On the Mickle-Rado covering theorems

by

A. R. Bednarek (Akron, Ohio)

1. Introduction. In [1], E. J. Mickle and T. Rado derive a general covering theorem that is expressed in terms of a pair of binary relations. This covering theorem generalizes Theorem 3.3 in A. P. Morse's paper [2]. The Morse theorem is realized by a particularization of the relations. Moreover, Mickle and Rado show that their theorem is equivalent to Zorn's lemma [5]. The logical equivalence of this covering theorem and one of its modified forms (involving a single relation) is implicit in their argument. The purpose of this note is to present some equivalent, though structurally different, formulations of the Mickle-Rado theorems, thus adding to the list of equivalents to the axiom of choice. Some of the set-theoretic maximality principles are immediate consequents of our formulations.

2. Definitions. Let R be a binary relation over the non-empty set X; this is denoted by $(X;R)$. We say that $x, y \in X$ are *R-comparable (R-incomparable)* if and only if xRy or yRx [$x \operatorname{non} Ry$ and $y \operatorname{non} Rx$].

A set $S \subset X$ is called *R-scattered (R-coherent)* if for every pair of distinct $x, y \in S$, x and y are R-incomparable (R-comparable). An R-scattered (R-coherent) subset of X that is not a proper subset of any other R-scattered (R-coherent) subset of X is called a *maximal R-scattered (maximal R-coherent)* subset of X.

If $x \in X$, then $R(x) = \{y \mid y \in X \text{ and } yRx\}$. Given $S \subset X$, let $R(S) = \bigcup_{x \in S} R(x)$. Given two binary relations R and R^* over X; that is, $(X; R, R^*)$, we define the set $N(x)$, for $x \in X$, as follows:

$$N(x) = R(x) \cap R^*(x) = \{y \mid y \in X,\ yRx, \text{ and } yR^*x\}.$$

Let $N(S) = \bigcup_{x \in S} N(x)$. Although $N(E) \subset R(E)$ and $N(E) \subset R^*(E)$, $N(E)$ does not equal $R(E) \cap R^*(E)$.

Given $(X; R^*)$ and $E \subset X$, an element $x' \in E$ is called an *R^*-dominant element* if $E \subset R^*(x')$.

3. The Mickle-Rado covering theorems. The Mickle-Rado theorems can be stated in the following form:

MR-I. *If R is a reflexive and symmetric relation over X, then there exists an R-scattered subset S of X such that $X = R(S)$.*

MR-II. *Given $(X; R, R^*)$ and*

(1) *R is reflexive and symmetric over X, and*

(2) *every non-empty subset E of X contains an R^*-dominant element, then there exists an R-scattered subset S of X such that $X = N(S)$.*

Since each set $\{x\}$, for $x \in X$, must, by (2), contain an R^*-dominant element, R^* is a reflexive relation over X.

In general, our formulations are derived by characterizing the R-scattered sets in the conclusions of MR-I and MR-II.

4. Derivation of equivalents to the Mickle-Rado theorems. In [1], Mickle and Rado observe that the R-scattered set in the conclusion of MR-I is a maximal R-scattered set. The following theorem shows this maximality is both necessary and sufficient.

THEOREM 1. *If R is a reflexive and symmetric relation over X, and S is an R-scattered subset of X, then $X = R(S)$ if and only if S is a maximal R-scattered set.*

Proof. Let S be an R-scattered subset of X and $X = R(S)$. Assume S is not a maximal R-scattered set; that is, assume there exists an R-scattered set $S^* \subset X$ such that $S \subset S^*$ and $S^* - S \neq \emptyset$.

Let $t \in S^* - S$. Since $X = R(S)$, there is an $s \in S$ such that tRs. But $S \subset S^*$. Therefore $t, s \in S^*$, $t \neq s$, and tRs. This contradicts the R-scatteredness of S^*. Consequently, S is a maximal R-scattered subset of X.

Let S be a maximal R-scattered subset of X, and assume $X - R(S) \neq \emptyset$. Let $S^* = \{t\} \cup S$, where $t \in X - R(S)$. Suppose tRs for $s \in S$. This implies $t \in R(s) \subset R(S)$; but this is a contradiction. Therefore t non Rs for all $s \in S$, and since S is R-scattered, the set S^* is R-scattered. This is a contradiction of the maximality of S; hence, $X = R(S)$.

