so that j is the integer r associated with u, 0, and n as in our definition of translation continuity. Thus for u in E_i

$$f(u+2\pi r/n) = f_i(u+2\pi r/n) = 1 . \tag{10}$$

It is known [1] that $|E_i|$, the measure of E_i, exceeds $2\pi - C\gamma_i$ so that $|\liminf E_i| = 2\pi$. Thus for almost every u, the equation (10) occurs infinitely often.

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On Vietoris mapping theorem and its inverse

by

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0. We shall use Čech cohomology groups with coefficients in an arbitrary abelian group A. Let X, Y, T be compact Hausdorff spaces and let $f\colon X\to Y$, $g_1\colon X\to T$, $g_2\colon Y\to T$ be continuous onto maps such that $g_2 f = g_1$. In this paper we prove (Theorem 1) that if f induces isomorphisms of ith cohomology groups for $i = 0, 1, \ldots, n$ and a monomorphism of the $(n+1)$st cohomology groups of fibres $g_2^{-1}(x)$, $g_1^{-1}(x)$, for all $x \in T$, then f induces isomorphisms of ith cohomology groups for $i = 0, 1, \ldots, n$ and a monomorphism of the $(n+1)$st cohomology groups of spaces Y and X. The result generalizes the well-known Vietoris-Begle theorem [1].

On the other hand, we show (Theorem 2) that if there exists a totally disconnected subset $T_1 \subset T$ such that the fibres $g_1^{-1}(x)$, $g_2^{-1}(x)$ are $(n+1)$-acyclic for $x \in T - T_1$ and f induces isomorphisms of ith cohomology groups of Y and X for $i = 0, 1, \ldots, n$, then f induces isomorphisms of ith cohomology groups of fibres $g_2^{-1}(x)$, $g_1^{-1}(x)$ for all $x \in T$ and $i = 0, 1, \ldots, n$. In the last part we give some applications of the theorems connected with an Eilenberg-Kuratowski theorem [2].

1. All topological spaces considered here are Hausdorff. Let T denote a topological compact space. $\mathcal{F}, \mathcal{G}, \mathcal{J}$ will denote sheaves of abelian groups. If $\mathcal{F}$ is a sheaf over T and $x \in T$, then $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ over x. For any abelian group A, A^T denotes the constant sheaf over T with stalks A. If $U \subset T$, then $\Gamma(U, \mathcal{F})$ denotes the group of all cross-sections over U into $\mathcal{F}$. If $d \in \Gamma(U, \mathcal{F})$, then $\bar{d}$ denotes the carrier of d, i.e., the subset of U composed of all $x \in U$ such that $d(x) \neq 0$. $\bar{d}$ is a closed subset of U. We shall write $\Gamma(\mathcal{F})$ instead of $\Gamma(T, \mathcal{F})$. We say that $\mathcal{F}$ has its support in $U \subset T$ if $\mathcal{F}_x = 0$ for any $x \in T - U$. $\mathcal{F}|U$ will denote the restriction of $\mathcal{F}$ to U. It is known that, if $\mathcal{F}$ is an injective sheaf and U is a closed subset of T, then $\mathcal{F}|U$ is a soft sheaf. The ith cohomology group of X with coefficients in $\mathcal{F}$ is denoted by $H^i(X, \mathcal{F})$, $i = 0, 1, 2, \ldots$

F, I, J will denote positive cochain complexes of sheaves. The mth sheaf of F will be denoted by F_m, $m = 0, 1, \ldots$ If we write $F_m \to F_{m+1}$, we always have in mind the complex map of F_m into F_{m+1}. For any complex F of sheaves over T and $U \subset T$, $F|U$ denotes the restriction of F to U. $\Gamma(F)$ denotes the cochain complex of cross-sections corresponding to F. We denote by $\mathcal{H}^m(F)$ and $H^m(F)$ the mth cohomology group of F and $\Gamma(F)$, respectively.

Let $\mathcal{F}$ be a sheaf and let F be a complex of sheaves over T. We shall say that ε is an augmentation of $\mathcal{F}$ into F if $\varepsilon\colon \mathcal{F} \to F_0$ is a homomorphism and $0 \to \mathcal{F} \to F_0 \to F_1$ is exact. In this case ε induces isomorphisms $\mathcal{F} \to \mathcal{H}^0(F)$, $\Gamma(\mathcal{F}) \to H^0(F)$ and we shall often identify $\mathcal{F}$ and $\mathcal{H}^0(F)$, $\Gamma(\mathcal{F})$ and $H^0(F)$ under these isomorphisms.

If $\varepsilon\colon \mathcal{F} \to F_0$ is an augmentation, then the triple $(\mathcal{F}, F, \varepsilon)$ will be called an *augmented complex*. If $(\mathcal{F}', F', \varepsilon')$, $(\mathcal{F}'', F'', \varepsilon'')$ are two augmented complexes, then any pair (α, α') composed of a homomorphism $\alpha\colon \mathcal{F}' \to \mathcal{F}''$ and a homomorphism $\alpha'\colon F' \to F''$ such that

$$\begin{array}{ccc} \mathcal{F}' & \to & F_0' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha'} \\ \mathcal{F}'' & \to & F'' \end{array}$$

is commutative will be called a *homomorphism of* $(\mathcal{F}', F', \varepsilon')$ *into* $(\mathcal{F}'', F'', \varepsilon'')$.

