Weak products of spaces and complexes

by

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Weak products of spaces and complexes. The Cartesian product of two locally finite CW complexes is also a CW complex. A well-known example due to C. H. Dowker [2] shows that "locally finite" cannot be removed from this statement. In this paper, a result in the opposite direction is proved; we construct a "product space" of arbitrarily many spaces, which is a CW complex whenever each of its factors is a locally finite CW complex. This "product" is, of course, not the usual topological product: for this there is substituted a straightforward generalization of the notion of a weak vector space. We also calculate the homotopy groups of such a "weak product" in terms of those of its factors.

§ 1. Introduction: notations and lemmas. Kelley's notation $\underset{\alpha\in A}{\times} E_\alpha$ will be used for the product set of the collection of sets $\{E_\alpha|\ \alpha\in A\}$, the sign $\prod$ being reserved for the usual topological product ([5], p. 89), or the Cartesian product of groups. All spaces will be assumed to be provided with a basepoint (and so in particular, no space will be empty); basepoints will be denoted systematically by asterisks. The letter X will be used as an abbreviation for a Cartesian product like $\underset{\alpha\in A}{\times} E_\alpha$. The projection maps from such a product will be written $\mathrm{pr}_\beta\colon X\to E_\beta$. The "injection" maps defined by using the basepoints will be written $\mathrm{inj}_\beta\colon E_\beta\to X$. Thus, for any point z of E_β,

$$\mathrm{pr}_\alpha \mathrm{inj}_\beta z = *_\alpha \text{ if } \alpha\neq\beta, \quad \text{and} \quad \mathrm{pr}_\beta \mathrm{inj}_\beta z = z\,.$$

If $x\in X$, the notation x_α will be used for $\mathrm{pr}_\alpha x$; and x will then be written (x_α), the round brackets being used to indicate an indexed set.

The phrase "all but finitely many" will be abbreviated to "abfm".

Suppose that $\{Y_\alpha|\ \alpha\in A\}$ is a collection of sets, indexed by some set A, and that each Y_α contains a specified "basepoint" $*_\alpha$. If x is a point of X, then we denote by $\|x\|$ the subset of A defined by

$$\alpha\in\|x\| \quad \text{iff} \quad x_\alpha\neq *_\alpha\,.$$

We call that subset of X consisting of the points x for which $\|x\|$ is finite, the *restricted product* of the given collection of sets: we denote the restricted product by L. Further, if N is a finite subset of A, we denote by L_N the set of points x of L with $\|x\| \subset N$. There is a natural one-to-one correspondence between L_N and $\mathop{\times}_{\alpha\in N} Y_\alpha$: we shall when necessary identify the two sets under this correspondence. If N has just one member β, then L_N is the range of the injection map inj_β.

Now let each Y_α be a topological space. For every finite subset N of A, we endow L_N with the topology of $\prod_{\alpha\in N} Y_\alpha$. Evidently, these topologies agree on the intersections $L_N \cap L_M$. We now give L the "weak topology" [1] with respect to its covering by all the L_N. Explicitly,

G is open in L iff $G \cap L_N$ is open in L_N, for each finite subset N of A.

It is easy to check that the topology induced from L onto its subspace L_N is the same as has been given to L_N already; and that if A is finite, L has the topology of $\prod_{\alpha\in A} Y_\alpha$.

DEFINITION 1. The space L defined above is the *weak product* of its *factors* Y_α. Its *basepoint* is $(*_\alpha)$ (which we shall write as $*$). We shall also use the notation $\mathop{L}_{\alpha\in A} Y_\alpha$, or $L(Y_\alpha, *_\alpha)$ when it is necessary to emphasise the basepoint. The sets L_N where N is finite will be called *finite subproducts* of L.

We see first that L does not depend essentially on its basepoint. That is, if x is a point of L, then the pair (L, x) is homeomorphic to the space-with-basepoint $\mathop{L}_{\alpha\in A}(Y_\alpha, x_\alpha)$. For the underlying sets are the same, since $x_\alpha = *_\alpha$ for abfm α. The identity map on $\mathop{\times}_{\alpha\in A} Y_\alpha$, restricted to the set involved, plainly takes x to the basepoint of $\mathop{L}_{\alpha\in A}(Y_\alpha, x_\alpha)$; also, this map is continuous both ways, for each finite subproduct of $L(Y_\alpha, x_\alpha)$ is contained in a finite subproduct of $L(Y_\alpha, *_\alpha)$, and *vice versa*.

