On the independence of continuous functions

by

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We shall investigate the inter-relation of two notions of independence of continuous functions. One of these notions has an algebraical character and is associated with a composition algebra (cf. definition below) on the given set of functions; it depends on the choice of this algebra in which it is naturally defined according to the scheme of independence in abstract algebras proposed by E. Marczewski [2]. The second notion expresses a more intrinsic, topological property of the functions involved. Some textbooks on function theory give it in their course (cf. e.g. [1]).

We shall consider continuous real valued functions $f(x_1, \ldots, x_n)$ defined on an arbitrary open subset D of the Euclidean space E^n, where $n \geqslant 2$. D will be fixed and the above class of functions will be denoted by $\boldsymbol{F}$. By C^0 we shall denote the class of all continuous real valued functions defined over the whole E^n. Every $g \in C^0$ can be regarded as an operation associating with every n-tuple $(f_1, \ldots, f_n)$, $f_i \in \boldsymbol{F}$ the function $g(f_1, \ldots, f_n) \in \boldsymbol{F}$ defined by

$$(1) \qquad g(f_1, \ldots, f_n)(x_1, \ldots, x_n) = g\big(f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)\big).$$

For any $\boldsymbol{H} \subset C^0$, we shall call the pair $\mathcal{A} = (\boldsymbol{F}, \boldsymbol{H})$ a *composition algebra*. In this algebra, $\boldsymbol{F}$ is the set of elements and every $g \in \boldsymbol{H}$ is an operation acting on F according to formula (1). We say, following E. Marczewski [2], that $f_1, \ldots, f_m \in \boldsymbol{F}$ are *independent* in the algebra $\mathcal{A}$ if every mapping of the set $\{f_1, \ldots, f_m\}$ into $\boldsymbol{F}$ can be extended to a homomorphism of the subalgebra generated by $f_1, \ldots, f_m$ into $\mathcal{A}$. We shall compare this notion of independence with another notion (cf. [1], p. 156) according to which $f_1, \ldots, f_m \in \boldsymbol{F}$ are called independent if the set

$$(f_1, \ldots, f_m)(D) = \big\{\big(f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)\big) \colon (x_1, \ldots, x_n) \in D\big\}$$

is dense in some open subset of E^m. To avoid misunderstanding we shall say that the functions $f_1, \ldots, f_m$ are *independent* if and only if they are independent in the sense of this latter definition and we shall call them $\mathcal{A}$*-free* if they are independent in a composition algebra $\mathcal{A}$. It is well known [1] that if $f_1, \ldots, f_n$ are differentiable and their Jacobian does not vanish identically on D, then these functions are independent.

We denote by C^k the class of all k times differentiable functions on E^n and we define $C^\infty = \bigcap_{n=0}^{\infty} C^k$. Then our main result may be stated as follows

THEOREM. *There exists a composition algebra $\mathcal{A} = (\boldsymbol{F}, \boldsymbol{H})$ such that arbitrary n functions $f_1, \ldots, f_n \in \boldsymbol{F}$ are $\mathcal{A}$-free if and only if they are independent. Moreover $\boldsymbol{H} \subset C^\infty$, hence, for every k, $(\boldsymbol{F} \cap C^k, \boldsymbol{H})$ is a subalgebra of $\mathcal{A}$.*

The question remains open whether there is a composition algebra $\mathcal{A}_0$ such that, for every k, any $f_1, \ldots, f_k \in F$ are $\mathcal{A}_0$-free if and only if they are independent. For the algebra $\mathcal{A}$ constructed below this is not the case since every $k < n$ functions are $\mathcal{A}$-free but there exist $k < n$ dependent functions in $\boldsymbol{F}$, e.g. $f, 2f, \ldots, kf$ (f arbitrary). On the other hand, there is a class χ of operations on $\boldsymbol{F}$ (which are not of the form (1)) such that $(\boldsymbol{F}, \chi)$-freeness coincides with independence (cf. [3]).

The essential part of our proof is the construction of a class $\boldsymbol{H} \subset C^0$ such that:

(i) $\boldsymbol{H} \subset C^\infty$.

(ii) *The constant function identically equal 0 belongs to $\boldsymbol{H}$.*

(iii) *The 'projections' e_i $(i = 1, \ldots, n)$, defined by $e_i(x_1, \ldots, x_n) = x_i$, for every $x_1, \ldots, x_n$ belong to $\boldsymbol{H}$.*

(iv) *If $h_0, h_1, \ldots, h_n \in \boldsymbol{H}$, then $h_0(h_1, \ldots, h_n) \in \boldsymbol{H}$.*

(v) *If $h_1, \ldots, h_n \in \boldsymbol{H}$ are different from each other and none of these functions is identically equal zero, then these functions are independent on every open subset $V \subset E^n$, or equivalently, (by (i)), the Jacobian*

$$\partial(h_1, \ldots, h_n)/\partial(x_1, \ldots, x_n)$$

does not vanish identically on any open set.

(vi) *If S is an arbitrary nowhere dense subset of E^n, then there is an $h \in \boldsymbol{H}$ such that h does not vanish identically and h is vanishing on S.*

Sufficiency of (i), ..., (vi). Suppose that $\boldsymbol{H}$ satisfies (i), ..., (vi) and define $\mathcal{A} = (\boldsymbol{F}, \boldsymbol{H})$. In view of (iii) and (iv) the functions $f_1, \ldots, f_n$ are $\mathcal{A}$-free if and only if

$$g(f_1, \ldots, f_n) \neq h(f_1, \ldots, f_n) \quad \text{when} \quad g, h \in \boldsymbol{H} \text{ and } g \neq h. \tag{2}$$

Let $f_1, \ldots, f_n$ be independent. Then $(f_1, \ldots, f_n)(D)$ is dense in some open subset of E^n and since $g, h \in \boldsymbol{H}$ cannot coincide on any open set, if they are different, by (v), we have (2). Conversely, if $f_1, \ldots, f_n$ are dependent, i.e. $S = (f_1, \ldots, f_n)(D)$ is nowhere dense, then $h(f_1, \ldots, f_n) = 0$

holds for that function h which is given by (vi). Since h does not vanish identically and (ii), we have an instance of non (2), hence $f_1, \ldots, f_n$ are not $\mathcal{A}$-free.

1. The definition of $\boldsymbol{H}$. Let T be a set of power $2^{\aleph_0}$. Suppose that $\{S_\tau\}$, $\tau \in T$, is the family of all nowhere dense closed subsets of E^n and let

$$S_\tau^0 = S_\tau \cup \{(x_1, \ldots, x_n)\colon x_i = 0 \text{ or } x_i = x_j \text{ for some } i, j;\ i \neq j\}.$$

It is well known that for every S_τ^0 there exists a family of disjoint cubes

$$M_{\tau,i} = \{(x_1, \ldots, x_n)\colon -a_j^i < x_j - b_j^i < a_j^i,\ j = 1, \ldots, n\}; \qquad i = 1, 2, \ldots$$

with rational a_j^i, b_j^i such that $\bigcup_{i=1}^{\infty} M_{\tau,i}$ is dense in E^n and disjoint to S_τ^0. Moreover, there are rational constants $r_i > 0$ such that, if we define

$$\bar{h}_{\tau,i}(x_1, \ldots, x_n) = r_i \exp\Big\{-\prod_{j=1}^{n} [(a_j^i)^2 - (x_j - b_j^i)^2]^{-1}\Big\}, \tag{3}$$

$$h_{\tau,i}(p) = \begin{cases} \bar{h}_{\tau,i}(p) & \text{when} \quad p \in M_{\tau,i}, \\ 0 & \text{otherwise,} \end{cases}$$

and $h_\tau = \sum_{i=1}^{\infty} h_{\tau,i}$, then $h_\tau \in C^\infty$. (For a similar construction see [1], p. 158.) Obviously the function h_τ does not vanish identically on any open set and we have $h_\tau(p) = 0$ when $p \in S_\tau^0$.

Now let $\{c_{l_1,\ldots,l_n}^\tau\}$, $\tau \in T$, $l_1, \ldots, l_n = 0, 1, 2, \ldots$ be a set composed of irrational algebraically independent numbers such that, for every τ, the series

$$s_\tau(x_1, \ldots, x_n) = \sum_{l_1,\ldots,l_n} c_{l_1,\ldots,l_n}^\tau x_1^{l_1} x_2^{l_2} \ldots x_n^{l_n} \tag{4}$$

converges and its sum belongs to C^∞.