Theorem 1 yields the following proposition which is equivalent to the Mickle-Rado theorems and, therefore, equivalent to the axiom of choice.

MR-E_1. *If R is a reflexive and symmetric relation over X, then there exists a maximal R-scattered subset S of X.*

The proposition below is obviously equivalent to MR-I. The proof is omitted.

MR-$E_{3/2}$. *If R is a reflexive and symmetric relation over X, and A is a subset of X, there exists an R-scattered set $S \subset A$ such that $A \subset R(S)$.*

We use MR-E_1 and MR-$E_{3/2}$ to generate the following equivalent to the Mickle-Rado theorems.

MR-E_2. *If R is a reflexive and symmetric relation over X, then every R-scattered subset S of X is contained in a maximal R-scattered subset S^* of X.*

Proof. Suppose that $X - R(S) \neq \emptyset$. By MR-$E_{3/2}$, there is an R-scattered subset $S_1 \subset X - R(S)$ such that $X - R(S) \subset R(S_1)$. Let $S^* = S \cup S_1$. Then $X \subset R(S \cup S_1)$; and since, by definition, $R(S \cup S_1) \subset X$, we have $X = R(S \cup S_1)$.

The sets S and S_1 are both R-scattered. Assert $S \cup S_1$ is R-scattered. Deny! Assume there exists an $s \in S$ and $s_1 \in S_1$ such that sRs_1. Since R is symmetric, s_1Rs. Therefore $s_1 \in R(s) \subset R(S)$. This is a contradiction, since $s_1 \in S_1 \subset X - R(S)$. Thus $S \cup S_1$ is R-scattered.

Hence, by Theorem 1 and the above, $S^* = S \cup S_1$ is a maximal R-scattered set, and $S \subset S^*$.

We observe that since each set $\{x\}$, for $x \in X$, is R-scattered, the family of maximal R-scattered sets covers X.

We now characterize the R-scattered sets satisfying the conclusion of MR-II.

THEOREM 2. *Given $(X; R, R^*)$ such that*

(1) *R is a reflexive and symmetric relation over X, and*

(2) *every non-empty subset E of X contains an R^*-dominant element,*

a given R-scattered set S satisfies $X = N(S)$ if and only if

(3) *S is a maximal R-scattered set, and*

(4) *$X - N(S-x) \subset N(x)$ for every $x \in S$.*

Proof. Let S be an R-scattered set and $X = N(S)$. Therefore $X = R(S)$, and, by Theorem 1, S is a maximal R-scattered subset of X. Let $x' \in X - N(S-x)$. Thus $x' \in X$, but there is no $s \in S-x$ such that $x'Rs$ and $x'R^*s$. However, since $S = N(S)$, there must be an element $s' \in S$ such that $x'Rs'$ and $x'R^*s'$. From the above $s' \notin S-x$; therefore, $s' = x$. Thus $x' \in N(x)$, and $x - N(S-x) \subset N(x)$ for every $x \in S$.

Assume that the R-scattered set S has properties (3) and (4) and that $X - N(S) \neq \emptyset$. Let $x' \in X - N(S)$. Since S is a maximal R-scattered set, it follows from Theorem 1 that $X = R(S)$. Therefore, there is an $s' \in S$ such that $x'Rs'$. But $X - N(S) \subset X - N(S-s') \subset N(s')$; that is, $x' \in N(s') \subset N(S)$. This is a contradiction; therefore $X = N(S)$.

This characterization and MR-II yield

MR-E_3. *Given $(X; R, R^*)$ such that*

(1) *R is reflexive and symmetric over X, and*

(2) *every non-empty subset E of X contains an R^*-dominant element,*

then there exists an R-scattered subset of X such that

(3) *S is a maximal R-scattered set, and*

(4) $X - N(S-x) \subset N(x)$ *for every* $x \in S$.

Theorem 3, below, contains another characterization of R-scattered sets satisfying the conclusion of MR-II.