If I is a complex of injective (soft) sheaves, $\mathcal{H}^i(I) = 0$ for $i > 0$, and $\varepsilon\colon \mathcal{G} \to I$ is an augmentation, then $(\mathcal{G}, I, \varepsilon)$ is called an *injective (soft) resolution* of $\mathcal{G}$. If $\alpha\colon \mathcal{G} \to \mathcal{F}$ is a homomorphism, I^1 is a complex of injective sheaves, $(\mathcal{F}, I^1, \varepsilon_1)$ is an augmented complex and $(\mathcal{G}, I, \varepsilon)$ is a soft resolution of $\mathcal{G}$, then there exists a homomorphism $\alpha'\colon I \to I^1$ such that (α, α') is a homomorphism of $(\mathcal{G}, I, \varepsilon)$ into $(\mathcal{F}, I^1, \varepsilon_1)$. α' is determined uniquely up to homotopy. Hence α induces unique homomorphisms

$$H^i(I) \to H^i(I^1), \quad i = 0, 1, \ldots$$

If $(\mathcal{F}, I, \varepsilon)$ is an injective or soft resolution of $\mathcal{F}$, then $H^i(I) = H^i(T, \mathcal{F})$ for $i = 0, 1, \ldots$

In the sequel, we shall often denote a homomorphism and the maps induced by the homomorphism by the same letter. Moreover, we shall often identify objects if a canonical isomorphism between them has been exhibited.

Let X, Y be compact spaces and let $f\colon X \to Y$ be a continuous map of X onto Y. For every sheaf $\mathcal{F}$ on X, $\mathfrak{f}(\mathcal{F})$ denotes the direct image (see, e.g., [3], p. 171) of $\mathcal{F}$ under f. Let Y_1 be a closed subset of Y and let $X_1 = f^{-1}(Y_1)$. Then

(1) $\mathfrak{f}(\mathcal{F})|Y_1 = \mathfrak{f}(\mathcal{F}|X_1)$ as functors on the category of sheaves over X;

(2) there exists a canonical isomorphism

$$\gamma_f\colon \Gamma\big(Y_1, \mathfrak{f}(\mathcal{F})\big) \to \Gamma(X_1, \mathcal{F}).$$

Hence, if F is a complex, then

(3) there exists a canonical isomorphism

$$\gamma_f\colon H^i\big(\mathfrak{f}(F)|Y_1\big) \to H^i(F|X_1), \quad i = 0, 1, \ldots$$

In particular, taking $Y_1 = (x)$, where $x \in Y$, we find that

(4) there exists a canonical isomorphism

$$\gamma_f\colon \mathcal{H}^i\big(\mathfrak{f}(F)\big)_x \to H^i\big(F|f^{-1}(x)\big), \quad \text{for} \quad i = 0, 1, \ldots$$

Let $\mathcal{F}$ be a sheaf over X, let $\mathcal{G}$ be a sheaf over Y and let $\alpha\colon \mathcal{G} \to \mathfrak{f}(\mathcal{F})$ be a homomorphism. Let $(\mathcal{F}, I, \varepsilon_1)$ and $(\mathcal{G}, J, \varepsilon_2)$ be injective resolutions of $\mathcal{F}$ and $\mathcal{G}$, respectively. It is known (see, e.g. [3], p. 172) that $\mathfrak{f}$ is left exact and that $\mathfrak{f}(\mathcal{J})$ is injective whenever $\mathcal{J}$ is injective. Therefore $\big(\mathfrak{f}(\mathcal{F}), \mathfrak{f}(I), \mathfrak{f}(\varepsilon_1)\big)$ is an augmented complex, $\mathfrak{f}(I)$ is a complex of injective sheaves and hence there exists a homomorphism $\alpha'\colon J \to \mathfrak{f}(I)$ such that (α, α') is a homomorphism of $(\mathcal{G}, J, \varepsilon_2)$ into $\big(\mathfrak{f}(\mathcal{F}), \mathfrak{f}(I), \mathfrak{f}(\varepsilon_1)\big)$. In fact, α' is determined uniquely up to homotopy.

Let Y_1 be, as above, a closed subset of Y and $X_1 = f^{-1}(Y_1)$. Then (α, α') defines a homomorphism of $(\mathcal{G}|Y_1, J|Y_1, \varepsilon_2)$ into $\big(\mathfrak{f}(\mathcal{F})|Y_1, \mathfrak{f}(I)|Y_1, \mathfrak{f}(\varepsilon_1)\big)$, i.e., by (1) a homomorphism $(\mathcal{G}|Y_1, J|Y_1, \varepsilon_2) \to \big(\mathfrak{f}(\mathcal{F}|X_1), \mathfrak{f}(I|X_1), \mathfrak{f}(\varepsilon_1)\big)$.

The homomorphism induces maps

(5)
$$\begin{array}{ccc} \alpha\colon \mathcal{H}^i(J|Y_1) & \to & \mathcal{H}^i\big(\mathfrak{f}(I|X_1)\big) \\ \| & & \| \\ \mathcal{H}^i(J)|Y_1 & & \mathcal{H}^i\big(\mathfrak{f}(I)|Y_1\big) \end{array} \quad i = 0, 1, \ldots$$

(6) $$\alpha\colon H^i(J|Y_1) \to H^i\big(\mathfrak{f}(I|X_1)\big) \quad \text{for} \quad i = 0, 1, \ldots,$$

which do not depend on the choice of α'.