We associate with every $\tau \in T$ a function h_τ and a function s_τ defined above and we call the functions

$$s_\tau(x_1, \ldots, x_n) h_\tau(x_1, \ldots, x_n), \qquad \tau \in T, \tag{5}$$

fundamental. Now we define $\boldsymbol{H}$ as the smallest class which contains the projections $e_1, \ldots, e_n$ (cf. (iii)), every fundamental function, and is closed under composition as required by (iv). Given a fundamental function (5) we shall call the functions $s_\tau h_{\tau,i}$ the *atoms* of (5).

We have to verify that $\boldsymbol{H}$ satisfies (i), ..., (vi). Of these conditions only (v) needs a proof, the rest follows easily. (Condition (ii) follows by $h(h, \ldots, h) = 0$ for any fundamental function h.) We shall give the proof of (v) in section 4 after having obtained in sections 2 and 3 some auxiliary results concerning the structure of $\boldsymbol{H}$.

2. Compositions. We shall now define a class Γ of operations acting on C^0. They will be called *compositions*. The unique unary composition will be denoted by E and it is defined by the condition $E(f) = f$, for every $f \in C^0$. Suppose that m is a positive integer and that all q-ary composition operations with $q < m$ have been defined. Then an m-ary operation R on C^0 is called a *composition* if and only if

$$(6)\quad R(f_1, \dots, f_m) = f_m\big(Q_1(f_{m_1+1}, f_2, \dots, f_{m_2}), Q_2(f_{m_2+1}, \dots, f_{m_3}), \dots, Q_n(f_{m_n+1}, \dots, f_{m_{n+1}})\big),$$

where $0 = m_1 < m_2 < m_3 < \dots < m_{n+1} = m-1$ and Q_i are compositions already defined. It is clear that if $R \in \Gamma$ is m-ary, then $m = 1$ or $m > n$. It is also easily seen that the representation (6) is unique, i.e. given any m-ary $R \in \Gamma$ with $m > n$, the operations $Q_1, \dots, Q_n$ satisfying (6) for every $f_1, \dots, f_m \in C^0$, are uniquely determined.

Let us denote by E_i the constant operations on C^0 such that $E_i(f) = e_i$ for every $f \in C^0$, where e_i are the projections defined in (iii). Given an m-ary $R \in \Gamma$, we shall associate with every $k = 1, \dots, m$ an ordered n-tuple $[R, k]$ of operations on C^0 defined as follows. If $m = 1$, i.e. $R = E$, then we put

$$[E, 1] = (E_1, \dots, E_n),$$

and assuming that we have defined the n-tuples $[Q, k]$ for all q-ary $Q \in \Gamma$ with $1 \leqslant k \leqslant q < m$, we take an m-ary $R \in \Gamma$, represent it uniquely in the form (6) and set

$$[R, m] = (Q_1, \dots, Q_n),$$

defining, for $k < m$,

$$[R, k] = [Q_i, k - m_i],$$

where i is the unique integer such that $m_i < k \leqslant m_{i+1}$. We see that $[R, k]$ is either an n-tuple of operations belonging to Γ or it is the n-tuple $(E_1, \dots, E_n)$. For every n-tuple $[R, k]$ we define an operation $[R, k]^*$ which associates with every $f_1, \dots, f_m \in C^0$ an n-tuple $[R, k]^*(f_1, \dots, f_m)$ of functions belonging to C^0. If $R = E$, we put

$$[E, 1]^*(f) = \big(E_1(f), \dots, E_n(f)\big).$$

Assuming that we have defined $[Q, k]^*$ for all q-ary $Q \in \Gamma$ with $q < m$, we take an m-ary $R \in \Gamma$, represent it in the form (6) and define

$$(7)\quad [R, m]^*(f_1, \dots, f_m) = \big((Q_1(f_{m_1+1}, \dots, f_{m_2}), \dots, Q_n(f_{m_n+1}, \dots, f_{m_{n+1}})\big)$$

and for $k < m$ we take the i satisfying $m_i < k \leqslant m_{i+1}$ and put

$$(8)\quad [R, k]^*(f_1, \dots, f_m) = [Q_i, k - m_i]^*(f_{m_i+1}, \dots, f_{m_{i+1}}).$$

It is not difficult to see that if $R \in \Gamma$ is m-ary and $[R, k] = (\Omega_1, \dots, \Omega_n)$, then there are numbers $0 \leqslant s_1 < s_2 < \dots < s_{n+1} < m$ such that, for every $f_1, \dots, f_m \in C^0$,

$$(9)\quad [R, k]^*(f_1, \dots, f_m) = \big(\Omega_1(f_{s_1+1}, \dots, f_{s_2}), \dots, \Omega_n(f_{s_n+1}, \dots, f_{s_{n+1}})\big).$$

We shall prove now a lemma concerning the operations $[R, k]^*$. For $a > 0$ and $p \in E^n$, we shall denote by $K_a p$ the interior of the sphere with radius a and centre at p.

LEMMA 1.2. *Let $f_1, \dots, f_m \in C^0$, $p_0 \in E^n$, and let $R \in \Gamma$ be m-ary. We denote $p_k = [R, k]^*(f_1, \dots, f_m)(p_0)$. Then, for every $a > 0$, there is a $\beta > 0$ such that for every $\gamma > 0$ there is a $\delta > 0$ with the property that whenever $g_1, \dots, g_m \in C^0$ are such that for every $k = 1, \dots, m$, $|g_k - f_k| < \delta$ holds inside $K_a p_k$, then, for every $p \in K_\beta p_0$,*

$$(10)\quad [R, k]^*(g_1, \dots, g_m)(p) \in K_a p_k,$$

$$(11)\quad |R(g_1, \dots, g_m)(p) - R(f_1, \dots, f_m)(p)| < \gamma.$$

Proof. If $m = 1$, i.e. $R = E$, then clearly it is enough to put $\beta = a$ and $\delta = \gamma$. Now take the inductive assumption that the lemma is proved for all q-ary compositions with $q < m$. Let $R \in \Gamma$ be m-ary and consider the compositions $Q_1, \dots, Q_n$ appearing in (6). Since the lemma holds for each Q_i, we have that to a correspond numbers β_i, $(i = 1, \dots, n)$ having the property stated in the lemma with respect to Q_i and $f_{m_i+1}, \dots, f_{m_{i+1}}$. Let η satisfy $0 < \eta < a$. From the continuity of $f_1, \dots, f_m$ follows the existence of a number β_0 such that

$$(12)\quad [R, m]^*(f_1, \dots, f_m)(p) \in K_{a-\eta} p_m \quad \text{for} \quad p \in K_{\beta_0} p_0.$$

We define $\beta = \min(\beta_0, \beta_1, \dots, \beta_n)$. Now suppose that γ is given. The function f_m is uniformly continuous on bounded sets, hence there is an $\varepsilon > 0$ such that

$$(13)\quad |f_m(q) - f_m(p)| < \gamma/2 \quad \text{if} \quad q \in K_a p_m \cap K_\varepsilon p.$$

We can assume that $\varepsilon < \eta$. Let μ be any number which is small enough so that a cube with sides equal 2μ is contained in a sphere of radius ε (in the space E^n). Recollecting that the hypothesis of the lemma holds for each Q_i and $f_{m_i+1}, \dots, f_{m_{i+1}}$ with the values of a and β fixed above, we consider this hyphotesis with μ in place of γ. Then we have, for each $i = 1, \dots, m$, that there is a number δ_i with the property that whenever $g_{m_i+1}, \dots, g_{m_{i+1}}$ are such that, for every $k = m_i+1, \dots, m_{i+1}$, $|g_k - f_k| < \delta_i$ holds inside the sphere

$$K_a[Q_i, k - m_i]^*(f_{m_i+1}, \dots, f_{m_{i+1}})(p_0),$$

i.e. inside $K_a p_k$, by (8), then, for every $p \in K_\beta p_0$,

$$(14)\quad [Q_i, k - m_i]^*(g_{m_i+1}, \dots, g_{m_{i+1}})(p) \in K_a p_k$$

and

(15) $|Q_i(g_{m_i+1}, \ldots, g_{m_{i+1}})(p) - Q_i(f_{m_i+1}, \ldots, f_{m_{i+1}})(p)| < \mu$.

We now assert that $\delta = \min(\gamma/2, \delta_1, \ldots, \delta_m)$ satisfies the hypothesis of the lemma. Indeed, if $k < m$, then (10) holds by (14) and (8). Moreover, if $p \in K_\beta p_0$, then, by (15) and by our definition of μ,

(16) $[R, m]^*(g_1, \ldots, g_m)(p) \in K_\varepsilon [R, m]^*(f_1, \ldots, f_m)(p)$

and hence, by (12) and by $\varepsilon < \eta$,

(17) $[R, m]^*(g_1, \ldots, g_m)(p) \in K_\alpha p_m$.

