and R_i depends on the k-th variable under fixed P_i. Now let $D_1, \ldots, D_n$ be numbers which satisfy the system of equations

(52) $$\sum_{i=1}^{n} D_i \left[\frac{\partial}{\partial x_j} Q_i^m\right](p_0) = \varepsilon_j; \quad j = 1, \ldots, n,$$

where $\varepsilon_m = 1$ and $\varepsilon_j = 0$ for $j \neq m$. This system has a solution by (48). Since the Jacobian in (51) is of the form $W\left(\frac{\partial}{\partial x_1} R_1, \ldots, \frac{\partial}{\partial x_m} R_m\right)$ where W is a polynomial with rational coefficients, we are in position to apply Lemma 3.3. Let $\tilde{h}_1^i, \ldots, \tilde{h}_{q_i}^i$ be the functions which satisfy (31), (32) and (33). By (31), (33) and (52) we have

$$\left[\frac{\partial}{\partial x_m} \tilde{R}_m\right](p_0) = 1 \quad \text{and} \quad \left[\frac{\partial}{\partial x_j} \tilde{R}_m\right](p_0) = 0 \quad \text{for} \quad j = 1, \ldots, m-1.$$

Hence

(53) $$[\partial(\tilde{R}_1, \ldots, \tilde{R}_m)/\partial(x_1, \ldots, x_m)](p_0) = [\partial(\tilde{R}_1, \ldots, \tilde{R}_{m-1})/\partial(x_1, \ldots, x_{m-1})](p_0),$$

and by (34), (51)

(54) $$[\partial(\tilde{R}_1, \ldots, \tilde{R}_m)/\partial(x_1, \ldots, x_m)](p_0) = 0,$$

and by (32)

(55) $$[\partial(\tilde{R}_1, \ldots, \tilde{R}_{m-1})/\partial(x_1, \ldots, x_{m-1})](p_0) = [\partial(R_1, \ldots, R_{m-1})/\partial(x_1, \ldots, x_{m-1})](p_0).$$

It is clear that (53), (54) and (55) contradict (50). Therefore (51) cannot hold and we have an open set $V_m \subset V_m^0$ such that

$$[\partial(R_1, \ldots, R_m)/\partial(x_1, \ldots, x_m)](p) \neq 0 \quad \text{for every} \quad p \in V_m.$$

This completes the proof.

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On level sets of a continuous nowhere monotone function

by

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Let $f(x)$ be a real function, defined and continuous in a real closed interval I. Let $f(I)$ denote the interval of values taken by $f(x)$ in I. For any $y \in f(I)$, let $f^{-1}(y)$ denote the set of points in I where $f(x)$ takes the value y. The set $f^{-1}(y)$ is known as the *set of level* y of $f(x)$, or, briefly, as a *level set* of $f(x)$. Evidently, $f^{-1}(y)$ is closed for every y.

K. Padmavally [6] proved in 1953 that:

(*) "If $f(x)$ is continuous but monotonic in no interval, then $f^{-1}(y)$ has the power of the continuum for a set of values of y which is of the second category."

S. Marcus ([4], p. 102) improved this result in 1958 into the following form:

(**) "Given a real function $f(x)$, defined and continuous in I. A necessary and sufficient condition so that $f(x)$ may not be monotone in any interval contained in I is that, for every interval $J \subset I$, the values y for which the set $\{x; f(x) = y, x \in J\}$ is unenumerable form a set of the second category in $(-\infty, \infty)$ and residual in $f(J)$."

Let a function $f(x)$ be *nowhere monotone* in I if it is not monotone in any subinterval of I. Let, further, a nowhere monotone function $f(x)$ be of the *second species* in I in case the function $f(x)+rx$ remains nowhere monotone in I for every real value of r [1].

We prove in § 1 that for a continuous nowhere monotone function $f(x)$ in I, the level set $f^{-1}(y)$ is non-void and perfect for a set of values of y which is residual in $f(I)$. In case of a continuous nowhere monotone function of the second species in I, we investigate in § 2 the sets that are obtained by the intersection of the curve $y = f(x)$ with different straight lines $y = mx + c$.

[1] A detailed study of the Dini derivatives of nowhere monotone functions has been made by the author; see Garg [13], [14]. These investigations are further continued. It may be remarked here that non-differentiable functions constitute a particular case of nowhere monotone functions of the second species.