Thus we have defined a product space for each collection of topological spaces. The products of different collections are related in a natural way. For suppose that $\{Z_\alpha \mid \alpha\in A\}$ is another set of spaces, and that for each α there is a continuous map f_α from Y_α to Z_α, preserving basepoints. Let us write f for the transformation from $\mathop{\times}_{\alpha\in A} Y_\alpha$ to $\mathop{\times}_{\alpha\in A} Z_\alpha$ given by $f(x_\alpha) = (f_\alpha x_\alpha)$. Then

LEMMA 1. *f is a continuous map from $\mathop{L}_{\alpha\in A} Y_\alpha$ to $\mathop{L}_{\alpha\in A} Z_\alpha$, and preserves basepoints.*

Proof. Plainly, $\|(f_\alpha x_\alpha)\| \subset \|x\|$, so that f takes the restricted product into the restricted product, and preserves basepoints. Now, let G be an open set in $\mathop{L}_{\alpha\in A} Z_\alpha$, and let us consider the subset L_N of $\mathop{L}_{\alpha\in A} Y_\alpha$ and the corresponding subset, L'_N say, of $\mathop{L}_{\alpha\in A} Z_\alpha$. Then plainly $fL_N \subset L'_N$. Hence

$$L_N \cap f^{-1}G = L_N \cap f^{-1}(G \cap L'_N);$$

and the latter set is the same as $(f|L_N)^{-1}(G \cap L'_N)$.

But $G \cap L'_N$ is open in L'_N, and the map $f|L_N$ is (up to the natural identification of L_N with $\mathop{\times}_{\alpha\in N} Y_\alpha$) merely the product $\prod_{\alpha\in N} f_\alpha$, and is thus a continuous map from L_N to L'_N. Hence $(f|L_N)^{-1}(G \cap L'_N)$ is open in L_N; that is, $L_N \cap f^{-1}G$ is open in L_N, for each finite subset N of A. So $f^{-1}G$ is open, and therefore f is continuous.

LEMMA 2. *If B is a subset of A and for each α in $A-B$ the space Y_α contains just one point, then $\mathop{L}_{\alpha\in A} Y_\alpha$ and $\mathop{L}_{\alpha\in B} Y_\alpha$ are naturally homeomorphic.*

Proof. For then $\|x\| \subset B$ for all x. Thus the underlying sets are the same. For the topology, we remark that if N is finite, the subproduct L_N of $\mathop{L}_{\alpha\in A} Y_\alpha$ is homeomorphic to the subproduct $L_{N\cap B}$ of $\mathop{L}_{\alpha\in B} Y_\alpha$. Thus a set which is open in L_B is open also in L_A; and the converse is trivial.

LEMMA 3. *Each injection inj_β: $Y_\beta \to \mathop{L}_{\alpha\in A} Y_\alpha$ is continuous.*

Proof. Let us write Z_α for the set $\{*_\alpha\} \subset Y_\alpha$ if $\alpha \neq \beta$, and Z_β for Y_β. Then inj_β is the product of the inclusion maps $Z_\alpha \subset Y_\alpha$ for $\alpha \neq \beta$, with the identity $Y_\beta \to Y_\beta$. Thus inj_β is continuous from $\mathop{L}_{\alpha\in A} Z_\alpha$ to $\mathop{L}_{\alpha\in A} Y_\alpha$, by Lemma 1. However, by Lemma 2, $\mathop{L}_{\alpha\in A} Z_\alpha$ is homeomorphic to Y_β.

LEMMA 4. *Each projection pr_β: $\mathop{L}_{\alpha\in A} Y_\alpha \to Y_\beta$ is continuous.*

Proof. Let us write Z_α for the same spaces as before, f_α for the unique map $Y_\alpha \to Z_\alpha$ if $\alpha \neq \beta$, and f_β for the identity on Y_β. Then pr_β is the product of the maps f_α, and Lemma 1 shows that pr_β: $\mathop{L}_{\alpha\in A} Y_\alpha \to \mathop{L}_{\alpha\in A} Z_\alpha$ is continuous. However, $\mathop{L}_{\alpha\in A} Z_\alpha$ is homeomorphic to Y_β, by Lemma 2.

LEMMA 5. *Each projection pr_β is an open map.*

Proof. It has already been remarked (just after Definition 1) that the space L is independent of its basepoint. Thus it suffices to show that if U is an open set in L containing $*$, then $\mathrm{pr}_\beta U$ is a neighbourhood of $*_\beta$. But U contains the subset $U \cap \mathrm{inj}_\beta Y_\beta$, which is open in $\mathrm{inj}_\beta Y_\beta$; and $\mathrm{pr}_\beta|\mathrm{inj}_\beta Y_\beta$ is a homeomorphism. Thus the subset $\mathrm{pr}_\beta(U \cap \mathrm{inj}_\beta Y_\beta)$ of $\mathrm{pr}_\beta U$ contains $*_\beta$ and is open in Y_β.