THEOREM 3. *Given* $(X; R, R^*)$ *such that*

(1) *R is reflexive and symmetric over X, and*

(2) *every non-empty subset E of X contains an* R^**-dominant element,*

a given R-scattered set S satisfies $X = N(S)$ *if and only if*

(3) *S is a maximal R-scattered set, and*

(4) $X - N(S-X) \subset R^*(x)$ *for all* $x \in S$.

Proof. Let S be an R-scattered set and $X = N(S)$. By Theorem 2, S is a maximal R-scattered set, and $X - N(S-x) \subset N(x)$ for all $x \in S$. But $N(x) = R(x) \cap R^*(x)$; therefore, $N(x) \subset R^*(x)$. Hence $X - N(S-x) \subset R^*(x)$ for all $x \in S$.

Assume that an R-scattered set S has properties (3) and (4), and that $X - N(S) \neq \Phi$. Let $x' \in X - N(S)$. As in Theorem 2, S being a maximal R-scattered set implies $X = R(S)$; therefore, there is an $s' \in S$ such that $x'Rs'$. But $X - N(S) \subset X - N(S-s') \subset R^*(s')$; that is, $x' \in R^*(s')$. This is a contradiction; for $x' \in X - N(S)$ implies x' cannot be in the R and R^* relationship to the same element in S. Therefore $X = N(S)$.

In view of Theorem 3 and MR-II, the following proposition is equivalent to the Mickle-Rado theorems.

MR-E_4. *Given* $(X; R, R^*)$ *such that*

(1) *R is reflexive and symmetric over X, and*

(2) *every non-empty subset E of X contains an* R^**-dominant element,*

then there exists an R-scattered subset S of X such that

(3) *S is a maximal R-scattered set, and*

(4) $X - N(S-x) \subset R^*(x)$ *for all* $x \in S$.

5. Some maximality principles. Some of the known set-theoretic maximality principles are derived from our formulations.

VAUGHT'S PRINCIPLE [3]. *Every family* $\mathcal{S}$ *contains a maximal subfamily* $\mathcal{S}'$ *of disjoint sets.*

Proof. Define the binary relation R over $\mathcal{S}$ as follows: $S'RS''$ if and only if (1) $S' = S''$, or (2) $S' \neq S''$ and $S' \cap S'' \neq \Phi$. A subfamily $\mathcal{S}'$ of $\mathcal{S}$ is a disjoint subfamily if and only if $\mathcal{S}'$ is an R-scattered set. The conclusion follows from MR-E_1.

In [3], Vaught provides an interesting demonstration of the equivalence of his principle and the axiom of choice. In view of the above, Vaught's principle and MR-E_1 are equivalent. Similarly, Wallace [4] indicates the equivalence of his principle, stated below, and the axiom

of choice. We show that Wallace's principle is an immediate consequent of MR-E_2, and we observe, furthermore, that Wallace's principle is equivalent to MR-E_2.

WALLACE'S PRINCIPLE [4]. *If* R^* *is an arbitrary relation over the space X, then every* R^**-coherent subset of X is contained in a maximal* R^**-coherent subset of X.*

Proof. Define the binary relation R over X as follows: xRy if and only if either (1) $x = y$, or (2) $x \neq y$ and x, y are R^*-incomparable. A set $C \subset X$ is an R^*-coherent set if and only if C is an R-scattered set. The conclusion follows from MR-E_2.

References

[1] E. J. Mickle and T. Rado, *On covering theorems*, Fund. Math. 45 (1957), pp. 325-331.

[2] A. P. Morse, *A theory of covering and differentiation*, Trans. Amer. Math. Soc. 55 (1944), pp. 205-235.

[3] R. L. Vaught, *On the equivalence of the axiom of choice and a maximal principle*, Bull. Amer. Math. Soc. 58 (1952), p. 66.

[4] A. D. Wallace, *A substitute for the axiom of choice*, Bull. Amer. Math. Soc, 50 (1944), p. 278.

[5] M. Zorn, *A remark on method in transfinite algebra*, Bull. Amer. Math. Soc. 41 (1935), pp. 667-670.

UNIVERSITY OF BUFFALO
UNIVERSITY OF AKRON

Reçu par la Rédaction le 20. 8. 1962