W. Sierpiński ([8], p. 370) proved in 1926 that for a continuous function $f(x)$ the set H of values y for which $f^{-1}(y)$ is perfect is a $F_{\sigma\delta}$. We prove in § 3 that in case $f(x)$ has no lines of invariability, the set H resembles a set G_δ in the sense that it is residual in each interval in which it is everywhere dense.

1. Nowhere monotone functions. We shall prove the following

THEOREM 1 (²). *A function $f(x)$, continuous in I, is nowhere monotone in I if, and only if, the level set $f^{-1}(y)$ is non-dense for every y, and is non-void and perfect for a set of values of y residual in $f(I)$.*

Let us first prove the following two lemmas:

LEMMA 1. *If $f(x)$ is continuous and nowhere monotone in I, there exists a set H residual in $f(I)$ such that $f(x)$ is oscillating (³), on one side at least, at every point x for which $f(x) \in H$.*

LEMMA 2. *Let $f(x)$ be continuous in I and be such that $f^{-1}(y)$ is non-dense for every $y \in f(I)$. Then, if the set H of values y for which $f^{-1}(y)$ is perfect is everywhere dense in $f(I)$, $f(x)$ is nowhere monotone in I.*

Proof of Lemma 1. Let $f(x)$ be continuous and nowhere monotone in a finite closed interval $I = [a, b]$, and let E denote the set of points in I where $f(x)$ is neither 0_+ nor 0_-. We have to prove that $f(E)$ is of the first category.

If $t \in E$, there exists a maximum real number $h_t > 0$ such that one of the following four conditions is satisfied:

(c1) $f(x) \geqslant f(t)$ for $t < x < t+h_t$,
and $f(x) \leqslant f(t)$ for $t-h_t < x < t$;

(c2) $f(x) \leqslant f(t)$ for $t < x < t+h_t$,
and $f(x) \geqslant f(t)$ for $t-h_t < x < t$;

(c3) $f(x) \geqslant f(t)$ for $t-h_t < x < t+h_t$;

(c4) $f(x) \leqslant f(t)$ for $t-h_t < x < t+h_t$.

For each of $i = 1, 2, 3, 4$, let E_i consist of those points of E where (ci) holds. Then

$$f(E) = \sum_{i=1}^{4} f(E_i),$$

(²) Since a non-void perfect set has the power of the continuum, the theorem (*) of Padmavally and the necessity part of theorem (**) of Marcus are both deducible from the above theorem 1.

(³) A function $f(x)$ is *oscillating on the right* (*left*) of t, or is 0_+ (0_-) at t, if every right (left) nbd. of t contains points where $f(x) > f(t)$ as well as points where $f(x) < f(t)$. For the definitions see R. Vaidyanathaswamy [12], p. 71, and for the notations see Á. Császár [2], p. 14.

and it suffices to prove that each of the sets $f(E_i)$ ($i = 1$ to 4) is of the first category.

Since the points of E_3 and E_4 are respectively points of maxima and minima of $f(x)$, each of the sets $f(E_3), f(E_4)$ is enumerable (⁴), and so is of the first category. It shall therefore suffice to prove that $f(E_1)$ is of the first category, for it will then follow that $f(E_2)$ is of the first category just on considering the function $-f(x)$.

Let E_{1n} ($n = 1, 2, \ldots$) denote the set consisting of those points t of E_1 for which $h_t > 1/n$. Clearly,

$$E_1 = \sum_{n=1}^{\infty} E_{1n}, \quad f(E_1) = \sum_{n=1}^{\infty} f(E_{1n}),$$

and the lemma shall follow if we show that $f(E_{1n})$ is non-dense for every n.

Let J be an arbitrary subinterval of $f(I)$. We can determine an open interval $J' \equiv (u, v) \subset J$, which contains neither of the values $f(a)$ and $f(b)$. The set

(1) $$f^{-1}(J') = \{x;\ f(x) \in J'\}$$

is then open, and so consists of a finite or enumerable sequence of non-overlapping open intervals

(2) $$\{I_i \equiv (a_i, b_i),\ (i = 1, 2, \ldots)\}.$$

Clearly, for each i, $f(a_i)$ and $f(b_i)$ are each equal to u or v.

