Functions with sets of points of discontinuity belonging to a fixed ideal

by

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Many theorems concern description of global properties of sets or functions from their local properties. According to C. Kuratowski ([20], p. 34), if **P** is a property of sets in a topological space T, then a set A has this property locally at a point t of T if there exists a neighborhood G of this point such that $G \cap A$ has the property **P**. In spaces satisfying the second axiom of countability inference from local belonging to a class to global belonging is often trivial and efforts are made to extend this to more general cases. E.g. D. Montgomery [26] and C. Kuratowski [19] have investigated this in non-separable metric spaces and so did E. Michael [25] and S. Mrówka [26a] for paracompact and uniform spaces; Banach ([2], see also [20], p. 49) has proved that if a set is locally of the first category at all of its points, then it is globally of the first category. (This theorem has many applications, e.g. it is the crucial point in the proof of completeness of the quotient Boolean algebra of Baire sets modulo sets of the first category.) Among others, let us also mention Bourbaki's integration theory in locally compact spaces the requirement of which is that locally negligible sets be of measure zero, and Brelot's results on locally polar sets in the general potential theory [5].

The subject of this paper is an investigation of some quotient spaces of function algebras considered modulo an ideal of sets. Specifically, there are considered spaces $\mathcal{U}(T, \mathfrak{N})$ of bounded real-valued functions on T with sets of points of discontinuity belonging to a σ-ideal $\mathfrak{N}$ of boundary subsets of T, with identification of functions x, y such that $\{t \in T\colon x(t) \neq y(t)\} \in \mathfrak{N}$. The typical example is the space of Riemann-integrable functions with identification of functions equal almost everywhere. By a well-known theorem of Lebesgue, a bounded function is Riemann integrable if and only if the set of its points of discontinuity is of measure 0. The question of characterization of functions equivalent (i.e. equal a.e.) to some Riemann-integrable functions has been solved by Carathéodory [7]. He proved, for any bounded function, the existence of the least upper semicontinous function $u_f^\dagger$ such that $u_f^\dagger \geqslant f$ a.e. and of the greatest lower semicontinuous function $u_f^{\downarrow}$ such that $u_f^{\downarrow} \leqslant f$ a.e.; the equality $u_f^{\uparrow} = u_f^{\downarrow}$ a.e. characterizes the functions equivalent to Riemann-integrable functions. The method of Carathéodory can be easily applied to any σ-ideal $\mathfrak{N}$ in a space T satisfying the second axiom of countability, but the general case needs the assumption that any set which is locally in $\mathfrak{N}$ is globally in $\mathfrak{N}$. Such ideals are considered in this paper and it turns out that actually there are two non-equivalent notions:

(i) *Strong Banach's localization property* means that if A belongs to $\mathfrak{N}$ locally at each of its points, then $A \in \mathfrak{N}$ (this is just Banach's statement about the ideal of the sets of the first category).

(ii) *Weak Banach's localization property* means that if A belongs to $\mathfrak{N}$ locally at each point of its closure, then $A \in \mathfrak{N}$.

Condition (i) and sometimes condition (ii) enable us to generalize theorems which follow simply from σ-additivity of $\mathfrak{N}$ in the case when countable open coverings can be found.

The second part of the paper concerns the structure space $\Omega(T, \mathfrak{N})$ of the ring $\mathcal{U}(T, \mathfrak{N})$. Theorem 2 states that the density character of $\Omega(T, \mathfrak{N})$ is equal to that of βT. This implies, in particular, existence of a countable dense subset in the Stone space of the Boolean algebra of Baire sets of an interval considered modulo sets of the first category. Since such a countable dense subset cannot exist in the Stone space of the Boolean algebra of μ-measurable sets modulo sets of μ-measure zero whenever μ is an atomless measure on a set, this contributes a little to the discussion of resemblances and differences between measure and category. In particular, this gives a new proof of non-existence of measures vanishing just on the sets of the first category.

Concluding remarks concern Gleason's irreducible map from an extremally disconnected space onto $\Omega(T, \mathfrak{N})$. It turns out that if T is a fixed completely regular space, then all spaces $\Omega(T, \mathfrak{N})$ have the same minimal extremally disconnected resolution independent of $\mathfrak{N}$, which is just $\Omega(\beta T, \mathfrak{B})$ where $\mathfrak{B}$ is the ideal of subsets of βT of the first category.