We know that

(7)
$$H^i(J|Y_1) = H(Y_1, \mathcal{G}|Y_1),$$
$$H^i\big(\mathfrak{f}(I|X_1)\big) \underset{\gamma_f}{\overset{\approx}{\to}} H^i(I|X_1) = H^i(X_1, \mathcal{F}|X_1)$$

since $J|Y_1$, $I|X_1$ are soft resolutions of $\mathcal{G}|Y_1$, $\mathcal{F}|X_1$, respectively. Hence (6) and (7) define a homomorphism

(8) $$h(\alpha, f)\colon H^i(Y_1, \mathcal{G}|Y_1) \to H^i(X_1, \mathcal{F}|X_1) \quad \text{for} \quad i = 0, 1, \ldots$$

determined uniquely by α and f.

Now let T be another compact space, let $g\colon Y \to T$ be a continuous map of Y onto T and let $\mathfrak{g}$ be the direct image functor defined on sheaves over Y and corresponding to g. Then $\mathfrak{g}(\alpha')$ is a homomorphism $\mathfrak{g}(J) \to \mathfrak{g}\mathfrak{f}(I)$ and induces a homomorphism

(9) $$\mathfrak{g}(\alpha)\colon \mathcal{H}^i\big(\mathfrak{g}(J)\big) \to \mathcal{H}^i\big(\mathfrak{g}\mathfrak{f}(I)\big), \quad i = 0, 1, \ldots$$

The induced homomorphism does not depend on the choice of a' and is determined by a and g.

The following diagram is commutative:

$$\text{(10)}\quad \begin{array}{ccccc} \mathcal{H}^i(\mathfrak{g}(J))_x & \xrightarrow{\gamma_g} & H^i(J|g^{-1}(x)) & = & H^i(g^{-1}(x), \mathcal{G}|g^{-1}(x)) \\ \downarrow{\scriptstyle \mathfrak{g}(a)} & & \downarrow{\scriptstyle a'} & & \\ & & H^i(\mathfrak{f}(I)|g^{-1}(x)) & & \downarrow{\scriptstyle h(a,f)} \\ \mathcal{H}^i(\mathfrak{g}\mathfrak{f}(I))_x & & \downarrow{\scriptstyle \gamma_f} & & \\ & \searrow{\scriptstyle \gamma_{gf}} & H^i(I|f^{-1}g^{-1}(x)) & = & H^i(f^{-1}g^{-1}(x), \mathcal{F}|f^{-1}g^{-1}(x)) . \end{array}$$

Consider the second right spectral hyperhomology functor (see, e.g., [3], p. 146 and p. 173) of $\Gamma\mathfrak{g}\big((\mathrm{II}\Gamma\mathfrak{g})_r^{pq}(F), (\mathrm{II}\Gamma\mathfrak{g})^i(F)\big)$, $r \geqslant 2$. To simplify notation we shall denote $(\mathrm{II}\Gamma\mathfrak{g})_r^{pq}$ by G_r^{pq} and $(\mathrm{II}\Gamma\mathfrak{g})^i$ by G^i. The spectral sequences $\big(G_r^{pq}(J), G^i(J)\big)$, $\big(G_r^{pq}(\mathfrak{f}(I)), G^i(\mathfrak{f}(I))\big)$ and the homomorphism $\big(G_r^{pq}(J), G^i(J)\big) \to \big(G_r^{pq}(\mathfrak{f}(I)), G^i(\mathfrak{f}(I))\big)$ induced by a' are determined uniquely by $\mathcal{F}, \mathcal{G}, f, g$ and a since I, J, a' are determined uniquely up to homotopy. We shall denote the homomorphism of spectral sequences by a^*.

Notice that, $G^i(f(I))$ and $G^i(J)$ being as abelian groups (filtration not considered) we have

$$\text{(11)}\qquad \begin{aligned} G^i(\mathfrak{f}(I)) &= H^i(\mathfrak{g}\mathfrak{f}(I)) \underset{\gamma_{gf}}{\overset{\approx}{\to}} H^i(X, \mathcal{F}) \text{ and} \\ G^i(J) &= H^i(\mathfrak{g}(J)) \underset{\gamma_g}{\overset{\approx}{\to}} H^i(Y, \mathcal{G}) , \end{aligned}$$

since $\mathfrak{g}\mathfrak{f}(I)$, $\mathfrak{g}(J)$ are complexes of injective sheaves (see, e.g., [3], p. 148, Remark 3).

Moreover, the following diagram is commutative:

$$\text{(12)}\qquad \begin{array}{ccc} G^i(J) & \underset{\gamma_g}{\overset{\approx}{\to}} & H^i(Y, \mathcal{G}) \\ \downarrow{\scriptstyle a^*} & & \downarrow{\scriptstyle h(a,f)} \\ G^i(\mathfrak{f}(I)) & \underset{\gamma_{gf}}{\overset{\approx}{\to}} & H^i(X, \mathcal{F}) . \end{array}$$

On the other hand,

$$\begin{aligned} G_2^{pq}(\mathfrak{f}(I)) &= H^p\big(T, \mathcal{H}^q(\mathfrak{g}\mathfrak{f}(I))\big), \\ G_2^{pq}(J) &= H^p\big(T, \mathcal{H}^q(\mathfrak{g}(J))\big) \end{aligned}$$

(13) and a^*: $G_2^{pq}(J) \to G_2^{pq}(\mathfrak{f}(I))$ coincides with the map induced by $\mathfrak{g}(a)$: $\mathcal{H}^q(\mathfrak{g}(J)) \to \mathcal{H}^q(\mathfrak{g}\mathfrak{f}(I))$ defined in (9).