This completes the proof of (10). Moreover, by (13), (16) and (17) we have

(18) $\big|f_m\big([R, m]^*(g_1, \ldots, g_m)(p)\big) - f_m\big([R, m]^*(f_1, \ldots, f_m)(p)\big)\big| < \gamma/2$

for $p \in K_\beta p_0$. Thus by (17) and by $|g_m - f_m| < \delta \leqslant \gamma/2$ on $K_\alpha p_m$, we have that (18) implies (11).

COROLLARY. *Let $R, f_1, \ldots, f_m, p_0, p_k$ be as in the Lemma 1.2. Then, for every $\alpha > 0$, there is a $\beta > 0$ with the property that whenever $g_1, \ldots, g_m$ are functions such that, for $k = 1, \ldots, m$, $g_k = f_k$ holds on $K_\alpha p_k$, then*

(19) $R(g_1, \ldots, g_m) = R(f_1, \ldots, f_m)$ *holds on* $K_\beta p_0$.

To prove the corollary it is enough to take a β given by Lemma 1.2 and let $\gamma \to 0$. Since obviously $|g_k - f_k| < \delta$ holds on $K_\alpha p_k$ for every $\delta > 0$, hence (11) implies (19).

To state the next lemma we need another definition. Suppose $R \in \Gamma$ is m-ary and let $i_1, \ldots, i_q$ be a sequence composed of some of the numbers $1, \ldots, m$. Let $P = f_{i_1}, \ldots, f_{i_q}$ be a fixed sequence of functions. We say that R depends on the k-th variable $(1 \leqslant k \leqslant m)$ under fixed P if there are functions $g_1, \ldots, g_k, \ldots, g_m, g'_k$ such that $g_i = f_i$ for $i = i_1, \ldots, i_q$ and

$$R(g_1, \ldots, g_k, \ldots, g_m) \neq R(g_1, \ldots, g'_k, \ldots, g_m).$$

LEMMA 2.2. *Let $R, f_1, \ldots, f_m, p_0, p_k$ be as in Lemma 1.2 and let P be the subsequence of $f_1, \ldots, f_m$ composed of all projections among these functions. Assume that every f_k which is not a projection is vanishing identically on each hyperplane $x_j = 0$, $j = 1, \ldots, n$, and let $R(f_1, \ldots, f_m)(p_0) \neq 0$. Then we have $f_k(p_k) \neq 0$ for all those f_k which are not projections and are such that R depends on the k-th variable under fixed P.*

Proof. The lemma holds trivially if $R = E$. Now take the inductive assumption that it is true for every q-ary composition with $q < m$. Let $R \in \Gamma$ be m-ary of the form (6). Suppose first that f_m is a projection, say $f_m = e_i$. Then $R(f_1, \ldots, f_m) = Q_i(f_{m_i+1}, \ldots, f_{m_{i+1}})$. If f_k is not a projection and R depends on the k-th variable under fixed P, then obviously $m_i < k \leqslant m_{i+1}$ and Q_i depends on the $(k - m_i)$-th variable. We have $p_k = [Q_i, k - m_i]^*(f_{m_i+1}, \ldots, f_{m_{i+1}})(p_0)$, and, by the inductive assumption, $f_k(p_k) \neq 0$.

Now let f_m be not a projection. Then, by $f_m(p_m) = R(f_1, \ldots, f_m)(p_m)$, we have $f_m(p_m) \neq 0$ and moreover p_m does not lie on any of the hyperplanes $x_i = 0$. Thus $Q_i(f_{m_i+1}, \ldots, f_{m_{i+1}})(p_0) \neq 0$ holds for $i = 1, \ldots, n$. If, for some $k < m$, f_k is not a projection, and R depends on the k-th variable, then, for this i which satisfies $m_i < k \leqslant m_{i+1}$ we have that Q_i depends on the $(k - m_i)$-th variable. As before, it follows by the inductive assumption that $f_k(p_k) \neq 0$.

LEMMA 3.2. *Let $f_1, \ldots, f_m \in C^0$, $V \subset E^n$ and let $R \in \Gamma$ be an m-ary composition such that, for a certain function $g \in C^0$ and a number $l \leqslant m$, we have*

$$g(p) = f_l\big([R, l]^*(f_1, \ldots, f_m)(p)\big) \quad \textit{for every} \quad p \in V.$$

Then there is a number $0 \leqslant k < l$ and a composition $Q \in \Gamma$ such that

$$R(f_1, \ldots, f_m)(p) = Q(f_1, \ldots, f_k, g, f_{l+1}, \ldots, f_m)(p) \quad \textit{for every} \quad p \in V.$$

We omit the simple proof (induction on m).

3. Atoms and compositions.

LEMMA 1.3. *Let $R_i \in \Gamma$ $(i = 1, 2)$ be q_i-ary compositions and let $h_1^i, \ldots, h_{q_i}^i$ be two sequences composed of atoms and projections such that, for some $p_0 \in E^n$,*

(20) $R_i(h_1^i, \ldots, h_{q_i}^i)(p_0) > 0$.

Assume that there is a neighbourhood V of p_0 such that

(21) $R_1(h_1^1, \ldots, h_{q_1}^1)(p) = R_2(h_1^2, \ldots, h_{q_2}^2)(p)$ *for every* $p \in V$.

Let us denote $p_k^i = [R_i, k]^*(h_1^i, \ldots, h_{q_i}^i)(p_0)$. Then, for arbitrary series $s_k^i = \sum_{r_1, \ldots, r_n} c_{r_1, \ldots, r_n}^{i,k} x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}$ such that

(π) $s_k^i = h_k^i$ *whenever* h_k^i *is a projection,*

(ρ) $s_k^i = s_l^j$ *whenever* $h_k^i = h_l^j$ *and* $p_k^i = p_l^j$,

we have

(22) $R_1(s_1^1, \ldots, s_{q_1}^1)(p) = R_2(s_1^2, \ldots, s_{q_2}^2)(p)$ *for every* $p \in E^n$.

COROLLARY. *In particular (22) holds for any series s_k^i which satisfy (π) and are such that $s_k^i = s_l^j$ whenever $h_k^i = h_l^j$.*

Proof. Let P_i be the subsequence of all the projections among the $h_1^i, \ldots, h_{q_i}^i$. It follows from (π) that $h_k^i \to s_k^i$ are substitutions with fixed P_i. Thus we need to consider only these atoms h_k^i for which R_i depends on the k-th variable under fixed P_i. Let the following be all these atoms

(23) $h_b^a = s_\varrho h_{\varrho,m}, \ldots, h_d^c = s_\sigma h_{\sigma,r}, \ldots, h_w^u = s_\mu h_{\mu,\nu}$,

where $s_\varrho, \ldots, s_\mu$ are of the form (4). Let $\bar{h}_{\varrho,m}, \ldots, \bar{h}_{\mu,\nu}$ be given by (3). Define

(24) $\bar{h}_b^a = s_\varrho \bar{h}_{\varrho,m}, \ \ldots, \ \bar{h}_d^c = s_\sigma \bar{h}_{\sigma,r}, \ \ldots, \ \bar{h}_w^u = s_\mu \bar{h}_{\mu,\nu}; \ \bar{h}_k^i = h_k^i$ for all other h_k^i.

We have, in view of (20) and by Lemma 2.2 that $h_b^a(p_b^a) > 0, \ldots, h_w^u(p_w^u) > 0$. Clearly there is a number $a > 0$ such that $h_b^a = \bar{h}_b^a, \ldots, h_w^u = \bar{h}_w^u$ hold in the spheres $K_a p_b^a, \ldots, K_a p_w^u$ respectively. Thus, by the Corollary to Lemma 1.2, we have that there is an open set $U \subset V$ such that

$$R_i(h_1^i, \ldots, h_{q_i}^i)(p) = R_i(\bar{h}_1^i, \ldots, \bar{h}_{q_i}^i)(p) \quad \text{for every} \quad p \in U.$$

Obviously, identically on E^n

$$R(\bar{h}_1^i, \ldots, \bar{h}_{q_i}^i)(x_1, \ldots, x_n) = \sum_{k_1,\ldots,k_n} w_{k_1,\ldots,k_n}^{(i)}(c_{l_1,\ldots,l_n}^{\varrho}, \ldots, c_{t_1,\ldots,t_n}^{\mu}) x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n},$$

where $w_{k_1,\ldots,k_n}^{(i)}$ are certain polynomials, with rational coefficients, of the algebraically independent irrational numbers $c_{l_1,\ldots,l_n}^{\varrho}, \ldots, c_{t_1,\ldots,t_n}^{\mu}$ appearing in (4). Since (21) holds on U, it follows from the above that

$$[w_{k_1,\ldots,k_n}^{(1)} - w_{k_1,\ldots,k_n}^{(2)}](c_{l_1,\ldots,l_n}^{\varrho}, \ldots, c_{t_1,\ldots,t_n}^{\mu}) = 0.$$

