k \cup A_{k-1},$$

where

$$(50) \qquad E_k = \bigcup_{s=1}^{\infty} p_{k;s,n_k},$$

and n_k is given by (32). Obviously the set $E(F_{Pk})$ is countable.

According to (46)

$$(51) \qquad \tau(F_{Pk}) = k .$$

Since the length of every polygonal line $P_{k;s}$ is greater than n_k and $n_k \to \infty$ together with $s \to \infty$, it follows by C1, for $\varepsilon = 1,5$ and $\eta = 0,5$, that

$$(52) \qquad F_{Pk} \text{ is not uniformly arcwise connected.}$$

Denote by F_{Pk}^* a fan symmetric to F_{Pk} with respect to point O and put for an arbitrary natural number $n > 1$ and for every $i = 0, 1, ..., n-1$

$$(53) \qquad D_i = F_{Hn+i} \cup F_{Pn-i}^* .$$

Hence by T29

(54) $$\tau(D_i) = n + i \,.$$

Two mapping properties. In order to show the incomparability of the fans D_i, consider firstly the Cantor discontinuum C on the arc $0 \leqslant \varphi \leqslant 1$ of the circumference $\varrho = 1$, i.e. the set of points $p = (1, \varphi)$ where $\varphi = \sum\limits_{l=0}^{\infty} 2c_l/3^l$ and $c_l = 0$ or 1. The union

$$F_C = \bigcup_{p \in C} \overline{Op}$$

is a fan homeomorphic with the Cantor fan. By C2

(55) F_C is uniformly arcwise connected.

Since the set A_k defined by (35) and (38) has dimension 0 by virtue of (42), it can be homeomorphically imbedded in C (see [3], § 21, IV, Theorem VI, p. 173). Consequently, the fan F_{Hk} consisting by (36), (38) and (43) of straight segments $\overline{Op_{j;s}}$ where $j = 0, 1, \ldots, k$, it is easy to see that there exists a homeomorphism

(56) $h\colon F_{Hk} \to F_C$,

e.g. in such a manner that $h(A_k) \subset C$.

Further, let $F \subset F_C$ be an arbitrary subcontinuum of the fan F_C containing the top O. Hence $F \cup F^*_{Pk}$ is a fan.

Now the continuous mappings of dendroids onto the fans F_{Pk} have the following two properties:

P1. *If f is a continuous mapping of a dendroid X containing F onto F_{Pk}, then for every subset S of $E_k \subset E(F_{Pk})$ satisfying the inequality*

(57) $F \cap f^{-1}(y) \neq 0$ *for every* $y \in S$,

there exists a number m such that $n_k < m$ for every $y \in S$.

Proof. Recall that the points $y \in S \subset E_k$ are by (50) of the form $y = p_{k;s,n_k}$ and that n_k, defined by (32), is by (48) a half of the number of straight segments forming the polygonal line $P_{k;s}$ of F_{Pk}.

Suppose that there exists a set $S \subset E_k$ satisfying (57) but containing a sequence of points $\{y_r\}$ for which $n_k(r) > m$ no matter how great m is. Thus the irreducible continuum $I(S)$ contains a sequence of polygonal lines $\{P_{k;s,r}\}_{r=1,2,\ldots}$ with lengths infinitely increasing. As for the whole F_{Pk} by (52), we hence state that

(58) $I(S)$ is not uniformly arcwise connected.

However, the dendroid $F \subset F_C$ is uniformly arcwise connected by (55) and C3. Thus the dendroid $f(F) \subset F_{Pk}$ is also uniformly arcwise

connected by C4, and thereby likewise $I(S)$ by C3, since $I(S)$ is a dendroid and $I(S) \subset f(F)$. But this contradicts (58).

P2. *If f is a continuous mapping of $F \cup F^*_{Pk_1}$ onto F_{Pk_2}, then $k_2 \leqslant k_1$.*

Proof. Let E' be the set of all points $y \in E_{k_2} \subset F_{Pk_2}$ for which $F \cap f^{-1}(y) \neq 0$, and let $E'' = E_{k_2} - E'$. Thus

(59) $f^{-1}(E'') \subset F^*_{Pk_1}$.

By virtue of property P1 recently proved the set E' contains no point set S with boundless n_k. Therefore all such sets are contained in E''. Thus the continuum $I(E'')$ contains all limit points of these sets. Consequently $N(F_{Pk_2}) \subset I(E'')$. Hence by D4 and (51)

(60) $\tau\big(I(E'')\big) = \tau(F_{Pk_2}) = k_2$.

Further, it follows from (59) that $E'' \subset f(F^*_{Pk_1})$. Then, by the irreducibility of $I(E'')$ and by the arcwise connectedness of $F^*_{Pk_1}$

(61) $I(E'') \subset f(F^*_{Pk_1})$.