Consider the case where $\mathcal{F} = A^X$, $\mathcal{G} = A^Y$ and a is the canonical map (see, e.g., [5], p. 151, Corollary) a: $A^Y \to \mathfrak{f}(A^X)$. Then $A^X|Y_1 = A^{Y_1}$, $A^X|X_1 = A^{X_1}$, where $X_1 = f^{-1}(Y_1)$, and a restricted to $A^Y|Y_1$ gives the canonical map $A^{Y_1} \to \mathfrak{f}(A^{X_1})$. Let f^* be the homomorphism $H^i(Y_1, A) \to H^i(X_1, A)$ $i = 0, 1, \ldots$, induced by f. It is known that

(14) $H^i(Y_1, A) = H^i(Y_1, A^{Y_1})$; $H^i(X_1, A) = H^i(X_1, A^{X_1})$, for $i = 0, 1, \ldots$, and f^* coincides with $h(a, f)$.

2. Lemma 1. *Let (E_r^{pq}, G^i), (F_r^{pq}, H^i), $r \geqslant 2$, be two cohomological spectral sequences, let β be a homomorphism of (E_r^{pq}, G^i) into (F_r^{pq}, H^i) and let n be an integer. Suppose that β: $E_2^{pq} \to F_2^{pq}$ is an isomorphism for $p \leqslant 2(n+1-q)$, $q \neq n+1$ and a monomorphism for $p = 0$, $q = n+1$. Then β: $G^i \to H^i$ is an isomorphism for $i = 0, 1, \ldots, n$ and a monomorphism for $i = n+1$.*

Proof. We shall prove by induction on r that, for $r \geqslant 2$,

$$\text{(15)}\quad \beta\colon E_r^{pq} \to F_r^{pq} \text{ is an isomorphism for } p \leqslant n+1-q+\frac{n+1-q}{r-1},$$
$$q \neq n+1 \text{ and a monomorphism for } p = 0,\ q = n+1.$$

If $r = 2$, then this follows from our assumptions. Assume that β: $E_k^{pq} \to F_k^{pq}$ is an isomorphism for $p \leqslant n+1-q+\dfrac{n+1-q}{k-1}$, $q \neq n+1$ and a monomorphism for $p = 0$, $q = n+1$, where $k \geqslant 2$. We shall prove that β: $E_{k+1}^{pq} \to F_{k+1}^{pq}$ is an isomorphism for $p \leqslant n+1-q+\dfrac{n+1-q}{k}$, $q \neq n+1$ and a monomorphism for $p = 0$, $q = n+1$. It is easy to see that β: $E_{k+1}^{0n+1} \to F_{k+1}^{0n+1}$ is a monomorphism. In order to prove the remaining part it suffices to show that $p \leqslant n+1-q+\dfrac{n+1-q}{k}$ implies that

$$p+k \leqslant n+1-(q-k+1)+\frac{n+1-(q-k+1)}{k-1}, \qquad q-k+1 \neq n+1 ,$$

$$p-k \leqslant n+1+(q-k-1)+\frac{n+1-(q+k-1)}{k-1}, \qquad q+k-1 \neq n+1 .$$

But if $p \leqslant n+1-q+(n+1-q)/k$ then

$$\begin{aligned} p+k &\leqslant n+1-(q-k+1)+\frac{n+1-q}{k}+1 \leqslant n+1-(q-k+1)+\frac{n+1-q}{k-1}+1 \\ &= n+1-(q-k+1)+\frac{n+1-(q-k+1)}{k-1} . \end{aligned}$$

The inequality $q-k+1 \neq n+1$ is obvious.

$$\begin{aligned} p-k &\leqslant n+1-(q+k-1)+\frac{n+1-q}{k}-1 \leqslant n+1-(q+k-1)+\frac{n+1-q}{k-1}-1 \\ &= n+1-(q+k-1)+\frac{n+1-(q+k-1)}{k-1} . \end{aligned}$$

On the other hand, $q+k-1 = n+1$ implies $p-k = 0$. But it is easy to check that $p = k$, $q = n+2-k$ do not satisfy the inequality $p \leqslant n+1-q+(n+1-q)/k$. Thus the proof of (15) is complete. Therefore, for all $r \geqslant 2$, $\beta\colon E_r^{pq} \to F_r^{pq}$ is an isomorphism for $p \leqslant n+1-q$, $q \neq n+1$, and a monomorphism for $p = 0$, $q = n+1$. Hence $\beta\colon G^i \to H^i$ is an isomorphism for $i = 0, 1, \ldots, n$ and a monomorphism for $i = n+1$.

THEOREM 1. *Let X, Y, T be compact spaces and let $f\colon X \to Y$, $g\colon Y \to T$ be continuous onto maps. Let A be an abelian group. Suppose that, for every $x \in T$, $f^*\colon H^i(g^{-1}(x), A) \to H^i(f^{-1}g^{-1}(x), A)$ is an isomorphism for $i = 0, 1, \ldots, n$, and a monomorphism for $i = n+1$. Then $f^*\colon H^i(Y, A) \to H^i(X, A)$ is an isomorphism for $i = 0, 1, \ldots, n$ and a monomorphism for $i = n+1$.*

Proof. Take $\mathcal{F} = A^X$, $\mathcal{G} = A^Y$ and let α be the canonical monomorphism $\alpha\colon A^Y \to \mathfrak{f}(A^X)$. We shall use the notation of part 1. Consider spectral sequences $\big(G_r^{pq}(J), G^i(J)\big)$, $\big(G_r^{pq}(\mathfrak{f}(I)), G^i(\mathfrak{f}(I))\big)$ and the homomorphism α^* induced by α. Then

(16) $\alpha^*\colon G_2^{pq}(J) \to G_2^{pq}(\mathfrak{f}(I))$ *is an isomorphism for* $p = 0, 1, \ldots$, $q = 0, 1, \ldots, n$ *and a monomorphism for* $p = 0$, $q = n+1$.