It is not difficult to see that L may be characterised in terms of Lemmas 1 and 2 and the remark just before Definition 1. In fact, if P is any product with these properties, then there is a naturally-defined map from $\underset{\alpha\in A}{L} Y_\alpha$ to $\underset{\alpha\in A}{P} Y_\alpha$ which is one-to-one and continuous. That is, L is the smallest space with these properties, and has the largest topology.

We remark at this point that the product L is not associative; that is, if the index set A is partitioned into subsets A_β, and β ranges over B, then $L = \underset{\alpha\in A}{L} Y_\alpha$ and $R = \underset{\beta\in B}{L}\ \underset{\alpha\in A_\beta}{L} Y_\alpha$ are, in general, different spaces (on the same underlying set). An example to show this is given after Theorem 3.

If E is any subset of L, we shall write $\|E\|$ for $\bigcup_{x\in E}\|x\|$. Thus, $\|E\|$ is finite if and only if E is a subset of a finite subproduct of L.

LEMMA 6. *Let C be a countably compact subset of L. If each factor space Y_α is T_1, then $\|C\|$ is finite.*

Proof. Suppose that, on the contrary, $\|C\|$ is infinite. Then we can choose a sequence of distinct points $x^{(1)}, x^{(2)}, \ldots$ of C, and a sequence of values $(1, 2, \ldots,$ say) of α, such that

$$x_i^{(i)} \neq *_i \quad \text{and} \quad x_j^{(i)} = *_j \quad \text{for all} \quad j > i\,.$$

Now, for each i there is an open neighbourhood U_i of $*_i$ excluding $*_i^{(i)}$. We define subsets $U^{(i)}$ of L as follows.

$$y \in U^{(i)} \quad \text{iff} \quad y_j \in U_j \quad \text{for all} \quad j \geqslant i\,.$$

Plainly, each $U^{(i)}$ is open in the topology of L. Now, suppose that $i > k$. Then, for all $j \geqslant i$, we have

$$x_j^{(k)} = *_j \in U_j\,, \quad \text{and so} \quad x^{(k)} \in U^{(i)}\,.$$

However, if $k \geqslant i$, then $x^{(k)}$ does not lie in $U^{(i)}$; for $x_k^{(k)} \notin U_k$. That is, $U^{(i)}$ contains just those $x^{(k)}$ for which $k < i$.

However, the set consisting of all the $x^{(i)}$ is closed in L. For if z is any other point of L, then $\|z\|$ is finite. If $z_i = *_i$, we can choose a neighbourhood of z_i to exclude $x_i^{(i)}$. For each of the finitely many i in $\|z\|$, there is some α for which $z_\alpha \neq x_\alpha^{(i)}$, and we pick a neighbourhood of z_α excluding $x_\alpha^{(i)}$. For each value of α we have now chosen in all only finitely many neighbourhoods of z_α (possibly none at all); let V_α be the intersection of these neighbourhoods. Then the set V defined by

$$y \in V \quad \text{iff} \quad y_\alpha \in V_\alpha \quad \text{for each} \quad \alpha$$

is plainly a neighbourhood of z; and none of the points $x^{(i)}$ belongs to V. Thus $\{x^{(i)} |\ \text{all}\ i\}$ is closed in L.

Let W be the complement of this set, and consider the countable open covering of C by W and all the sets $U^{(j)}$. W contains no $x^{(i)}$, and each $U^{(j)}$ contains only finitely many, so the covering has no finite subcover; this contradicts the countable compactness of C.

COROLLARY. *In fact, this proof goes through with the, usually much stronger, "box topology" on the product* ([5], problem 3 V).

§ 2. Homotopy groups of L. We now consider the homotopy groups of various spaces. The basepoint of a homotopy group is always at a given basepoint of the space, so we shall suppress explicit mention of the former. For convenience in writing the proof of Theorem 2, we write all groups, abelian or not, in the additive notation.

DEFINITION 2. If $\{G_\alpha |\ \alpha \in A\}$ is a set of groups, we denote by $\prod_{\alpha\in A} G_\alpha$ the "unrestricted direct product" of the G_α; that is, the Cartesian product of the sets G_α with the group structure of "coordinatewise multiplication".

DEFINITION 3. The "restricted direct product" of the G_α, that is, that subgroup of $\prod_{\alpha\in A} G_\alpha$ consisting of the elements with abfm coordinates equal to zero, will be denoted here by $\sum_{\alpha\in A} G_\alpha$.

THEOREM 1. *We have*

$$\pi_n\big(\prod_{\alpha\in A} Y_\alpha\big) \cong \prod_{\alpha\in A} \pi_n(Y_\alpha)\,.$$

The proof of this is omitted; it is true for any spaces Y_α, and any index set A. It is proved in [4], Ch. IV, Th. 6.1, when A has two members and each factor space is Hausdorff. The latter assumption is not used; and the proof requires only minor modifications to apply to an arbitrary set A.