We conclude that each of the polynomials $w_{k_1,\ldots,k_n}^{(1)} - w_{k_1,\ldots,k_n}^{(2)}$ is vanishing whatever numbers $\tilde{c}_{l_1,\ldots,l_n}^{\varrho}, \ldots, \tilde{c}_{t_1,\ldots,t_n}^{\mu}$ we substitute for the $c_{l_1,\ldots,l_n}^{\varrho}, \ldots, c_{t_1,\ldots,t_n}^{\mu}$ provided equal numbers are substituted for equal ones. Hence, if we replace all the series $s_\varrho, \ldots, s_\mu$ appearing in (24) by arbitrary series $\tilde{s}_\varrho, \ldots, \tilde{s}_\mu$ so that identical series are replaced by identical ones, and we set

(25) $\tilde{h}_b^a = \tilde{s}_\varrho \bar{h}_{\varrho,m}, \ \ldots, \ \tilde{h}_d^c = \tilde{s}_\sigma \bar{h}_{\sigma,r}, \ \ldots, \ \tilde{h}_w^u = \tilde{s}_\mu \bar{h}_{\mu,\nu};$
$\tilde{h}_k^i = h_k^i$ for the remaining h_k^i,

then

(26) $$R_1(\tilde{h}_1^1, \ldots, \tilde{h}_{q_1}^1) = R_2(\tilde{h}_1^2, \ldots, \tilde{h}_{q_1}^2) \text{ on } E^n.$$

We may assume that a is a sufficiently small number such that the functions $\bar{h}_{\varrho,m}, \ldots, \bar{h}_{\mu,\nu}$ are positive valued in the spheres $K_a p_b^a, \ldots, K_a p_w^u$ respectively and moreover any two of these spheres are either disjoint or coinciding. It follows by Lemma 1.2 that there is a number $\delta > 0$ such that for any functions $\bar{s}_k^i$ satisfying $|\bar{s}_k^i - h_k^i| < \delta$ on $K_a p_k^i$, $(i = 1, 2;\ k = 1, \ldots, q_i)$, we have, by (10),

(27) $$[R_i, k]^*(\bar{s}_1^i, \ldots, \bar{s}_{q_i}^i)(p_0) \in K_a p_k^i.$$

Since there is a dense set of irrational algebraically independent numbers, there are series $\bar{s}_b^a, \ldots, \bar{s}_w^u$ such that

a) $|\bar{s}_b^a - h_b^a| < \delta, \ldots, |\bar{s}_w^u - h_w^u| < \delta$ hold in $K_a p_b^a, \ldots, K_a p_w^u$ respectively;

b) Two of these series, say $\bar{s}_d^c$ and $\bar{s}_w^u$, are identical if and only if the corresponding atoms h_d^c and h_w^u are identical and $p_d^c = p_w^u$;

c) The sets of coefficients of different series are mutually disjoint;

d) The coefficients of all series $\bar{s}_b^a, \ldots, \bar{s}_w^u$ form a set of irrational algebraically independent numbers.

We set $\bar{s}_k^i = h_k^i$ for pairs (i, k) such that h_k^i does not appear in (23). We denote $\bar{p}_k^i = [R_i, k]^*(\bar{s}_1^i, \ldots, \bar{s}_{q_i}^i)(p_0)$. Then, by (27), there is a number $\varepsilon > 0$ such that $K_\varepsilon \bar{p}_k^i \subset K_a p_k^i$ for every p_k^i. We note that in the spheres $K_\varepsilon \bar{p}_b^a, \ldots, K_\varepsilon \bar{p}_w^u$ the corresponding functions $\bar{h}_{\varrho,m}, \ldots, \bar{h}_{\mu,\nu}$ (cf. (23)) are positive. Suppose we now wish to find series $\tilde{s}_\varrho, \ldots, \tilde{s}_\mu$ such that the functions (25) approximate the series $\bar{s}_b^a, \ldots, \bar{s}_w^u$ in the spheres $K_\varepsilon \bar{p}_w^u, \ldots, K_\varepsilon \bar{p}_b^a$ respectively. Remembering that two of the series $\tilde{s}_\varrho, \ldots, \tilde{s}_\sigma, \ldots, \tilde{s}_\mu$, say $\tilde{s}_\sigma$ and $\tilde{s}_\mu$, have to be taken identical if and only if $s_\sigma = s_\mu$, we see that such approximation is always possible provided that $s_\sigma = s_\mu$ and $K_\varepsilon \bar{p}_d^c \cap K_\varepsilon \bar{p}_w^u \neq 0$ implies $\bar{s}_d^c = \bar{s}_w^u$. Now this is always the case, as follows from our construction: Indeed, we then have $\sigma = \mu$ and $p_d^c = p_w^u$. Moreover, by $h_{\sigma,r}(p_d^c) > 0$, $h_{\mu,\nu}(p_w^u) > 0$, the supports of $h_{\mu,r}$ and $h_{\mu,\nu}$ intersect, hence $r = \nu$. From $\sigma = \mu$, $r = \nu$, we have $h_d^c = h_w^u$ and thus, by b), $\bar{s}_d^c = \bar{s}_w^u$.

It now follows that, for every $\delta > 0$, we can find functions (25) satisfying $|\tilde{h}_b^a - \bar{s}_b^a| < \delta, \ldots, |\tilde{h}_w^u - \bar{s}_w^u| < \delta$ in $K_\varepsilon \bar{p}_b^a, \ldots, K_\varepsilon \bar{p}_w^u$ respectively. We obtain, by Lemma 1.2, that there is a $\beta > 0$ with the property that $|R_i(\tilde{h}_1^i, \ldots, \tilde{h}_{q_i}^i) - R_i(\bar{s}_1^i, \ldots, \bar{s}_{q_i}^i)|$ can be arbitrarily small on $K_\beta p_0$ at suitable selections of the $\tilde{h}_1^i, \ldots, \tilde{h}_{q_i}^i$ of the form (25). Thus, by (26)

(28) $$R_1(\bar{s}_1^1, \ldots, \bar{s}_{q_1}^1) = R_2(\bar{s}_1^2, \ldots, \bar{s}_{q_2}^2) \text{ on } K_\beta p_0.$$

Since the functions in (28) are analytic, we have that (28) holds identically on E^n, and moreover, by d), (28) will remain true if we replace $\bar{s}_b^a, \ldots, \bar{s}_w^u$ by arbitrary series $s_b^a, \ldots, s_w^u$ provided that identical series $\bar{s}_k^i$ are replaced by identical series s_k^i. This proves the lemma.

LEMMA 2.3. *Let $R \in \Gamma$ be m-ary, let $h_1, \ldots, h_m$ be atoms and projections and let $p_0 \in E^n$. We define $p_k = [R, k]^*(h_1, \ldots, h_m)(p_0)$. We write R for $R(h_1, \ldots, h_m)$ and if in R the functions $h_1, \ldots, h_m$ are replaced by $\tilde{h}_1, \ldots, \tilde{h}_m$, then we denote the resulting function by $\tilde{R}$. Then, for any functions $\tilde{h}_1, \ldots, \tilde{h}_m$ such that*

(29) $$\tilde{h}_k(p_k) = h_k(p_k) \quad \textit{and} \quad \left[\frac{\partial}{\partial x_j}\tilde{h}_k\right](p_k) = \left[\frac{\partial}{\partial x_j}h_k\right](p_k)$$

for $k = 1, \ldots, m$; $j = 1, \ldots, n$ we have

(30) $$\tilde{R}(p_0) = R(p_0) \quad \textit{and} \quad \left[\frac{\partial}{\partial x_j}\tilde{R}\right](p_0) = \left[\frac{\partial}{\partial x_j}R\right](p_0); \quad j = 1, \ldots, n.$$

Proof. The lemma is trivial if $R = E$. We take the inductive assumption that it is true for all q-ary composition operations with $q < m$. Let $R \in \Gamma$

be m-ary of the form (6). We have, by (29) and by the inductive hypothesis that $\tilde{Q}_i(p_0) = Q_i(p_0)$; $i = 1, \ldots, n$. Hence $p_m = [R, m]^*(\tilde{h}_1, \ldots, \tilde{h}_m)(p_0)$ and thus, by $\tilde{h}_m(p_m) = h_m(p_m)$, we have the first equation in (30). Further

$$\left[\frac{\partial}{\partial x_j} R\right](p_0) = \sum_{i=1}^{n} \left[\frac{\partial}{\partial x_i} h_m\right](p_m) \cdot \left[\frac{\partial}{\partial x_j} Q_i\right](p_0)$$

and there is an analogous equation with R, h_m and Q_i replaced by $\tilde{R}$, $\tilde{h}_m$ and $\tilde{Q}_i$. Since we have

$$\left[\frac{\partial}{\partial x_i} \tilde{h}_m\right](p_m) = \left[\frac{\partial}{\partial x_i} h_m\right](p_m)$$

and, by the inductive hypothesis,

$$\left[\frac{\partial}{\partial x_j} \tilde{Q}_i\right](p_0) = \left[\frac{\partial}{\partial x_j} Q_i\right](p_0),$$

the remaining equalities in (30) follow.