The function $f(x)$, being continuous in I, is uniformly continuous in I, and so there exists a real number $\delta > 0$ such that

$$|f(x') - f(x'')| < v - u \quad \text{whenever} \quad |x' - x''| < \delta,\ x' \in I,\ x'' \in I.$$

Consider now an interval I_i of the family $\{I_i\}$ for which

(3) $$\text{Length of } I_i = b_i - a_i < \min(\delta, 1/n).$$

We then have

$$|f(b_i) - f(a_i)| < v - u,$$

and so it is not possible that $f(a_i) = u$ and $f(b_i) = v$, or *vice versa*. Hence either

(i) $$f(a_i) = f(b_i) = u,$$

or

(ii) $$f(a_i) = f(b_i) = v.$$

Let $t \in I_i$. We then have

$$a_i < t < b_i, \quad u < f(t) < v.$$

(⁴) See W. Sierpiński [7], p. 236.

The point t cannot belong to E_{1n}; for, in case (i),

$$f(b_i) = u < f(t), \quad \text{whereas} \quad 0 < b_i - t < b_i - a_i < 1/n,$$

and in case (ii),

$$f(a_i) = v > f(t), \quad \text{whereas} \quad 0 < t - a_i < b_i - a_i < 1/n.$$

Hence,

$$I_i \cdot E_{1n} = 0. \tag{4}$$

Thus no interval of $\{I_i\}$ which is in length $< \min(\delta, 1/n)$ contains any point of E_{1n}. Denoting by G_1 the union of such intervals we have

$$G_1 \cdot E_{1n} = 0. \tag{5}$$

The remaining intervals of $\{I_i\}$, viz. those which are in length $\geqslant \min(\delta, 1/n)$, being non-overlapping, and since I is finite, are finite in number. If any of them is in length $\geqslant 1/n$, let it be splitted into two or more parts such that the length of each part is $< 1/n$. The number of intervals, clearly, still remains finite. On adding with each interval thus obtained its end-points, we get a finite number of closed intervals, say

$$I'_i \equiv [a'_i, b'_i] \quad (i = 1, 2, \ldots, k).$$

Clearly,

$$f^{-1}(J') = \sum_i I_i \subset G_1 + \sum_{i=1}^{k} I'_i,$$

which gives with the help of (5),

$$f^{-1}(J') \cdot E_{1n} \subset \sum_{i=1}^{k} I'_i \cdot E_{1n}.$$

Hence we have

$$J' \cdot f(E_{1n}) \subset \sum_{i=1}^{k} J' \cdot f(I'_i \cdot E_{1n}). \tag{6}$$

Consider now the interval I'_1. In case $J' \not\subset f(I'_1)$, since J' is an open interval, where as $f(I'_1)$ is a closed interval, there exists an open interval $J_1 \subset J' - f(I'_1)$. We then clearly have

$$J_1 \subset J', \quad J_1 \cdot f(I'_1 \cdot E_{1n}) = 0. \tag{7}$$

In case $J' \subset f(I'_1)$, since J' is open, the set

$$\{x;\ f(x) \in J',\ x \in I'_1\}$$

contains a non-degenerate interval I''_1. Thus

$$I''_1 \subset I'_1, \quad f(I''_1) \subset J'. \tag{8}$$

Since $f(x)$ is nowhere monotone in I, it cannot be non-decreasing in I''_1, and so there exist points $c, d \in I''_1$, $c < d$, such that

$$f(c) > f(d).$$

Let J_1 denote the open interval $(f(d), f(c))$. Then, by (8),

$$J_1 \subset f(I''_1) \subset J', \tag{9}$$

and

$$(c, d) \subset I''_1 \subset I'_1 = [a'_1, b'_1],$$

so that

$$a'_1 \leqslant c < d \leqslant b'_1. \tag{10}$$

Let t be a point of I'_1 such that $f(t) \in J_1$. We have

$$a'_1 \leqslant t \leqslant b'_1, \quad f(d) < f(t) < f(c),$$

so that, in case $a'_1 \leqslant t < d$,

$$f(d) < f(t) \quad \text{although} \quad 0 < d - t \leqslant b'_1 - a'_1 < 1/n,$$

and in case $d \leqslant t \leqslant b'_1$,

$$f(c) > f(t) \quad \text{although} \quad 0 < t - c \leqslant b'_1 - a'_1 < 1/n,$$

and so in either case t cannot be a point of E_{1n}. Hence,

$$J_1 \cdot f(I'_1 \cdot E_{1n}) = 0. \tag{11}$$

The relations (9) and (11) prove that in case $J' \subset f(I'_1)$, there still exists an open interval $J_1 \subset J'$ for which the relations (7) hold.