THEOREM 2. *We have*

$$\pi_n\big(\underset{\alpha\in A}{L} Y_\alpha\big) \cong \sum_{\alpha\in A} \pi_n(Y_\alpha)$$

for any T_1 spaces Y_α, and any index set A.

To establish this theorem, we use the characterization of $\sum$ given in [6], pp. 119 and 122. This is:

LEMMA 7. *If a group G contains subgroups G_α such that*

(i) *elements of distinct G_α commute, and*

(ii) *each non-zero element of G has a representation as a sum of one non-zero element from each of finitely many of the G_α, and this representation is unique apart from reordering of the summands,*

then G is isomorphic to $\sum_{\alpha\in A} G_\alpha$.

Proof of Theorem 2. Let the projection $\mathrm{pr}_\alpha\colon L \to Y_\alpha$ induce the map $p_\alpha\colon \pi_n(L) \to \pi_n(Y_\alpha)$ and let the injection $\mathrm{inj}_\alpha\colon Y_\alpha \to L$ induce $i_\alpha\colon \pi_n(Y_\alpha) \to \pi_n(L)$. Then, as $\mathrm{pr}_\alpha \mathrm{inj}_\alpha$ is the identity map on Y_α, its induced map $p_\alpha i_\alpha$ is the identity on $\pi_n(Y_\alpha)$. Thus, i_α is a monomorphism, and so $i_\alpha \pi_n(Y_\alpha)$ is an isomorphic copy of $\pi_n(Y_\alpha)$ in $\pi_n(L)$. We write this subgroup of $\pi_n(L)$ as G_α, and use Lemma 7.

(i) If $n > 1$, it is trivial that elements of distinct G_α commute, for all the groups involved are then abelian. If $n = 1$, $\lambda \in G_\beta$ and $\mu \in G_\gamma$, then we consider the injection map of $Y_\beta \times Y_\gamma$ into $\bigtimes_{\alpha \in A} Y_\alpha$. This plainly induces a monomorphism (say, j) of $\pi_1(Y_\beta \times Y_\gamma)$ into $\pi_1(L)$, and the elements λ and μ lie in the image of j. But, by Theorem 1, $\pi_1(Y_\beta \times Y_\gamma) \cong \pi_1(Y_\beta) \times \pi_1(Y_\gamma)$, so that in particular, $j^{-1}\lambda$ and $j^{-1}\mu$ commute. Thus so do λ and μ.

(ii) Now let ν be an element of $\pi_n(L)$, represented by the map $f\colon S^n, * \to L, *$. The n-sphere is compact, hence so is the image of f, and Lemma 6 shows that $\|fS^n\|$ is finite—that is, for abfm α, $\mathrm{pr}_\alpha f S^n = *_\alpha$. Hence abfm of the elements $i_\alpha p_\alpha \nu$ are equal to the zero of $\pi_n(L)$, and for each α, $i_\alpha p_\alpha \nu$ lies in G_α. Thus the element

$$\tau = \nu - \sum_{\alpha \in A} i_\alpha p_\alpha \nu$$

is well-defined. (The summation is meaningful because abfm summands are zero; and it is unambiguous because the summands commute.) Moreover, for each β,

$$p_\beta \tau = p_\beta \nu - \sum_{\alpha \in A} p_\beta i_\alpha p_\alpha \nu = p_\beta \nu - p_\beta \nu ,$$

so that $p_\beta \tau$ is the zero of $\pi_n(Y_\beta)$. Let $g\colon S^n, * \to L, *$ be a map representing τ. Lemma 6 shows that $\|gS^n\|$ is a finite set, say N. Thus, for $\alpha \notin N$, $\mathrm{pr}_\alpha g$ is the constant map to $*_\alpha$. Moreover, for $\alpha \in N$, $\mathrm{pr}_\alpha g$ is a representative of $p_\alpha \tau$, and hence is null-homotopic. Suppose that this is so by the homotopy

$$G_\alpha\colon I \times S^n \to Y_\alpha ,$$

and for each $\alpha \notin N$, let G_α be the constant map of $I \times S^n$ to $*_\alpha$. Then the product map $G\colon I \times S^n \to L$ of all the G_α is continuous, for it is essentially the same as $\prod_{\alpha \in N} G_\alpha\colon I \times S^n \to \prod_{\alpha \in N} Y_\alpha = L_N$. But G is a null-homotopy of g. Thus the homotopy class τ of g is the zero of $\pi_n(L)$. That is,

$$\nu = \sum_{\alpha \in A} i_\alpha p_\alpha \nu ,$$

and each element of $\pi_n(L)$ is represented as the sum of elements from the subgroups G_α, as required by Lemma 7. It remains to be seen that this representation is unique.