Indeed, $G_2^{pq}(J) = H^p\big(\mathcal{H}^q(\mathfrak{g}(J))\big)$, $G_2^{pq}(\mathfrak{f}(I)) = H^p\big(\mathcal{H}^q(\mathfrak{g}\mathfrak{f}(I))\big)$ and (by (13)) $\alpha^*\colon G_2^{pq}(J) \to G_2^{pq}(\mathfrak{f}(I))$ is induced by $\mathfrak{g}(\alpha)\colon \mathcal{H}^q(\mathfrak{g}(J)) \to \mathcal{H}^q(\mathfrak{g}\mathfrak{f}(I))$ defined in (9). But, by (10), for $x \in T$, we have the commutative diagram

$$\begin{array}{ccc} \mathcal{H}^q\big(\mathfrak{g}(J)\big)_x & \xrightarrow{\mathfrak{g}(\alpha)} & \mathcal{H}^q\big(\mathfrak{g}\mathfrak{f}(I)\big)_x \\ \wr\!\wr\downarrow \gamma_g & & \wr\!\wr\downarrow \gamma_{gf} \\ H^q\big(g^{-1}(x), A\big) & \xrightarrow[h(\alpha,f)]{} & H^q\big(f^{-1}g^{-1}(x), A\big) \end{array}$$

and $h(\alpha, f) = f^*$ is an isomorphism for $q = 0, 1, \ldots, n$ and a monomorphism for $q = n+1$. Hence $\mathfrak{g}(\alpha)\colon \mathcal{H}^q(\mathfrak{g}(J)) \to \mathcal{H}^q(\mathfrak{g}\mathfrak{f}(I))$ is an isomorphism for $q = 0, 1, \ldots, n$ and a monomorphism for $q = n+1$. This implies (16). Now the theorem follows from Lemma 1, (11) and (12).

Remark. Using a similar method one can prove the following theorem:

THEOREM 1′. *Let X, Y, T be compact spaces, let $f\colon X \to Y$, $g\colon Y \to T$ be continuous onto maps and let A be an abelian group. Suppose that, for every $x \in T$, $f^*\colon H^i(g^{-1}(x), A) \to H^i(f^{-1}g^{-1}(x), A)$ is an isomorphism, for $i = 0, 1, \ldots, n$. Then $f^*\colon H^i(Y, A) \to H^i(X, A)$ is an isomorphism for $i = 0, 1, \ldots, n$.*

3. LEMMA 2. *Let T be a compact space, let T_1 be a totally disconnected subset of T and let $\mathcal{F}$ be a sheaf over T that has its support in T_1. Then $H^i(T, \mathcal{F}) = 0$ for $i > 0$.*

Proof. Let $\{U_i\}$ be a finite covering of T and let c be an i-cochain of this covering with coefficients in $\mathcal{F}$. Let $\{V_j\}$ be a covering of T such that $\overline{V}_j \subset U_j$. Then $\overline{V_{j_0} \cap \ldots \cap V_{j_i}} \subset U_{j_0} \cap \ldots \cap U_{j_i}$, $c(j_0, \ldots, j_i) \in \Gamma(U_{j_0} \cap \ldots \cap U_{j_i}, \mathcal{F})$, its carrier $\bar{c}(j_0, \ldots, j_i)$ is closed in $U_{j_0} \cap \ldots \cap U_{j_i}$ and hence $\bar{c}(j_0, \ldots, j_i) \cap \overline{V_{j_0} \cap \ldots \cap V_{j_i}} \subset T_1$ is closed in T. Let c^* denote the i-cochain of the covering $\{V_j\}$ which corresponds to c. Then the closure (in T) of the carrier of $c^*(j_0, \ldots, j_i)$ is contained in T_1. Let T_0 be the union of carriers of all cross sections $c^*(j_0, \ldots, j_i)$. Then T_0 is 0-dimensional and we may choose a covering $\{W_m\}$ of T refining $\{V_j\}$ and such that $W_k \cap W_l \cap T_0 = \emptyset$ for every $k \neq l$. It is easy to see that, if $i > 0$, then the cochain of the covering $\{W_m\}$ corresponding to c is the zero cochain. Thus $H^i(T, \mathcal{F}) = 0$ for all $i > 0$.

LEMMA 3. *Let T, T_1 be as in Lemma 2. Let $\mathcal{F}, \mathcal{G}$ be two sheaves over T and let $\beta\colon \mathcal{F} \to \mathcal{G}$ be a homomorphism which induces an isomorphism for stalks over $x \in T - T_1$. Then the induced map $\beta\colon H^i(T, \mathcal{F}) \to H^i(T, \mathcal{G})$ is an isomorphism for $i > 1$ and an epimorphism for $i = 1$.*

Proof. First consider two cases

(a) β is a monomorphism. Then $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{F}/\beta(\mathcal{G}) \to 0$ is exact and we have an exact sequence $H^1(T, \mathcal{F}) \to H^1(T, \mathcal{G}) \to H^1\big(T, \mathcal{F}/\beta(\mathcal{G})\big) \to H^2(T, \mathcal{F}) \to \ldots \to H^i(T, \mathcal{F}) \to H^i(T, \mathcal{G}) \to H^i\big(T, \mathcal{F}/\beta(\mathcal{G})\big) \to H^{i+1}(T, \mathcal{F}) \to \ldots$ But $\mathcal{F}/\beta(\mathcal{G})$ has its support in T_1. Hence it follows from Lemma 2 that $H^i\big(T, \mathcal{F}/\beta(\mathcal{G})\big) = 0$, for $i > 0$. Thus $\beta\colon H^i(T, \mathcal{F}) \to H^i(T, \mathcal{G})$ is an isomorphism for $i > 1$ and an epimorphism for $i = 1$.