Thus we can always determine an open interval $J_1 \subset J'$ such that the relations (7) are satisfied.

In a similar manner we can determine an open interval $J_2 \subset J_1$ such that

$$J_2 \cdot f(I'_2 \cdot E_{1n}) = 0.$$

Continuing in this manner we get non-degenerate open intervals $J_3, J_4, \ldots, J_k$ such that

$$J \supset J' \supset J_1 \supset J_2 \supset \ldots \supset J_k \neq 0, \tag{12}$$

and

$$J_i \cdot f(I'_i \cdot E_{1n}) = 0 \quad \text{for} \quad i = 1, 2, \ldots, k. \tag{13}$$

These relations imply that

$$J_k \cdot f(I'_i \cdot E_{1n}) = 0 \quad \text{for} \quad i = 1, 2, \ldots, k,$$

which with the help of (6) give

$$J_k \cdot f(E_{1n}) = 0. \tag{14}$$

We have thus proved that every interval $J \subset f(I)$ contains a subinterval J_k which contains no point of $f(E_{1n})$. Hence $f(E_{1n})$ is non-dense in $f(I)$.

This completes the proof of Lemma 1.

Proof of Lemma 2. Let $f(x)$ be a continuous function defined in I for which every level set $f^{-1}(y)$ is non-dense, and the set H of values y for which $f^{-1}(y)$ is perfect is everywhere dense in $f(I)$.

First, $f(x)$ cannot be constant in any subinterval of I; for, if $f(x)$ takes a constant value, say c, in $I' \subset I$, the level set $f^{-1}(c)$ contains the interval I', and so is not non-dense.

Further, if $f(x)$ is strictly monotone in any subinterval I' of I, $f(I')$ is a non-degenerate interval. For any y lying in the interior of $f(I')$, the level set $f^{-1}(y)$ contains one and only one point in the interior of I', and so is not perfect. Hence $f(x)$ cannot be strictly monotone in any subinterval of I.

This proves that $f(x)$ is not monotone in any subinterval of I, and so is nowhere monotone. Hence the Lemma 2.

Proof of Theorem 1. The sufficiency part of the theorem clearly follows from Lemma 2.

To prove the necessity part, let $f(x)$ be a function continuous and nowhere monotone in I. According to Lemma 1 there exists a set H residual in $f(I)$ such that, for every $y \in H$, $f(x)$ is 0_+, or 0_-, at every point $x \in f^{-1}(y)$.

If $f(x)$ is 0_+ at x, every right-nbd. of x contains points x_1, x_2 such that

$$f(x_1) > f(x)\,, \qquad f(x_2) < f(x)\,.$$

As the function $f(x)$, being continuous, possesses the property of Darboux, there exists a point x' in (x_1, x_2), or (x_2, x_1), such that

$$f(x') = f(x)\,.$$

Thus, if $f(x)$ is 0_+ at x, every right-nbd. of x contains a point x' such that $f(x') = f(x)$. Hence x is a limit point of the level set $f^{-1}\{f(x)\}$.

Similarly, in case $f(x)$ is 0_- at x, every left-nbd. of x contains a point x' such that $f(x') = f(x)$, and so x is again a limit point of the level set $f^{-1}\{f(x)\}$.

Hence, if $y \in H$, every point $x \in f^{-1}(y)$ is a limit point of $f^{-1}(y)$. This proves that $f^{-1}(y)$ is dense-in-itself.

But, since $f(x)$ is continuous, its every level set is closed. Hence there exists a residual set H in $f(I)$ such that for every $y \in H$, $f^{-1}(y)$ is perfect.

Further, since $H \subset f(I)$, and $f(x)$ takes each value $y \in f(I)$ at least once in I, the level set $f^{-1}(y)$ is non-void for every $y \in H$.

Moreover, since $f(x)$ is nowhere monotone, it has no lines of invariability, and so, for any $y \in f(I)$, $f^{-1}(y)$ contains no non-degenerate interval. As we have already observed that $f^{-1}(y)$ is closed, this proves that $f^{-1}(y)$ is non-dense for every $y \in f(I)$.