Suppose then that

$$\nu = \sum_{\alpha \in A} i_\alpha \varkappa_\alpha ,$$

where $\varkappa_\alpha \in \pi_n(Y_\alpha)$, and $\varkappa_\alpha = 0$ for abfm α.

Then

$$p_\beta \nu = \sum_{\alpha \in A} p_\beta i_\alpha \varkappa_\alpha = p_\beta i_\beta \varkappa_\beta = \varkappa_\beta ,$$

so that $\varkappa_\beta = p_\beta \nu$, and the representation of ν is the same as the one obtained above.

Thus, the conditions of Lemma 7 are satisfied, and so $\pi_n(L)$ is isomorphic to $\sum_{\alpha \in A} G_\alpha$; that is,

$$\pi_n(L) \cong \sum_{\alpha \in A} \pi_n(Y_\alpha) .$$

Remarks. (i) The isomorphism established here is plainly "natural". That is, if $\{Z_\alpha | \alpha \in A\}$ is another set of T_1-spaces, with maps $f_\alpha\colon Y_\alpha \to Z_\alpha$, then the product map f (as in Lemma 1) satisfies the commutativity relation

$$\varphi \circ \pi_n(f) = \big(\sum \pi_n(f_\alpha)\big) \circ \varphi\colon \pi_n\big(\underset{\alpha \in A}{L} Y_\alpha\big) \to \sum_{\alpha \in A} \pi_n(Z_\alpha)$$

where $\pi_n(h)$ is the map of the nth homotopy groups induced by h, and φ is the isomorphism established in Theorem 2.

(ii) Theorem 2 remains true for $n = 0$, without the group structure and without the hypothesis that the factor spaces are T_1. This specifies the path-components of L in terms of those of the factors. Explicitly, if the path-components of Y_α are $\{Y_\alpha^\lambda | \lambda \in \Lambda_\alpha\}$, then a typical path-component of L is obtained by picking for each α an element λ_α of Λ_α, forming the Cartesian product over A of the sets $Y_\alpha^{\lambda_\alpha}$, and taking its intersection with L.

(iii) Lemma 7 can also be used to show, without much difficulty, that if each factor space Y_α is T_1 and n—LC [7] then the product L is also n—LC.

§ 3. L as a CW complex. Suppose that for each α there is given a closure-finite abstract cell-complex P_α, with basepoint $*_\alpha$, a 0-cell of P_α. That is, P_α is a collection of "cells" c of specified dimension $\dim c$ (a non-negative integer), together with incidence numbers, written as $[c : c']$ (an integer) such that

(i) $[c : c'] = 0$ unless $\dim c = 1 + \dim c'$,

(ii) for each c, $[c : c'] = 0$ for abfm cells c', and

(iii) for each fixed pair of cells c, c'',

$$\sum [c : c'][c' : c''] = 0 ,$$

where the summation is over all cells $c' \in P_\alpha$. Suppose also that A is linearly ordered. Then a closure-finite abstract cell-complex is defined as follows. The collection of cells is the restricted product of the sets of cells of the P_α, with dimensions

$$\dim c = \sum_{\alpha \in A} \dim c_\alpha ,$$

and with incidence numbers specified by the rule:

$[c : c'] = 0$ unless $c'_\alpha = c_\alpha$ for all but one value of α, and if $\{\alpha |\ c_\alpha \neq c'_\alpha\} = \{\beta\}$, then

$$[c : c'] = (-1)^\sigma [c_\beta : c'_\beta] ,$$

where $\|c\| = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $\alpha_1 < \alpha_2 \ldots < \alpha_n$ in the ordering of A, $\dim c_{\alpha_i} = d_i$, $\beta = \alpha_b$, and $\sigma = \sum \{d_i |\ i < b\}$.

For the sum σ we shall use the notation $\sigma(c, b)$, and we shall write εk for $(-1)^k$. We shall also abbreviate $[c_\beta : c'_\beta]$ to $[c : c' : \beta]$.

Definition 5. Let this collection of cells, dimensions and incidence numbers be denoted by S.

Lemma 8. *S is a closure-finite abstract cell-complex.*

Proof. (i) is evident.

(ii) For each c, $\|c\|$ is finite, and if $\beta \in \|c\|$ then there are only finitely many cells c'_β of P_β with $[c : c' : \beta] \neq 0$.