(b) β is an epimorphism. Then $0 \to \ker\beta \to \mathcal{F} \to \mathcal{G} \to 0$ is exact, $\ker\beta$ has its support in T_1 and, considering the corresponding exact sequence of cohomology groups, we infer (as in (a)) that $\beta\colon H^i(T, \mathcal{F}) \to H^i(T, \mathcal{G})$ is an isomorphism for $i \geqslant 1$.

The general case follows from (a), (b) and the remark that if $\beta\colon \mathcal{F} \to \mathcal{G}$ is a homomorphism satisfying assumptions of the lemma, then β can be represented as a composition of an epimorphism and a monomorphism, both satisfying the assumptions.

LEMMA 4. *Let T and T_1 be as in Lemma 2. Let $\mathcal{F}, \mathcal{G}$ be two sheaves over T with support in T_1 and let $\beta\colon \mathcal{F} \to \mathcal{G}$ be a homomorphism. If the induced homomorphism $\beta\colon \Gamma(T, \mathcal{F}) \to \Gamma(T, \mathcal{G})$ is an isomorphism (monomorphism), then $\beta\colon \mathcal{F} \to \mathcal{G}$ is also an isomorphism (monomorphism).*

Proof. It suffices to prove that if $\mathcal{F}'$ is a sheaf over T with support in T_1, then for every $x \in T$ and $a \in \mathcal{F}_x$, $a \neq 0$, there exists a cross-section over T through a. For every $x \in T$, $a \in \mathcal{F}_x$ there exists a neighbourhood V of x such that there exists a cross-section d_0 through a over V. Let V_1 be a neighbourhood of x such that $\overline{V}_1 \subset V$. Then the intersection of the carrier $\bar{d}_0$ of d_0 and $\overline{V}_1$ is closed in T and contained in T_1, whence 0-dimensional. Moreover, if $a \neq 0$ then $x \in \bar{d}_0 \cap V_1$. Hence there exists a neighbourhood V_2 of x contained in V_1 and such that $\bar{V}_2 \cap \overline{(T-V_2)} \cap \bar{d}_0 = \emptyset$. Now define

$$d(y) = \begin{cases} d_0(y) & \text{for} \quad y \in V_2, \\ 0 & \text{for} \quad y \in T-V_2; \end{cases}$$

then d is a section over X through a.

LEMMA 5. *Let T, T_1 be as in Lemma 2. Let $\mathcal{F}$, $\mathcal{G}$ be two sheaves over T and let $\beta\colon \mathcal{F} \to \mathcal{G}$ be a monomorphism which induces an isomorphism of stalks over $x \in T-T_1$. If the induced homomorphism $\beta\colon H^0(T, \mathcal{F}) \to H^0(T, \mathcal{G})$ is an epimorphism and $\beta\colon H^1(T, \mathcal{F}) \to H^1(T, \mathcal{G})$ is a monomorphism, then $\beta\colon \mathcal{F} \to \mathcal{G}$ is an isomorphism.*

Proof. We have the following exact sequence: $0 \to H^0(T, \mathcal{F}) \to$ $\to H^0(T, \mathcal{G}) \to H^0(T, \mathcal{G}/\beta(\mathcal{F})) \to H^1(T, \mathcal{F}) \to H^1(T, \mathcal{G}) \to H^1(T, \mathcal{G}/\beta(\mathcal{F}))$. Thus $H^0(T, \mathcal{G}/\beta(\mathcal{F})) = 0$. But $\mathcal{G}/\beta(\mathcal{F})$ has its support in T_1, whence, from $\mathcal{G}/\beta(\mathcal{F}) = 0$, i.e., $\mathcal{G} = \beta(\mathcal{F})$ and β is an isomorphism.

LEMMA 6. *Let (E_r^{pq}, G^i), (F_r^{pq}, H^i), $r \geqslant 2$, be two cohomological spectral sequences such that $E_2^{pq} = F_2^{pq} = 0$ for $p, q = 1, 2, \ldots, n+1$, where n is a fixed integer. Let β be a homomorphism of (E_r^{pq}, G^i) into (F_r^{pq}, H^i).*

Suppose that $\beta\colon G^i \to H^i$ is an isomorphism for $i = 0, 1, \ldots, n$ and a monomorphism for $i = n+1$ and $\beta\colon E_2^{p0} \to F_2^{p0}$ is an isomorphism for $p = 2, 3, \ldots, n+2$ and an epimorphism for $p = 1$. Then $\beta\colon E_2^{0q} \to F_2^{0q}$ is an isomorphism for $q = 0, 1, \ldots, n$ and a monomorphism for $q = n+1$. Moreover $\beta\colon E_2^{1,0} \to F_2^{1,0}$ is, in fact, an isomorphism.