This completes the proof of Theorem 1.

As a consequence of Theorem 1 we have

COROLLARY 1 ([5]). *Given a real function $f(x)$ defined and continuous in I. If the values y for which $f^{-1}(y)$ is not perfect form a set of the second category, then there exists a subinterval of I in which $f(x)$ is monotone.*

If the values y for which $f^{-1}(y)$ is perfect in no subinterval of I form a set which is of the second category in every subinterval of $f(I)$, then there exists in I an everywhere dense family of intervals in each of which $f(x)$ is monotone.

2. Nowhere monotone functions of the second species. Let now $f(x)$ be a continuous nowhere monotone function of the second species in I. Let m be a given real number. Since the function

$$g(x) = f(x) - mx$$

is continuous and nowhere monotone in I, it follows from Theorem 1 that there exists a set H_m residual in $g(I)$ such that for every $c \in H_m$, the roots of the equation

$$f(x) - mx = c$$

form a non-dense and perfect set. That is, for every $c \in H_m$, the line $y = mx + c$ intersects the curve $y = f(x)$ in a non-dense perfect set of points. For a $c \notin g(I)$, the line $y = mx + c$ is evidently disjoint with the curve $y = f(x)$. Denoting by R the set of all real numbers, the set

$$H'_m = H_m + \{R - g(I)\}$$

is evidently residual in R. Hence we have

THEOREM 2. *If $f(x)$ be a continuous nowhere monotone function of the second species in I, given a real number m, there exists a residual set H_m of real numbers such that for every $c \in H_m$ the line $y = mx + c$ intersects the curve $y = f(x)$ in a non-dense perfect set (possibly void).*

([5]) The following result of S. Marcus ([4], p. 103) constitutes a particular case of the above corollary:

"Given a real function $f(x)$ defined and continuous in I. If the values y for which $f^{-1}(y)$ is at most enumerable form a set of the second category, then there exists a subinterval of I in which $f(x)$ is monotone.

If the values y for which $f^{-1}(y)$ is at most enumerable form a set residual in $f(I)$, then there exists in I an everywhere dense family of intervals in each of which $f(x)$ is monotone."

Giving to m, in the above theorem, an enumerable set of real values, e.g. all positive and negative rational numbers, since the intersection of an enumerable sequence of residual sets is in itself a residual set, we get

THEOREM 2'. *If $f(x)$ be a continuous nowhere monotone function of the second species in I, there exists a residual set H of real numbers such that for every $c \in H$, and for every rational number m, the line $y = mx + c$ intersects the curve $y = f(x)$ in a non-dense perfect set (possibly void).*

3. Continuous functions with no lines of invariability. From Lemma 2 and Theorem 1 we deduce

THEOREM 3. *If a continuous function $f(x)$ defined in I has no lines of invariability, then the set $Y_p(f)$ of values $y \in f(I)$ for which $f^{-1}(y)$ is perfect* ([6]) *is either non-dense, or is of the second category, residual in each subinterval of $f(I)$ in which it is everywhere dense.*

Proof. Let J be an open subinterval of $f(I)$ in which $Y_p(f)$ is everywhere dense. The set

$$f^{-1}(J) = \{x;\ f(x) \in J\}$$

consists of a finite or enumerable set of mutually disjoint open ([7]) intervals in I, say $\{I_n\}$.

Let us consider one of these intervals, say I_n. For any $y \in Y_p(f)$, the subset of $f^{-1}(y)$ contained in I_n is either void or perfect. But, since $f(x)$ has no lines of invariability, $f(I_n)$ is a non-degenerate interval, and since $f(I_n) \subset J$, the set $Y_p(f)$ is everywhere-dense in $f(I_n)$.

Hence, there exists in $f(I_n)$ an everywhere dense set of values of y for which the set

$$\{x;\ f(x) = y,\ x \in I_n\} \equiv f^{-1}(y) \cdot I_n$$

is perfect. Further, since $f(x)$ is a continuous function without any line of invariability, its every level set $f^{-1}(y)$ is closed and non-dense.

It, therefore, follows from Lemma 2 that $f(x)$ is nowhere monotone in I_n, and then in turn from Theorem 1 that there exists a residual set H_n in $f(I_n)$ such that for every $y \in H_n$, $f^{-1}(y) \cdot I_n$ is perfect.