Lemma 3.3. *Let $R_i \in \Gamma$ be q_i-ary and let $h_1^i, \ldots, h_{q_i}^i$ be sequences composed of atoms and projections satisfying* (20) *for a certain $p_0 \in E^n$ $(i = 1, \ldots, m)$. Define p_k^i and P_i as in Lemma* 1.3 *and assume that $h_{q_m}^m$ is an atom such that whenever $h_{q_m}^m = h_k^i$ for some $i \neq m$ or $k \neq q_m$ and R_i depends on the k-th variable under fixed P_i, then $p_{q_m}^m \neq p_k^i$. Let further $W(y_1, \ldots, y_n, \ldots, y_{mn})$ be a polynomial with rational coefficients such that in some neighbourhood of p_0 identically*

$$W\left(\frac{\partial}{\partial x_1} R_1, \frac{\partial}{\partial x_2} R_1, \ldots, \frac{\partial}{\partial x_n} R_1, \ldots, \frac{\partial}{\partial x_n} R_m\right) = 0,$$

where R_i stands for $R_i(h_1^i, \ldots, h_{q_i}^i)$. Then, denoting by $Q_1^m, \ldots, Q_n^m$ the operations appearing in the representation (6) *of R_m we have that for arbitrary numbers $D_1, \ldots, D_n$ there are functions $\tilde{h}_1^i, \ldots, \tilde{h}_{q_i}^i$ such that for $j = 1, \ldots, n$*

(31) $$\left[\frac{\partial}{\partial x_j} \tilde{h}_{q_m}^m\right](p_{q_m}^m) = D_j,$$

(32) $$\left[\frac{\partial}{\partial x_j} \tilde{R}_i\right](p_0) = \left[\frac{\partial}{\partial x_j} R_i\right](p_0); \quad i = 1, \ldots, m-1,$$

(33) $$\left[\frac{\partial}{\partial x_j} \tilde{Q}_i^m\right](p_0) = \left[\frac{\partial}{\partial x_j} Q_i^m\right](p_0), \quad \tilde{Q}_i^m(p_0) = Q_i^m(p_0), \quad (i = 1, \ldots, n),$$

(34) $$W\left(\frac{\partial}{\partial x_1} \tilde{R}_1, \frac{\partial}{\partial x_2} \tilde{R}_1, \ldots, \frac{\partial}{\partial x_n} \tilde{R}_1, \ldots, \frac{\partial}{\partial x_n} \tilde{R}_m\right)(p) = 0 \quad \textit{for each} \quad p \in E^n,$$

where $\tilde{R}_i$ and $\tilde{Q}_m^i$ denote the functions resulting from R_i, Q_i^m by replacing h_k^i by $\tilde{h}_k^i$.

Proof. Let (23) be defined as in the proof of Lemma 1.3. Then, as in the proof of that lemma, we have that (34) holds for any functions $\tilde{h}_k^i$ of the form (25). Thus, in view of Lemma 2.3 it is enough to show that there are functions $\tilde{h}_k^i$ of the form (25) which satisfy, for $j = 1, \ldots, n$,

(35) $$\left[\frac{\partial}{\partial x_j} \tilde{h}_{q_m}^m\right](p_{q_m}^m) = D_j,$$

(36) $$\tilde{h}_k^i(p_k^i) = h_k^i(p_k^i); \quad i = 1, \ldots, m; \ k = 1, \ldots, q_i,$$

(37) $$\left[\frac{\partial}{\partial x_j} \tilde{h}_k^i\right](p_k^i) = \left[\frac{\partial}{\partial x_j} h_k^i\right](p_k^i) \quad \text{if} \quad i \neq m \text{ or } k \neq q_m.$$

Since $h_{q_m}^m$ is not a projection, it is clear that this function appears in (23); we may assume that $h_{q_m}^m = h_w^u = s_\mu h_{\mu,\nu}$, i.e. $u = m$, $w = q_m$. By (20), we have $(s_\mu h_{\mu,\nu})(p_{q_m}^m) \neq 0$. Thus we may define

(38) $$D_j^* = \left\{D_j - \left[\frac{\partial}{\partial x_j} s_\mu \bar{h}_{\mu,\nu}\right](p_{q_m}^m)\right\}\{[s_\mu \bar{h}_{\mu,\nu}](p_{q_m}^m)\}^{-1}.$$

Let $s_b^a, \ldots, s_d^c, \ldots, s_{q_m}^m$ be series such that $s_k^i(p_k^i) = 1$ for every p_k^i, $\left[\frac{\partial}{\partial x_j} s_k^i\right](p_k^i) = 0$ for $(i, k) \neq (m, q_m)$ and $\left[\frac{\partial}{\partial x_j} s_{q_m}^m\right](p_{q_m}^m) = D_j^*$ for $j = 1, \ldots, n$. Moreover, let two of these series, say s_d^c and $s_{q_m}^m$, be identical if and only if the corresponding series s_σ and s_μ (cf. (23)) are identical. Such choice is always possible as we cannot have simultaneously $p_d^c = p_{q_m}^m$ and $s_\sigma = s_\mu$. Indeed, this implies that the atoms $h_d^c, h_{q_m}^m$ have intersecting supports and belong to the same fundamental function, hence $h_d^c = h_{q_m}^m$. But, by our assumption, $h_d^c = h_{q_m}^m$ implies $p_d^c \neq p_{q_m}^m$. Obviously also the following is an instance of (25)

$$\tilde{h}_b^a = s_b^a s_\varrho \bar{h}_{\varrho,m}, \ \ldots, \ \tilde{h}_d^c = s_d^c s_\sigma \bar{h}_{\sigma,r}, \ \ldots, \ \tilde{h}_{q_m}^m = s_{q_m}^m s_\mu \bar{h}_{\mu,\nu};$$
$$\tilde{h}_k^i = h_k^i \text{ for all other } h_k^i.$$

It is easily seen that these functions satisfy (36) and (37). To check (35) it is enough to apply (38). This completes the proof.

Lemma 4.3. *Let $R_i \in \Gamma$ be q_i-ary compositions, $q_i > n$, $i = 1, \ldots, m$. Suppose that $h_1^i, \ldots, h_{q_i}^i$ are sequences composed of atoms and projections such that $h_{q_i}^i$ are atoms and* (20) *holds for each i and every p_0 belonging to a certain open $V_0 \subset E^n$. Let P_i be the corresponding subsequence of projections. For $i, j \leqslant m$ we write $i \prec j$ if there is an $l < q_j$ such that R_j depends on the l-th variable under fixed P_j, $h_{q_i}^i = h_l^j$ and for some open $W \subset V_0$*

(39) $$[R_i, q_i]^*(h_1^i, \ldots, h_{q_i}^i)(p) = [R_j, l]^*(h_1^j, \ldots, h_{q_j}^j)(p)$$

holds for each $p \in W$. Then $\prec$ can be extended to an ordering relation on $\{1, \ldots, m\}$.

Proof. For $t = 1, \ldots, m$ and $r = 1, \ldots, q_t$ we define $s_r^t = x_1^2 + x_2^2 + \ldots + x_n^2$ if h_r^t is an atom and $s_r^t = h_r^t$ if h_r^t is a projection. It is easily seen that $R_t(s_1^t, \ldots, s_{q_t}^t)$ is a polynomial. We denote by $d(t)$ the degree of this polynomial. Now let $i \prec j$. It follows, by (39) and by $h_{q_i}^i = h_l^j$ that

$$R_i(h_1^i, \ldots, h_{q_i}^i)(p) = h_l^j\big([R_j, l]^*(h_1^j, \ldots, h_{q_j}^j)(p)\big)$$

for every $p \in W$. Hence, by Lemma 3.2, there is a $Q \in \Gamma$ and a number $0 \leqslant k < l$ such that

$$R_j(h_1^j, \ldots, h_{q_j}^j)(p) = Q\big(h_1^j, \ldots, h_k^j, R_i(h_1^i, \ldots, h_{q_i}^i), h_{l+1}^j, \ldots, h_{q_j}^j\big)(p)$$

for every $p \in W$. It follows by Lemma 1.3 (Corollary) that the above equality will hold identically in E^n if we replace each h_r^t by s_r^t. Since $s_{q_j}^j = x_1^2 + \ldots + s_n^2$, we see that the degree of $R_j(s_1^j, \ldots, s_{q_j}^j)$ is at least twice the degree of $R_i(s_1^i, \ldots, s_{q_i}^i)$, i.e. $2d(i) \leqslant d(j)$. Thus we have a mapping d of $\{1, \ldots, m\}$ into integers such that $i \prec j$ implies $d(i) < d(j)$. This is obviously sufficient for the existence of an ordering relation on $\{1, \ldots, m\}$ which is an extension of $\prec$.