Proof. Since $E_2^{pq} = F_2^{pq} = 0$, for $p, q = 1, \ldots, n+1$, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccccccc} 0 \to & E_2^{1,0} & \to & G^1 & \to & E_2^{0,1} & \to & E_2^{2,0} & \to & G^2 & \to & E_2^{0,2} & \to & E_2^{3,0} & \to \ldots \\ & {\scriptstyle\beta}\downarrow & & \downarrow\wr & & \downarrow & & \downarrow\wr & & \downarrow\wr & & \downarrow & & \downarrow\wr & \\ 0 \to & F_2^{1,0} & \to & H^1 & \to & F_2^{0,1} & \to & F_2^{2,0} & \to & H^2 & \to & F_2^{0,2} & \to & F_2^{3,0} & \to \ldots \end{array}$$

$$\begin{array}{ccccccccc} \ldots \to & E_2^{n+1,0} & \to & G^{n+1} & \to & E^{0,n+1} & \to & E_2^{n+2,0} \\ & {\scriptstyle\beta}\downarrow\wr & & \downarrow & & \downarrow & & \downarrow\wr \\ \ldots \to & F_2^{n+1,0} & \to & H^{n+1} & \to & F_2^{0,n+1} & \to & F_2^{n+2,0}. \end{array}$$

Hence $\beta\colon E_2^{0q} \to F_2^{0q}$ is an isomorphism for $q = 1, 2, \ldots, n$ and a monomorphism for $q = n+1$. Moreover, $\beta\colon E_2^{1,0} \to F_2^{1,0}$ is an isomorphism and $E_2^{0,0} = G^0$, $F_2^{0,0} = H^0$, whence $\beta\colon E_2^{0,0} \to F_2^{0,0}$ is also an isomorphism.

THEOREM 2. *Let X, Y, T be compact spaces and let $f\colon X \to Y$, $g\colon Y \to T$ be continuous onto maps. Let A be an abelian group. Let T_1 be a totally disconnected subset of T. Suppose that, for every $x \in T-T_1$, $H^i(g^{-1}(x), A)$ $= H^i(f^{-1}g^{-1}(x), A) = 0$ for $i = 1, 2, \ldots, n+1$ and $H^0(g^{-1}(x), A)$ $= H^0(f^{-1}g^{-1}(x), A) = A$. Moreover, assume that:*

$f^*\colon H^0(Y, A) \to H^0(X, A)$ *is an epimorphism,*

$f^*\colon H^i(Y, A) \to H^i(X, A)$ *is an isomorphism for $i = 1, \ldots, n$,*

$f^*\colon H^{n+1}(Y, A) \to H^{n+1}(X, A)$ *is a monomorphism.*

Then, for every $x \in T$, $f^*\colon H^i(g^{-1}(x), A) \to H^i(f^{-1}g^{-1}(x), A)$ is an isomorphism for $i = 0, 1, \ldots, n$ and a monomorphism for $i = n+1$. Moreover, $f^*\colon H^*(Y, A) \to H^0(X, A)$ is, in fact, an isomorphism.

Proof. We shall use the notation of part 1 and Theorem 1. It follows from our assumptions and (11), (12), (15) that

$\alpha^*\colon G^0(J) \to G^0(\mathfrak{f}(I))$ is an epimorphism,

$\alpha^*\colon G^i(J) \to G^i(\mathfrak{f}(I))$ is an isomorphism for $i = 1, \ldots, n$,

$\alpha^*\colon G^{n+1}(J) \to G^{n+1}(\mathfrak{f}(I))$ is a monomorphism.

Moreover, for every $x \in T-T_1$, $\mathcal{H}^i(g(J))_x = H^i(g^{-1}(x), A) = 0$ and $\mathcal{H}^i(\mathfrak{g}\mathfrak{f}(I))_x = H^i(f^{-1}g^{-1}(x), A) = 0$, for $i = 1, \ldots, n+1$. Hence the supports of $\mathcal{H}^i(\mathfrak{g}(J))$, $\mathcal{H}^i(\mathfrak{g}\mathfrak{f}(I))$ are in T_1 for $i = 1, \ldots, n+1$. From Lemma 2 and (13) we obtain $G_2^{pq}(J) = G_2^{pq}(\mathfrak{f}(I)) = 0$ for $q = 1, \ldots, n+1$; $p = 1, 2, \ldots$

On the other hand, for every $x \in T-T_1$, $\mathcal{H}^0(\mathfrak{g}(J))_x = H^0(g^{-1}(x), A)$ $= A$, $\mathcal{H}^0(\mathfrak{g}\mathfrak{f}(I))_x = H^0(f^{-1}g^{-1}(x), A) = A$ and by (10) and (14).

$\alpha^*\colon \mathcal{H}^0(g(J))_x \to \mathcal{H}^0(gf(I))_x$ is the identity isomorphism in every stalk ever $x \in T-T_1$. Therefore it follows from Lemma 3 and (13) that $\alpha^*\colon$ $G_0^{p0}(J) \to G_0^{p0}(\mathfrak{f}(I))$ is an epimorphism for $p = 1$ and an isomorphism for $p = 2, 3, \ldots$ Now, from Lemma 6 and (13) we infer that $\alpha^*\colon H^0(\mathcal{H}^q(\mathfrak{g}(J))) \to$ $\to H^0(\mathcal{H}^q(\mathfrak{g}\mathfrak{f}(I)))$ is an isomorphism for $q = 0, 1, \ldots, n$ and a monomorphism for $q = n+1$.