The continua $I(E'')$ and $f(F^*_{Pk_1})$ are hereditarily unicoherent by (61) and by the continuity of f as subcontinua of the dendroid F_{Pk_2}. Thus $\tau\big(I(E'')\big) \leqslant \tau\big(f(F^*_{Pk_1})\big)$ by T13 and (61), whence $\tau\big(I(E'')\big) \leqslant \tau(F^*_{Pk_1})$ by T18. Since $\tau(F^*_{Pk_1}) = \tau(F_{Pk_1}) = k_1$ by the symmetry with respect to the point O and by (51), it follows by (60) that $k_2 \leqslant k_1$.

Incomparability of the fans D_i. Let n be an arbitrary natural number, D_i the fans defined by (53) for $i = 0, 1, \ldots, n-1$, and f a continuous mapping of D_i into D_j where $i \neq j$. It ought to be proved that

(62) $f(D_i) \neq D_j$.

Assume first that $i < j$, i.e. that

(63) $n + i < n + j$.

We have, according to (54), $\tau(D_i) = n + i$ and $\tau(D_j) = n + j$; consequently (62) follows by (63) and T18.

Assume next that $j < i$, i.e. that

(64) $n - i < n - j$.

Consider the retraction

$$r(p) = \begin{cases} p & \text{when} \quad p \in F^*_{Pn-j}, \\ O & \text{when} \quad p \in F_{Hn+j}. \end{cases}$$

Then, if we suppose that (62) does not hold, the continuous mapping $g = rf$ maps D_i onto F^*_{Pn-j}:

(65) $g(D_i) = F^*_{Pn-j}$.

Let h be a homeomorphism which maps F_{Hn+i} into F_C according to (56), and let $F = h(F_{Hn+i})$. Putting $k_1 = n-i$, i.e. $F^*_{Pk_1} = F^*_{Pn-i}$, and $k_2 = n-j$, i.e. $F^*_{Pk_2} = F^*_{Pn-j}$, we have identically

$$(66) \qquad F \cup F^*_{Pk_1} = h(F_{Hn+i}) \cup F^*_{Pn-i}.$$

Further, let h_1 be a homeomorphism of $F_{Hn+i} \cup F^*_{Pn-i}$ defined as follows:

$$h_1(p) = \begin{cases} h(p) & \text{when} & p \,\epsilon\, F_{Hn+i}, \\ p & \text{when} & p \,\epsilon\, F^*_{Pn-i}. \end{cases}$$

Therefore we have by (66) $F \cup F^*_{Pk_1} = h_1(F_{Hn+i} \cup F^*_{Pn-i})$, whence $F \cup F^*_{Pk_1} = h_1(D_i)$. Thus $D_i = h_1^{-1}(F \cup F^*_{Pk_1})$ and, by (65), $gh_1^{-1}(F \cup F^*_{Pk_1}) = F^*_{Pk_2}$. In consequence of P2 we then have $k_2 \leqslant k_1$, i.e. $n-j \leqslant n-i$, contrary to (64).

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On the axiom of determinateness *

by

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1. Introduction. It is the purpose of this paper to study the consequences of a mathematical proposition introduced by H. Steinhaus and the author in [17]. This proposition, called *axiom of determinateness* and denoted by (A), is inconsistent with the axiom of choice but has many interesting implications. Several of them are opposite to the 'sad facts' following from the axiom of choice such as, e.g., paradoxical decompositions of the sphere. The actual state of knowledge permits to conjecture that replacing in the Zermelo-Fraenkel-Skolem set theory (ZFS) the axiom of choice by (A) we obtain a consistent theory.

The failure of the axiom of choice in this new theory is considered as a 'sad fact' by the author. He believes that the natural models of ZFS (see e.g. [14]) are real enough to prove the consistency of ZFS. The new theory does not present any such evidence of consistency. We can only hope that some submodels of the natural models of ZFS are models of the new theory [1]. In that case (A) may be considered as a limitation of the notion of a set excluding some 'patological' ZFS-sets [2]. From such a point of view (A) seems to be very successful.

Most of the results of this paper have a clear game-theoretical meaning. They are claiming that such and such constructions permit constructions of such and such not-determined infinite games with perfect information [3]. Of course their validity does not depend on the consistency of (A).

[1] I.e. a subclass of the class of all sets with the same membership relation. It would be still more pleasant if such a submodel contains all the real numbers of the natural model.

[2] Some other general remarks on (A) are given in [17]. Very few alternations to the axiom of choice were considered in literature. Some propositions of that kind are given by Church [3] and Specker [30]. Tarski has considered the proposition 'the set of real numbers is a denumerable union of denumerable sets' (by the results given below this is inconsistent with (A)). But none of them seem to have so many interesting consequences as (A).

[3] The definition and theory of such games is exposed in [15].