4. Proof of (v). We shall prove, for every d, the following two staments $(*)$ and $(**)$. It is easily seen that $(**)$ implies $(*)$ but it is required by our method of proof that, for each d, $(*)$ has to be assumed in order to obtain $(**)$. It is clear that $(**)$ implies (v). To see this, it is enough to observe that for each $h \in \boldsymbol{H}$ there is a sequence $h_1, \ldots, h_m$ composed of fundamental functions and projections such that for a certain m-ary $R \in \Gamma$ we have $h = R(h_1, \ldots, h_m)$.

$(*)$ *If $R_i \in \Gamma$ are q_i-ary compositions, $q_i \leqslant d$, $i = 1, 2$, and there are sequences $h_1^i, \ldots, h_{q_i}^i$ composed of fundamental functions and projections such that*

$$R_1(h_1^1, \ldots, h_{q_1}^1) = R_2(h_1^2, \ldots, h_{q_2}^2) \text{ holds on some open set } V,$$

then this equality holds identically on E^n.

$(**)$ *If $R_i \in \Gamma$ are q_i-ary compositions, $q_i \leqslant d$, $i = 1, \ldots, n$, and there are sequences $h_1^i, \ldots, h_{q_i}^i$ $(i = 1, \ldots, n)$ composed of fundamental functions and projections such that no two of the functions $R_i(h_1^i, \ldots, h_{q_i}^i)$ coincide on E^n and none is vanishing identically, then these functions are independent on every open set $V \subset E^n$, i.e. there is a $p \in V$ such that*

$$\big[\partial\big(R_1(h_1^1, \ldots, h_{q_1}^1), \ldots, R_n(h_1^n, \ldots, h_{q_n}^n)\big)/\partial(x_1, \ldots, x_n)\big](p) \neq 0 .$$

Proof of $(*)$ for $d = 1$. If $d = 1$, we have $R_1 = R_2 = E$. Thus condition $(*)$ means that there are two different functions, each of them being either a projection or a fundamental function which coincide on V. Applying the corollary to Lemma 1.3, we obtain a contradiction.

Proof of $(**)$ for $d = 1$. We have to show that if $h_1, \ldots, h_n$ are projections and fundamental functions, all distinct, then they are independent on every open set. Let the indices of these functions be so permuted that $h_1, \ldots, h_t$, $(t \leqslant n)$ are all the projections among $h_1, \ldots, h_n$ and moreover $h_i = e_i$ for $i = 1, \ldots, t$. If $t = n$, then there is nothing to prove. In the other case, let U be an open subset of V such that every fundamental function $h_{t+1}, \ldots, h_n$ is equal on U to some of its atoms, say $h_{t+1} = s_\varrho \overline{h}_{\varrho,m}, \ldots, h_n = s_\mu \overline{h}_{\mu,\nu}$. Supposing that $h_1, \ldots, h_n$ are dependent on U, we have

$$(40) \qquad \partial(x_1, \ldots, x_t, s_\varrho \overline{h}_{\varrho,m}, \ldots, s_a \overline{h}_{\mu,\nu} h)/\partial(x_1, \ldots, x_n) = 0 \quad \text{on} \quad U .$$

It is clear that this Jacobian is identical to a series and, using a similar argument as in the beginning of the proof of Lemma 1.3, we conclude that (40) will hold if we replace the series $s_\varrho, \ldots, s_\mu$ by arbitrary series $\tilde{s}_\varrho, \ldots, \tilde{s}_\mu$ which are convergent and differentiable on U. But if we define $\tilde{s}_\varrho = x_{t+1}\overline{h}_{\varrho,m}^{-1}, \ldots, \tilde{s}_\mu = x_n \overline{h}_{\mu,\nu}^{-1}$ (cf. (3)), then this Jacobian will take identically the value 1, hence a contradiction. Thus $h_1, \ldots, h_n$ are independent.

Induction step in the proof of $(*)$. We assume $(*)$ and $(**)$ for $d < d_0$ and we shall prove that $(*)$ holds for $d = d_0$. Suppose that $R_i \in \Gamma$, $q_i \leqslant d_0$, $(i = 1, 2)$ and

$$(41) \qquad R_1(h_1^1, \ldots, h_{q_1}^1) = R_2(h_1^2, \ldots, h_{q_2}^2) \text{ holds on some open } V \subset E^n.$$

Clearly we may assume that $h_{q_1}^1$ and $h_{q_2}^2$ are fundamental functions.

Suppose first that $R_1(h_1^1, \ldots, h_{q_1}^1) = 0$ on V. Write R_1 in the form (6) as

$$(42) \qquad R_1(h_1^1, \ldots, h_{q_1}^1) = h_{q_1}^1(Q_1, \ldots, Q_n) .$$

If the functions Q_i are all distinct and none of them is vanishing identically on E^n, then it follows by our inductive assumption that they are independent on every open set, in particular on V. This hovewer is impossible because it would imply, by (42), that $h_{q_1}^1$ is vanishing identically on some open set. Hence two of the functions Q_i are identical or one of them is vanishing identically, and thus, by the property of fundamental functions that they vanish on the hyperplanes $x_j = 0$, $x_i = x_j$, it follows, by (42), that $R_1(h_1^1, \ldots, h_{q_1}^1) = 0$ on E^n. By (41), the same is true for R_2 and thus (41) holds identically on E^n.

Before passing to the second part of our proof we have to consider a certain property of $[R_i, k]$. We denote by $\Omega_r^{i,k}$ the operations for which (cf. (9))

$$[R_i, k]^*(h_1^i, \ldots, h_{q_i}^i) = \big(\Omega_1^{i,k}(h_{s_1+1}^i, \ldots, h_{s_2}^i), \ldots, \Omega_n^{i,k}(h_{s_n+1}^i, \ldots, h_{s_{n+1}}^i)\big) ,$$

$$[R_j, l]^*(h_1^j, \ldots, h_{q_j}^j) = \big(\Omega_1^{j,l}(h_{\sigma_1+1}^j, \ldots, h_{\sigma_2}^j), \ldots, \Omega_n^{j,l}(h_{\sigma_n+1}^j, \ldots, h_{\sigma_{n+1}}^j)\big) .$$

We wish to show now that if for some $\Omega_r^{i,k}$, $\Omega_r^{j,l}$ identically on some open set

$$\Omega_r^{i,k}(h_{s_r+1}^i, \ldots, h_{s_{r+1}}^i) = \Omega_r^{j,l}(h_{\sigma_r+1}^j, \ldots, h_{\sigma_{r+1}}^j), \tag{43}$$

then (43) holds identically on E^n. Indeed, if $\Omega_r^{i,k}, \Omega_r^{j,l} \in \Gamma$, then this follows by our inductive assumption (*) for $d < d_0$. In the other case we have that at least one of these operations is equal to E_r, say $\Omega_r^{j,l} = E_r$. If also $\Omega_r^{i,k} = E_r$, then there is nothing to prove. If $\Omega_r^{i,k} \in \Gamma$, then (43) is equivalent to an equality

$$\Omega_r^{i,k}(h_{s_r+1}^i, \ldots, h_{s_{r+1}}^i) = \Omega_r^{i,k}(h_{s_r+1}^*, \ldots, h_{s_{r+1}}^*), \tag{44}$$

where $h_s^* = e_r$ for every s. By the inductive assumption, we have that (44) holds identically on E^n, hence (43) holds identically.