(iii) If $[c : c'][c' : c''] \neq 0$, it must happen that $c_\alpha = c'_\alpha$ for all but one value, β, of α, and $c'_\alpha = c''_\alpha$ for all but one value, γ, of α, and moreover that $\dim c_\beta = 1 + \dim c'_\beta$, and $\dim c'_\gamma = 1 + \dim c''_\gamma$. So $\sum, \sum = [c : c'][c' : c'']$, $= 0$ unless either

(a) c_α and c''_α differ for just two values β and γ of α, and

$$\dim c_\beta = 1 + \dim c''_\beta , \qquad \dim c_\gamma = 1 + \dim c''_\gamma ,$$

or

(b) c_α and c''_α differ for just one value (β, say) of α, and

$$\dim c_\beta = 2 + \dim c''_\beta .$$

We consider the two cases separately.

(a) Then $\sum = [c : d][d : c''] + [c : e][e : c'']$, where

$$d_\alpha = c_\alpha \quad \text{for} \quad \alpha \neq \beta \text{ and } d_\beta = c''_\beta ,$$
$$e_\alpha = c_\alpha \quad \text{for} \quad \alpha \neq \gamma \text{ and } e_\gamma = c''_\gamma ,$$

for all the other summands are necessarily zero. Suppose now that $\|c\| = \{\alpha_1, \ldots, \alpha_n\}$ in order, and

$$\beta = \alpha_b , \quad \gamma = \alpha_g , \quad b < g .$$

Then

$$\sum = \varepsilon\sigma(c, b)[c : c'' : \beta] \cdot \varepsilon\sigma(c', g)[c : c'' : \gamma] + \\ + \varepsilon\sigma(c, g)[c : c'' : \gamma] \cdot \varepsilon\sigma(c', b)[c : c'' : \beta] .$$

But

$$\sigma(c, b) = \sigma(c', b) - d_b + d'_b = \sigma(c', b) - 1 ,$$

and

$$\sigma(c', g) = \sigma(c, g) .$$

So

$$\varepsilon\sigma(c, b) \cdot \varepsilon\sigma(c', g) = -\varepsilon\sigma(c', b) \cdot \varepsilon\sigma(c, g) .$$

Thus

$$\sum = \varepsilon\sigma(c, g) \cdot \varepsilon\sigma(c', b) \cdot (-1 + 1)[c : c'' : \beta][c : c'' : \gamma] = 0 .$$

(b) In this case, $[c : c'][c' : c''] = 0$ unless $c'_\alpha = c_\alpha$ for all α except β. Thus

$$\sum = \sum \varepsilon\sigma \cdot [c : c' : \beta] \cdot \varepsilon\sigma' \cdot [c' : c'' : \beta]$$

where the sum is taken over $c'_\beta \in P_\beta$, and where $\sigma = \sigma(c, b)$ and $\sigma' = \sigma(c', b)$, so that $\sigma = \sigma'$. Thus $\varepsilon\sigma \cdot \varepsilon\sigma' = 1$, and

$$\sum = \sum [c : c' : \beta][c' : c'' : \beta] = 0 .$$

Thus S is indeed a closure-finite abstract cell-complex.

The incidence numbers in S depend not only on the complexes P_α but also on the linear ordering of A. The latter is, however, involved only inessentially.

Definition 6. Two abstract cell-complexes are *equivalent* if they have the same cells and dimensions, and their incidence numbers $[c : c']$ and $[\![c : c']\!]$ are related by

$$[\![c : c']\!] = \lambda c \cdot \lambda c' \cdot [c : c'] ,$$

where λ is a function defined on the cells of the complex and taking values ± 1.

Lemma 9. *The complexes S obtained from a given set $\{P_\alpha\}$ of complexes and two different orderings of A are equivalent. The details of the proof, which is not difficult, are omitted. The reorientation λc is given by*

$$\lambda c = \varepsilon \sum d_i d_j$$

where $\|c\| = \{\alpha_1, \ldots, \alpha_n\}$, $\dim c_{\alpha_i} = d_i$, and the summation is taken over all pairs i, j such that α_i and α_j are interchanged on passing from the one order to the other.

From now on, we take some fixed linear ordering of A, and the orientations in S defined by it.

S has been constructed as a closure-finite abstract complex from any closure-finite abstract complexes P_a. We now show that if each P_a is realised as a CW complex on the topological space $|P_a|$, then S also can be realised on the product $\underset{a\in A}{L}|P_a|$, provided that each P_a is *locally finite*.