Let

$$E_n = H_n + \{J - f(I_n)\}.$$

Then E_n is residual in J, and for each $y \in E_n$, $f^{-1}(y) \cdot I_n$ is either void or perfect.

([6]) For a continuous function $f(x)$, the set $Y_\infty(f)$ of values of y for which $f^{-1}(y)$ is *infinite* possesses a little weaker property. Viz. that if $Y_\infty(f)$ is unenumerable in every interval, then it is residual. For, it has been proved by K. Borsuk ([1], p. 278) that $Y_\infty(f)$ is of the form $G_\delta + E$, where E is enumerable.

([7]) If any interval I_n contains one of the end-points of I, then, although it is semi-closed, it is open relative to I.

In a similar manner, corresponding to each interval I_n we get a set E_n residual in J such that for each $y \in E_n$, $f^{-1}(y) \cdot I_n$ is either void or perfect.

Clearly, the set $E = \bigcap_n E_n$ is residual in J, and, for every $y \in E$, $f^{-1}(y)$ is perfect in $\sum_n I_n$. But, since $y \in E \subset J$, and $f^{-1}(J) = \sum_n I_n$, we have $f^{-1}(y) \subset \sum_n I_n$. Hence, for every $y \in E$ the set $f^{-1}(y)$, which is evidently non-void, is perfect in I.

Thus $E \subset Y_p(f)$. Since E is residual in J, this proves the Theorem.

Remark 1. W. Sierpiński proved ([8]) in 1926 that for a continuous function $f(x)$ the set $Y_p(f)$ is $F_{\sigma\delta}$. This evidently implies that $Y_p(f)$ possesses the Baire property (in the wider sense) ([9]), and so is of the form $G - P + Q$, where G is open and P and Q are sets of the first category.

The above Theorem 3 proves that in case the continuous function $f(x)$ has no lines of invariability, the set $Y_p(f)$ possesses a property stronger than that of Baire, viz. it is of the form $G - P + N$, where G is open, P is of the first category and N is non-dense. The set $Y_p(f)$, in fact, resembles in this case with a set G_δ, for a set G_δ is also residual in every interval in which it is everywhere dense.

Remark 2. For a continuous function $f(x)$ defined in I, the set $Y_p(f)$ can even be identical with $f(I)$ ([10]). For there exist continuous functions for which every level set is non-dense and perfect; e.g. the two classes of non-differentiable functions constructed by A. N. Singh in [9] (see [10], p. 91) and [11] (see p. 1).

Remark 3. The proof of Theorem 3 can be easily extended to functions which possess lines of invariability, provided the values that they take in the intervals of invariability form a non-dense set.

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On the structure of homogroups with applications to the theory of compact connected semigroups

by

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This work is, for the most part, devoted to the study and application of a certain type of semigroup called a *homogroup*. By a homogroup, we mean a semigroup having a two-sided minimal ideal which is a group.

In the first part, we obtain some conditions under which certain semigroups become homogroups, and introduce the notion of maximal sub-homogroup and other notions which will be useful in what follows.

In the second part, we apply these results to the study of topological semigroups. In particular, we shall study the structure of certain compact connected semigroups. The results in this area quite naturally depend upon the nature of the canonical endomorphism associated with a homogroup. Under suitable conditions, this endomorphism, in the topological case, is a monotone. As we shall see, this fact enables one to construct various sub-semigroups including arcs. In this connection we shall show that a compact connected abelian semigroup (which is not a group), having an identity 1 contains a non-degenerate compact connected sub-semigroup whose intersection with the maximal subgroup at 1 is precisely 1.

Another application of this canonical endomorphism is a natural description of certain semigroups as coordinate bundles with connected fibres.

§ 1. Homogroups. The term homogroup was introduced by G. Thierrin [29] who studied their regular equivalences and made a detailed study of a special homogroup called *resorbing* (*résorbant*).

Earlier, A. H. Clifford and D. D. Miller, [3], had studied homogroups under the title "*semigroups with zeroide elements*". Let us recall that an element x of a semigroup D is called *net* or *zeroid* if for any d there exist elements s and t such that $ds = x$ and $td = x$. Now Clifford and Miller show that K, the set of net elements, if non-vacuous, is a two-sided ideal

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