We give now the second part of our proof. It follows from the first part that, replacing V by a smaller set, if necessary, we may assume $R_i(h_1^i, \ldots, h_{q_i}^i) > 0$ on V. For $p \in V$ we denote by p_k^i the point $[R_i, k]^*(h_1^i, \ldots, h_{q_i}^i)(p)$. Since, by the above, (43) never holds on an open set if the functions on both sides of the equality are different, there is a $p \in V$ such that

$$p_k^i = p_l^j \quad \text{implies} \quad [R_i, k]^*(h_1^i, \ldots, h_{q_i}^i) = [R_j, l]^*(h_1^j, \ldots, h_{q_j}^j) \tag{45}$$

identically on E^n. Let P_i be the subsequence of $h_1^i, \ldots, h_{q_i}^i$ consisting of all the projections among these functions and let N_i be the subsequence of those fundamental functions h_k^i such that R_i does not depend on the k-th variable under fixed P_i. Since $R_i(h_1^i, \ldots, h_{q_i}^i)(p) > 0$, we have, by Lemma 2.1 that $h_k^i(p_k^i) > 0$ holds for every h which does not appear in N_i nor in P_i. We define now the functions $\bar{h}_k^i$ $(i = 1, 2;\ k = 1, \ldots, q_i)$ as follows

(j_1) *If h_k^i does not appear in N_i nor in P_i, then $\bar{h}_k^i$ is the unique atom of h_k^i such that $\bar{h}_k^i = h_k^i$ holds in some neighbourhood of p_k^i.*

(j_2) *If h_k^i appears in P_i, then $\bar{h}_k^i = h_k^i$.*

(j_3) *If h_k^i appears in N_i, then $\bar{h}_k^i$ is any atom of h_k^i.*

Applying Lemma 1.1 (Corollary) we see that p has a neighbourhood $V_0 \subset V$ such that

$$R_i(h_1^i, \ldots, h_{q_i}^i) = R_i(\bar{h}_1^i, \ldots, \bar{h}_{q_i}^i) \quad \text{holds on } V_0, \quad i = 1, 2.$$

Thus, by (41),

$$R_1(\bar{h}_1^1, \ldots, \bar{h}_{q_1}^1 h) = R_2(\bar{h}_1^2, \ldots, \bar{h}_{q_2}^2) \quad \text{on } V_0. \tag{46}$$

Since none of the functions $R_i(h_1^i, \ldots, h_{q_i}^i)$ does vanish on any open set, we shall prove that they are identical if we show that they coincide at all those points where they are positive. Let $t \in E^n$ be such that

$R_i(h_1^i, \ldots, h_{q_i}^i)(t) > 0$. Denote $\tilde{p}_k^i = [R_i, k]^*(h_1^i, \ldots, h_{q_i}^i)(t)$. Then, by Lemma 2.2, $h_k^i(\tilde{p}_k^i) > 0$ for these h_k^i which do not appear in N_i nor in P_i. We define $\tilde{h}_k^i$ by the conditions

(l_1) *If h_k^i does not appear in N_i nor P_i, then $\tilde{h}_k^i$ is the unique atom of h_k^i such that $h_k^i = \tilde{h}_k^i$ in some neighbourhood of $\tilde{p}_k^i$.*

(l_2) *If h_k^i appears in P_i, then $\tilde{h}_k^i = h_k^i$.*

(l_3) *If h_k^i appears in N_i, then $\tilde{h}_k^i = \bar{h}_k^i$.*

Clearly we have in (l_1) that if $h_k^i = h_l^j$ and $\tilde{p}_k^i = \tilde{p}_l^j$, then the atoms $\tilde{h}_k^i, \tilde{h}_l^j$ are identical. Since from $\bar{h}_k^i = \bar{h}_l^j$ it follows that $h_k^i = h_l^j$ (e.g. by (*) for $d = 1$) and, by (45), $p_k^i = p_l^j$ implies $\tilde{p}_k^i = \tilde{p}_l^j$, we have that the functions $\tilde{h}_k^i$ satisfy the condition

$$(\rho_0) \qquad \bar{h}_k^i = \bar{h}_l^j \ \text{and} \ p_k^i = p_l^j \quad \text{implies} \quad \tilde{h}_k^i = \tilde{h}_l^j.$$

Trivially the condition

$$(\pi_0) \qquad \text{If } \bar{h}_k^i \text{ is a projection, then } \tilde{h}_k^i = \bar{h}_k^i$$

holds. It follows that, on every bounded set, the functions $\tilde{h}_k^i$ can be uniformly approximated by series s_k^i satisfying the conditions (π) and (ρ) of Lemma 1.3 where h_k^i, h_l^j should be replaced by $\bar{h}_k^i, \bar{h}_l^j$. Applying this lemma we get, by (46),

$$R_1(s_1^1, \ldots, s_{q_1}^1) = R_2(s_1^2, \ldots, s_{q_2}^2) \quad \text{on } E^n,$$

for any such series. Hence

$$R_1(\tilde{h}_1^1, \ldots, \tilde{h}_{q_1}^1) = R_2(\tilde{h}_1^2, \ldots, \tilde{h}_{q_2}^2) \quad \text{on } E^n.$$

Further, by (l_1), (l_2) and (l_3) and by the corollary to Lemma 1.2, we have

$$R_i(h_1^i, \ldots, h_{q_i}^i) = R_i(\tilde{h}_1^i, \ldots, \tilde{h}_{q_i}^i) \quad \text{in some neighbourhood of } t.$$

Hence $R_1(h_1^1, \ldots, h_{q_1}^1)(t) = R_2(h_1^2, \ldots, h_{q_2}^2)(t)$.

Induction step in the proof of (**). We assume (**) for $d < d_0$ and (*) for $d = d_0$. Let R_i and h_k^i satisfy the assumptions of (**) where $d = d_0$. Clearly we can assume that $h_{q_i}^i$ are not projections. To simplify the notation, we shall sometimes abbreviate $R_i(h_1^i, \ldots, h_{q_i}^i)$ to R_i and similarly for other operations. Each R_i such that $q_i > n$, i.e. $R_i \neq E$, we represent in the form (6) as

$$R_i = h_{q_i}^i(Q_1^i, \ldots, Q_n^i). \tag{47}$$

We prove first that if R_i is of the form (47) then all derivatives $\dfrac{\partial}{\partial x_j} R_i$ $(j = 1, \ldots, n)$ cannot vanish identically on the open set V. Indeed, the functions $Q_1^i, \ldots, Q_n^i$ are different and none of them is vanishing identically, for otherwise R_i would be identically 0. It follows, by our inductive assumption that $Q_1^i, \ldots, Q_n^i$ are independent on V, and hence, by the continuity of the derivatives $\frac{\partial}{\partial x_j} Q^i$, there is an open set $V^0 \subset V$ such that

$$(48) \qquad [\partial(Q_1^i, \ldots, Q_n^i)/\partial(x_1, \ldots, x_n)](p) \neq 0 \quad \text{if} \quad p \in V^0.$$

For each $p \in V^0$ denote $p_{qi}^i = (Q_1^i, \ldots, Q_n^i)(p)$. It follows by the independence of $Q_1^i, \ldots, Q_n^i$ that the set B of all the points p_{qi}^i such that $p \in V^0$ is dense in some open subset of E^n. Now suppose that $\frac{\partial}{\partial x_j} R_i = 0$ $(j = 1, \ldots, n)$, identically on V^0. Hence, for every $p \in V^0$, by (47)

$$\left[\frac{\partial}{\partial x_j} R_i\right](p) = \sum_{l=1}^{n} \left[\frac{\partial}{\partial x_l} h_{qi}^i\right](p_{qi}^i) \cdot \left[\frac{\partial}{\partial x_j} Q_l^i\right](p) = 0, \qquad j = 1, \ldots, n.$$

This implies, by (48) that $\frac{\partial}{\partial x_j} h_{qi}^i = 0$ on B for $j = 1, \ldots, n$. From the continuity of these derivatives we conclude that they are vanishing on the open set in which B is dense. This however is impossible since we have assumed (**) for $d = 1$. Hence $\left[\frac{\partial}{\partial x_j} R_i\right](p) = 0$ cannot hold for every $p \in V^0$ and $j = 1, \ldots, n$.