DEFINITION 7. *$|P|$ is a CW complex realising the abstract complex P if*

(i) *$|P|$ is a Hausdorff space.*

(ii) *$|P|$ is the disjoint union of cells $|c|$, one for each cell c of P.*

(iii) *There is an attaching map $f_c: K\to|P|$, where $d=\dim c$ and K is the Euclidean d-cube*

$$\{(t_j)|\ 0\leqslant t_j\leqslant 1 \text{ for } 1\leqslant j\leqslant d\}.$$

(iv) *If bK denotes the "boundary" of K, that is, its boundary in Euclidean d-space if $d>0$, and $\emptyset$ if $d=0$, then*

$$f_c|(K-bK) \quad \text{is a homeomorphism onto } |c|.$$

(v) *f takes bK into the $(d-1)$-section $|P^{d-1}|$ of $|P|$, that is, the union of all the sets $|c'|$ for $\dim c'<d$.*

(vi) *If $|\bar{c}|$ denotes f_cK, then each set $|\bar{c}|$ is contained in a finite subcomplex $|F|$ of $|P|$; that is, $|F|$ is the union of finitely many cells of $|P|$, such that if $|c|\subset|F|$ then $|\bar{c}|\subseteq|F|$.*

(vii) *A subset G of $|P|$ is open if and only if $G\cap|F|$ is open in $|F|$ for each finite subcomplex $|F|$.*

(viii) *There is for each c, an orientation of the corresponding cube K such that the incidence number of two cells $|c|$ and $|c'|$ of $|P|$, as defined by using their attaching maps and the homology boundary of the triple $|P^d|$, $|P^{d-1}|$, $|P^{d-2}|$ (where $d=\dim c$) coincides with the incidence number $[c:c']$ originally specified in P.*

An orientation of K may be specified by ordering its d factors $I\times I\ldots\times I$. We shall not use the explicit definition of the incidence numbers, since the incidence of S is expressed in terms of that of the P_a, which we suppose to be known.

The complex $|P|$ is said to be *locally finite* if

(ix) *each point of $|P|$ is an interior point of some finite subcomplex.*

We shall need the following fact about CW complexes:

LEMMA 10. *If $[c:c']\neq 0$, then $|\bar{c}|\supset|\bar{c}'|$.*

Now suppose that each P_a is realised as a locally finite CW complex $|P_a|$. For each cell c of S, let $|c|$ be the subset of $\underset{a\in A}{L}|P_a|$ defined by

$$|c|=(|c_a|).$$

If $\|c\|=\{a_1,a_2,\ldots,a_n\}$, in the order of A, $\dim c_{a_i}=d_i$, and $\sum_{1\leqslant i\leqslant n} d_i=d$, then let f_c be the map of the d-cube defined by regarding it as the product of cubes $K_1\times K_2\times\ldots\times K_n$, with $\dim K_i=d_i$, and taking as f_c the product, in order, of the attaching maps of the factor cells c_{a_i}. Finally, let the orientation of K for the cell c be specified by the order: all factors of K_1 first, in the order in which they occur to orientate $|c_{a_1}|$, then the factors of K_2, and so on.

DEFINITION 8. Let $|S|$ denote the space $\underset{a\in A}{L}|P_a|$, with the cells, attaching maps and orientations described above.

LEMMA 11. *$|S|$ is a realisation of S as a CW complex.*

Proof. The proofs of (i) to (vi) are straightforward, and do not use the assumption that the factors are locally finite.

(vii) Let $\mathfrak{L}$ denote the topology of L, and $\mathfrak{S}$ the weak topology on finite subcomplexes of $|S|$. Plainly $\mathfrak{L}\subset\mathfrak{S}$, for a union of finitely many cells of $|S|$ must be contained in a finite subproduct of L. Now let G be a set open in $\mathfrak{S}$, and N a finite subset of A. The subproduct L_N of L is itself a subcomplex of $|S|$ (not of course in general finite). Consider any point $x\in G\cap L_N$. L_N has the product topology of the finite set of spaces $\{|P_a|\ a\in N\}$; but the product of finitely many locally finite CW complexes is locally finite. Thus, x is in the interior, relative to L_N, of some finite subcomplex $|H|$ of L_N. As $|H|$ is also a finite subcomplex of $|S|$, and $G\in\mathfrak{S}$, it follows that $G\cap|H|$ is open in $|H|$, and so x is in the interior, relative to $|H|$, of $G\cap|H|$. But if $D\subset E\subset F$, then $\mathrm{Int}_F E\cap\mathrm{Int}_E D\subset\mathrm{Int}_F D$, so that x is in the interior, relative to L_N, of $G\cap|H|$, and hence of $G\cap L_N$. That is, each point of $G\cap L_N$ is in its interior relative to L_N. Hence $G\in\mathfrak{L}$.

(viii) Suppose as usual that $\|c\|=\{a_1,\ldots,a_n\}$ and $c'_a=c_a$ except for $a=a_b$. (Plainly, unless this is so, the incidence of $|c'|$ with $|c|$ in $|S|$ must be zero, by Lemma 10.) Then

$$f_c=f_1\times f_2\times f_3 \quad\text{and}\quad f_{c'}=f_1\times f_4\times f_3,$$

where f_1 is the product in order of the attaching maps of the factors of c for $i<b$, f_3 is the same for $i>b$, f_2 is the attaching map of c_{a_b} and f_4 is the attaching map of c'_{a_b}.