By Lemma 4, $\mathfrak{g}(\alpha)\colon \mathcal{H}^q(\mathfrak{g}(J)) \to \mathcal{H}^q(\mathfrak{g}\mathfrak{f}(I))$ is an isomorphism for $q = 1, \ldots, n$ and a monomorphism for $q = n+1$. By the last part of Lemma 6, $\alpha^*\colon H^1(\mathcal{H}^0(\mathfrak{g}(J))) \to H^1(\mathcal{H}^0(\mathfrak{g}\mathfrak{f}(I)))$ is an isomorphism. Therefore, by Lemma 5, $\mathfrak{g}(\alpha)\colon \mathcal{H}^0(\mathfrak{g}(J)) \to \mathcal{H}^0(\mathfrak{g}\mathfrak{f}(I))$ is also an isomorphism. Thus we have proved that $g(\alpha)\colon \mathcal{H}^i(\mathfrak{g}(J)) \to \mathcal{H}^i(\mathfrak{g}\mathfrak{f}(I))$ is an isomorphism for $i = 0, 1, \ldots, n$ and a monomorphism for $i = n+1$. Therefore, for every $x \in T$, $\mathfrak{g}(\alpha)\colon \mathcal{H}^i(\mathfrak{g}(J))_x \to \mathcal{H}^i(\mathfrak{g}\mathfrak{f}(I))_x$ is an isomorphism for $i = 0, 1, \ldots, n$ and a monomorphism for $i = n+1$. This, by (10) and (14), gives the theorem.

COROLLARY 1. *Let U be a locally compact topological space, let X_1, X_2 be two compact extensions of U and let A be an abelian group. Let $f\colon X_1 \to X_2$ be a continuous map which is the identity on U. Then $f^*\colon H^i(X_2, A) \to$ $\to H^i(X_1, A)$ is an isomorphism for $i = 0, 1, \ldots, n$ and a monomorphism for $i = n+1$ if and only if, for any connected component Y_1 of $X_2 - U$,*

$f^*\colon H^i(Y_1, A)\to H^i(f^{-1}(Y_1), A)$ is an isomorphism for $i = 0, 1, \ldots, n$ and a monomorphism for $i = n+1$.

Proof. Let X_2^* denote the topological space obtained from X_2 by regarding each connected component of $X_2 - U$ as a single point.

Take $X = X_1$, $Y = X_2$, $T = X_2^*$, $f = f$, let g denote the canonical map $X_2 \to X_2^*$, let $T_1 = g(X_2 - U)$ and apply Theorem 1 and Theorem 2.

Let U, X_1, X_2 be as above, let βU be the Čech-Stone compactification of U and let f_1, f_2 be the canonical maps $f_1\colon \beta U \to X_1$, $f_2\colon \beta U\to X_2$. Eilenberg, Kuratowski [2] and Skliarenko [4] proved that *if*

$$f_1^*\colon H^0(X_1, A)\to H^0(\beta U, A),$$
$$f_2^*\colon H^0(X_2, A)\to H^0(\beta U, A) \quad \textit{are epimorphisms and}$$
$$f_1^*\colon H^1(X_1, A)\to H^1(\beta U, A),$$
$$f_2^*\colon H^1(X_2, A)\to H^1(\beta U, A) \quad \textit{are monomorphisms,}$$

then f_1 induces a homeomorphism of $(\beta U)^*$ onto X_1^* and f_2 induces a homeomorphism of $(\beta U)^*$ onto X_2^* (where $(\beta U)^*$, X_1^*, X_2^* denote the topological spaces obtained from $\beta U, X_1, X_2$, respectively, by regarding each connected component of $\beta U - U$, $X_1 - U$, $X_2 - U$ as a single point). In fact, this result can easily be derived from Corollary 1. Let us denote by h the canonical homeomorphism of X_1^* onto X_2^* obtained in this way. For U, as above, let $H_f^i(U, A)$ be the ith Čech cohomology group of U with coefficients in A defined by using finite open coverings.

Moreover, let U be normal. Then $H_f^i(U, A) = H^i(\beta U, A)$. The following proposition is easily obtained from the above considerations and Corollary 1.

PROPOSITION. *For U, X_1, X_2, h, as above, if the canonical homomorphisms $H^i(X_1, A)\to H_f^i(U, A)$, $H^i(X_2, A)\to H_f^i(U, A)$ are isomorphisms for $i = 1, \ldots, n$, monomorphisms for $i = n+1$ and epimorphisms for $i = 0$, then the connected components of $X_1 - U$, $X_2 - U$ corresponding under h have isomorphic i-th cohomology groups for $i = 0, 1, \ldots, n$.*

Remark. Using a similar method to that used in the proof of Theorem 2 one can prove the following theorem.

THEOREM 2'. *Let X, Y, T, f, A, T_1, be as in Theorem 2. Suppose that, for every $x\in T - T_1$, $H^i(g^{-1}(x), A) = H^i(f^{-1}g^{-1}(x), A) = 0$, for $i = 1, 2, \ldots, n+1$ and $H^0(g^{-1}(x), A) = H^0(f^{-1}g^{-1}(x), A) = A$. Moreover, assume that*

$$f^*\colon H^0(Y, A)\to H^0(X, A) \quad \textit{is an epimorphism,}$$
$$f^*\colon H^i(Y, A)\to H^i(X, A) \quad \textit{is an isomorphism for } i = 1, \ldots, n.$$

Then, for every $x\in T$, $f^*\colon H^i(g^{-1}(x), A)\to H^i(f^{-1}g^{-1}(x), A)$ is an isomorphism for $i = 0, 1, \ldots, n$. Moreover, $f^*\colon H^0(Y, A)\to H^0(X, A)$ is, in fact, an isomorphism.

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