To prove that R_i are independent on V, we take an open set $V^0 \subset V$ such that (48) holds for $i = 1, \ldots, n$ and we select a $p \in V^0$ such that $R_i(p) > 0$ for $i = 1, \ldots, n$. Such point p exists since we have assumed (*) for $d = d_0$, hence none of the functions R_i can be vanishing identically on any open set. Defining functions $\bar{h}_k^i$ by (j_1), (j_2) and (j_3), we have, as in the proof of (*) that $R_i(h_1^i, \ldots, h_{qi}^i) = R_i(\bar{h}_1^i, \ldots, \bar{h}_{qi}^i)$ holds on some neighbourhood V_0 of p. We can assume $V_0 \subset V^0$. Applying Lemma 4.3 (where the h_k^i in the lemma should be replaced now by $\bar{h}_k^i$) we consider the relation $\prec$ on those pairs (i, j) for which $q_i, q_j > n$. Let $\prec_1$ be an ordering relation which is an extension of $\prec$. Permuting the indices, if necessary, we may assume that $R_1, \ldots, R_t$ are all those among the operations $R_1, \ldots, R_n$ which are identical with E, $(t \geqslant 0)$, and for $i, j > t$, $i \prec_1 j$ is equivalent to $i < j$. We shall show that then, for each of the matrices

$$J_m = \begin{bmatrix} \frac{\partial}{\partial x_1} R_1 & \frac{\partial}{\partial x_2} R_1 & \cdots & \frac{\partial}{\partial x_n} R_1 \\ \frac{\partial}{\partial x_1} R_2 & \cdots\cdots & & \frac{\partial}{\partial x_n} R_2 \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial}{\partial x_1} R_m & \cdots\cdots & & \frac{\partial}{\partial x_n} R_m \end{bmatrix}, \qquad m = 1, \ldots, n,$$

there is an open set $V_m \subset V_0$ such that at each point of V_m the rank of J_m is m. This will clearly suffice to complete our proof, in fact, we then have, for $m = n$

$$|J_n| = \partial(R_1, \ldots, R_n)/\partial(x_1, \ldots, x_n) \neq 0 \quad \text{on } V_n.$$

The proof is by induction on m. Consider first $m = 1$. If $R_1 \neq E$ (i.e. $t = 0$), then, as we have shown above, we cannot have identically $\frac{\partial}{\partial x_j} R_1 = 0$ on an open set. Hence there is an open set $V_1 \subset V_0$ such that, for some j, $\left[\frac{\partial}{\partial x_j} R_1\right](p) \neq 0$ holds for all $p \in V_1$. Thus J_1 has rank 1 at every point of V_1. If $R_1 = R_2 = \ldots = R_t = E$, then it follows from (**), for $d = 1$, that V_0 contains an open set V_t such that at each point of V_t the rank of J_t is t.

Now take the inductive assumption that the matrix J_{m-1} has rank $m-1$ at each point of V_{m-1}, where $m > \max(1, t)$ and $V_{m-1} \subset V_0$. Let (i_j, k_j), $j = 1, \ldots, s$, be all those among the pairs (i, k) for which $\bar{h}_k^i = h_{q_m}^m$, R_i depends on the k-th variable under fixed P_i and $(i, k) \neq (m, q_m)$. For $t < i_j < m$ and $k_j < q_{i_j}$ we have, by $i_j \prec_1 m$, that $m \prec i_j$ does not hold, hence in every open subset of V_0 there is a point p such that

$$(49) \qquad [R_m, q_m]^*(\bar{h}_1^m, \ldots, \bar{h}_{q_m}^m)(p) \neq [R_{i_j}, k_j]^*(\bar{h}_1^{i_j}, \ldots, \bar{h}_{q_{i_j}}^{i_j})(p).$$

Since $m \prec m$ never holds, we have this property also for these j for which $i_j = m$ and $k_j < q_m$. It is clear that also in the case when $k_j = q_{i_j}$, every open subset of V_0 contains a point p at which (49) holds (otherwise R_m and R_{i_j} would be coinciding on an open subset of V_0, and, by property (*) for $d = d_0$, R_m and R_{i_j} would be identical on E^n). Since for $i_j \leqslant t$, we have $k_j = q_{i_j} = 1$, we conclude that, for every pair (i_j, k_j), $j = 1, \ldots, s$, every open subset of V_0 contains a point p at which (49) holds.

Renumerating the variables $x_1, \ldots, x_n$, if necessary, we can put our inductive assumption in the form

$$(50) \quad [\partial(R_1, \ldots, R_{m-1})/\partial(x_1, \ldots, x_{m-1})](p) \neq 0 \quad \text{for each} \quad p \in V_{m-1}.$$

Since $V_{m-1} \subset V_0$, there is an open set $V_m^0 \subset V_{m-1}$ such that (49) holds for each $p \in V_m^0$ and $j = 1, \ldots, s$. To complete our proof it is enough to show that it cannot be identically

$$(51) \qquad \partial(R_1, \ldots, R_m)/\partial(x_1, \ldots, x_m) = 0 \quad \text{on } V_m^0.$$

We shall show that, assuming (51), one obtains a contradiction. Suppose (51) holds and let $p_0 \in V_m^0$. Denoting by p_k^i the point $[R_i, k]^*(\bar{h}_1^i, \ldots, \bar{h}_{qi}^i)(p_0)$, we have, by (49), that $p_{q_m}^m \neq p_k^i$ whenever $\bar{h}_{q_m}^m = \bar{h}_k^i$, $i \leqslant m$, $(i, k) \neq (m, q_m)$

and R_i depends on the k-th variable under fixed P_i. Now let $D_1, \ldots, D_n$ be numbers which satisfy the system of equations

$$(52)\qquad \sum_{i=1}^{n} D_i \left[\frac{\partial}{\partial x_j} Q_i^m\right](p_0) = \varepsilon_j; \qquad j = 1, \ldots, n,$$

where $\varepsilon_m = 1$ and $\varepsilon_j = 0$ for $j \neq m$. This system has a solution by (48). Since the Jacobian in (51) is of the form $W\left(\frac{\partial}{\partial x_1} R_1, \ldots, \frac{\partial}{\partial x_m} R_m\right)$ where W is a polynomial with rational coefficients, we are in position to apply Lemma 3.3. Let $\tilde{h}_1^i, \ldots, \tilde{h}_{q_i}^i$ be the functions which satisfy (31), (32) and (33). By (31), (33) and (52) we have

$$\left[\frac{\partial}{\partial x_m} \tilde{R}_m\right](p_0) = 1 \quad \text{and} \quad \left[\frac{\partial}{\partial x_j} \tilde{R}_m\right](p_0) = 0 \quad \text{for} \quad j = 1, \ldots, m-1.$$

Hence

$$(53)\quad [\partial(\tilde{R}_1, \ldots, \tilde{R}_m)/\partial(x_1, \ldots, x_m)](p_0) = [\partial(\tilde{R}_1, \ldots, \tilde{R}_{m-1})/\partial(x_1, \ldots, x_{m-1})](p_0),$$

and by (34), (51)

$$(54)\qquad [\partial(\tilde{R}_1, \ldots, \tilde{R}_m)/\partial(x_1, \ldots, x_m)](p_0) = 0,$$

and by (32)

$$(55)\quad [\partial(\tilde{R}_1, \ldots, \tilde{R}_{m-1})/\partial(x_1, \ldots, x_{m-1})](p_0) = [\partial(R_1, \ldots, R_{m-1})/\partial(x_1, \ldots, x_{m-1})](p_0).$$

It is clear that (53), (54) and (55) contradict (50). Therefore (51) cannot hold and we have an open set $V_m \subset V_m^0$ such that

$$[\partial(R_1, \ldots, R_m)/\partial(x_1, \ldots, x_m)](p) \neq 0 \quad \text{for every} \quad p \in V_m.$$

This completes the proof.

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Reçu par la Rédaction le 29. 8. 1961

On level sets of a continuous nowhere monotone function

by

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Let $f(x)$ be a real function, defined and continuous in a real closed interval I. Let $f(I)$ denote the interval of values taken by $f(x)$ in I. For any $y \in f(I)$, let $f^{-1}(y)$ denote the set of points in I where $f(x)$ takes the value y. The set $f^{-1}(y)$ is known as the *set of level* y of $f(x)$, or, briefly, as a *level set* of $f(x)$. Evidently, $f^{-1}(y)$ is closed for every y.

K. Padmavally [6] proved in 1953 that:

(*) "If $f(x)$ is continuous but monotonic in no interval, then $f^{-1}(y)$ has the power of the continuum for a set of values of y which is of the second category."

S. Marcus ([4], p. 102) improved this result in 1958 into the following form:

(**) "Given a real function $f(x)$, defined and continuous in I. A necessary and sufficient condition so that $f(x)$ may not be monotone in any interval contained in I is that, for every interval $J \subset I$, the values y for which the set $\{x;\ f(x) = y,\ x \in J\}$ is unenumerable form a set of the second category in $(-\infty, \infty)$ and residual in $f(J)$."

Let a function $f(x)$ be *nowhere monotone* in I if it is not monotone in any subinterval of I. Let, further, a nowhere monotone function $f(x)$ be of the *second species* in I in case the function $f(x)+rx$ remains nowhere monotone in I for every real value of r (1).

We prove in § 1 that for a continuous nowhere monotone function $f(x)$ in I, the level set $f^{-1}(y)$ is non-void and perfect for a set of values of y which is residual in $f(I)$. In case of a continuous nowhere monotone function of the second species in I, we investigate in § 2 the sets that are obtained by the intersection of the curve $y = f(x)$ with different straight lines $y = mx + c$.

(1) A detailed study of the Dini derivatives of nowhere monotone functions has been made by the author; see Garg [13], [14]. These investigations are further continued. It may be remarked here that non-differentiable functions constitute a particular case of nowhere monotone functions of the second species.