But an incidence number depends only on the two cells involved, their attaching maps, and their orientations, not on the rest of the complex. Thus, $[c:c']$ is the same here as in the product of three finite complexes. However, we then have, with the method of orientation adopted,

$$[c_1\times c_2:c_1\times c_4]=\varepsilon\dim c_1\cdot[c_2:c_4]$$

and

$$[c_2\times c_3:c_4\times c_3]=[c_2:c_4].$$

Hence $[c : c'] = \varepsilon \dim K_1 \cdot [c : c' : \beta]$, where $\dim K_1 = \sum\{d_i | i < b\}$: and this is the incidence number originally given for S.

Thus we have proved

THEOREM 3. *If each Y_α is the space of a locally finite* CW *complex, then $\underset{\alpha \in A}{L} Y_\alpha$ is the space of a* CW *complex (not, of course, in general a locally finite one).*

Remarks. (i) This shows that a vector space with Dugundji's weak topology [3] is a CW complex. For in specifying the Dugundji topology, we may take a fixed basis of the space, and consider only those finite-dimensional subspaces spanned by subsets of the basis. The space thus becomes an L-product of real lines; and the real line is certainly a locally finite CW complex.

(ii) L is not associative. For if each Y_α is the unit interval, regarded as a CW complex with three cells, $B = \{0, 1\}$, A_0 is of cardinal $\aleph_0$ and A_1 is of cardinal c, then the subset

$$E = \{x|\ \|x\| \text{ has one member}\}$$

is a subcomplex of the CW complex $L = \underset{\alpha \in A}{L} Y_\alpha$ (and hence has CW topology in it); but as a subset of $R = \underset{\alpha \in A_0}{L} Y_\alpha \times \underset{\alpha \in A_1}{L} Y_\alpha$, E is just the Dowker counter example ([2], p. 563) which does not have the CW topology.

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Sur une propriété des ensembles partiellement ordonnés

par

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Soient E un ensemble quelconque, ϱ une relation binaire définie dans E.

On dit que la relation ϱ établit *un ordre partiel dans E* lorsqu'elle est:

1° *non-réflexive* (c'est-à-dire qu'on ait constamment non $(x \varrho x)$),

2° *transitive.*

On dit que ϱ établit *un ordre dans E* lorsque, en outre, la condition suivante est vérifiée:

3° *Quels que soient $x, y \in E$, $x \neq y$, on a soit $x \varrho y$, soit $y \varrho x$.*

N étant un ensemble partiellement ordonné par la relation σ, soit $f|E$ une fonction telle que $f(E) \subset N$.

Lorsque $f(x) \neq f(y)$ pour $x, y \in E$, $x \neq y$ et $x \varrho y \rightarrow f(x) \sigma f(y)$, nous appellerons f *transformation isomorphe de E dans N.*

Nous dirons alors que les ensembles E et $f(E)$ sont *semblables*, en symbole: $E \simeq f(E)$.

Soit $\varphi \geqslant \omega_0$ un nombre ordinal. Désignons par U_φ l'ensemble de toutes les suites de type φ formées de nombres 0 et 1, ordonné d'après le principe de premières différences, par U_φ^0 le sous-ensemble de U_φ se composant de toutes les suites de la forme $\{a_\xi\}_{\xi<\varphi}$ où $a_\gamma = 1$ pour un $\gamma \geqslant 0$ et $a_\xi = 0$ pour $\xi > \gamma$.

La relation qui établit l'ordre dans U_φ sera désignée, comme d'habitude, par le symbole $\prec$.

Remarquons que, lorsque $N = U_\varphi$, on a la propriété:

(i) *Si $f(x) = \{a_\xi^{(x)}\}_{\xi<\varphi}$, $x \in E$ et si l'on pose pour un $\psi > \varphi$: $g(x) = \{b_\xi^{(x)}\}_{\xi<\psi}$ où $b_\xi^{(x)} = a_\xi^{(x)}$ pour $\xi < \varphi$ et $b_\xi^{(x)} = 0$ pour $\varphi \leqslant \xi < \psi$, alors les transformations $f|E$, $f(E) \subset U_\varphi$, et $g|E$, $g(E) \subset U_\psi$, sont simultanément isomorphes où non.*

C'est une conséquence immédiate de la définition de la relation $\prec$.

Le but de cette Note est de démontrer le théorème suivant:

(T) *Tout ensemble partiellement ordonné de puissance $\aleph_\mu$ est semblable à un sous-ensemble de $U_{\omega_\mu}^0$.*