1. Preliminaries

1.1. Notation. Throughout this paper T will be any topological space (the finite sets will be assumed to be closed only). Next, we shall denote:

$t, t_0, u, v, \ldots$ —points of T, or one-point sets (we shall write for simplicity $A \cup t$ and $A \setminus t$ instead of $A \cup (t)$ and $A \setminus (t)$),

$x, x_0, y, z, \ldots$ —real-valued functions on T,

$A, B, \ldots$ —subsets of T, $A \setminus B$ — their difference,

$\mathfrak{G}$ —the family of all open sets in T,

$\mathfrak{G}(t) = \{G\colon G \in \mathfrak{G},\ t \in G\}$ —the family of neighbourhoods of t,

$\mathfrak{B}$ or $\mathfrak{B}_T$ —the family of the sets of the first category in T,

$D(x)$ —the set of points of discontinuity of x (in T),

$\chi_A(t)$ —the characteristic function of A,

$\mathcal{J}$ —the closed interval $\langle 0, 1\rangle$,

$\mathfrak{N}$ —the class of all subsets of $\mathcal{J}$ of (Lebesgue) measure zero,

$\mathcal{N}$ —the set of integers,

$\beta(E)$ —the Stone-Čech compactification of E,

$C^*(E)$ —the space of real bounded continuous functions on E,

$m(E)$ —the space of all bounded real functions on E,

$\mathfrak{N}$ —a σ-ideal of boundary subsets of T, i.e. a non-empty σ-additive and hereditary family containing no open non-void set,

$\Upsilon(\mathfrak{N}) = \{x \in m(T)\colon D(x) \in \mathfrak{N}\}$,

$\mathfrak{N}_A = \{B\colon B \in \mathfrak{N}, B \subset A\}$ —restricted ideal,

$\mathfrak{O}$ —the trivial ideal consisting of one set 0,

$\mathfrak{P}_A$ —the principal ideal of all subsets of A,

$T_{\mathfrak{N}} = \{t\colon t \in \mathfrak{N}\}$ —the set of all points of T such that the one-point set t belongs to $\mathfrak{N}$, i.e. the sum of all sets of $\mathfrak{N}$,

$A^{\mathfrak{N}} = \{t \in T\colon G \in \mathfrak{G}(t) \Rightarrow A \cap G \notin \mathfrak{N}\}$ —the set of all points at which A does not belong locally to $\mathfrak{N}$,

$M_a(x) = \{t \in T\colon x(t) > a\}$,

$\sup_{\mathfrak{N}A} x = \inf\{a\colon A \cap M_a(x) \in \mathfrak{N}\}$ —the $\mathfrak{N}$-*essential supremum* of x on A; it may be defined equivalently as $\inf\limits_{B \in \mathfrak{N}} \sup\limits_{A \setminus B} x(t)$; if $A \in \mathfrak{N}$, then $\sup_{\mathfrak{N}A} x = -\infty$ for any $x \in m(T)$; if $A = T$ we shall often drop the letter T, i.e. $\sup_{\mathfrak{N}} x = \sup_{\mathfrak{N}T} x$,

$\inf_{\mathfrak{N}A} x = -[\sup_{\mathfrak{N}A} - x]$ —the $\mathfrak{N}$-*essential infimum*,

$\overline{\lim}_{\mathfrak{N}\, u\to t}\, x(u)$ —the $\mathfrak{N}$-*essential limes superior* defined as $\inf\limits_{B \in \mathfrak{N}} \overline{\lim}\limits_{\substack{u\to t\\ u \in B}} x(u)$; it is equal to $\inf\{\sup\limits_{G \setminus t} x(u)\colon G \in \mathfrak{G}(t)\}$ if t is a limit point in T, and to $x(t)$ if t is isolated, and

$$\underline{\lim}_{\mathfrak{N}\, u\to t}\, x(u) = -[\overline{\lim}_{\mathfrak{N}\, u\to t} - x(u)]\,.$$

The symbol $x =_{\mathfrak{N}} y$ *on* A will mean that $\{t \in A\colon x(t) \neq y(t)\}$ belongs to $\mathfrak{N}$; if that is the case, we shall say that x is equal to y $\mathfrak{N}$-*almost everywhere*; $x/\mathfrak{N}$ will denote the class of functions that are equal to x $\mathfrak{N}$-almost everywhere.

$x \leqslant_{\mathfrak{N}} y$ on A will mean that $\{t \in A\colon x(t) > y(t)\} \in \mathfrak{N}$.

$$x^{\dagger}(t) = \begin{cases} \max[x(t)\,,\ \overline{\lim}_{\mathfrak{N}\, u\to t}\, x(u)] & \text{if} \quad t \in T \setminus T_{\mathfrak{N}}\,, \\ \overline{\lim}_{\mathfrak{N}\, u\to t}\, x(u) & \text{if} \quad t \in T_{\mathfrak{N}}\,. \end{cases}$$

Equivalently, $x^{\dagger}(t)$ may be defined as $\inf\{\sup_{\mathfrak{R}} x\colon G \in \mathfrak{G}(t)\}$, or as $\sup\{a\colon t \in M_a^{\mathfrak{R}}\}$, or as $\max[\sup_{\mathfrak{R}}_{\{t\}} x(u), \overline{\lim}_{\mathfrak{R}\, u\to t}\, x(u)]$. We shall also write $x^{\mathfrak{R}}$ instead of $x^{\dagger}$ if the ideal is not fixed, $x^{\downarrow} = -(-x)^{\dagger}$ has dual properties.

Let us resall some known properties of symbols written above (cf. Kuratowski [20], p. 35):

$$(A^{\mathfrak{R}})^{\mathfrak{R}} \subset A^{\mathfrak{R}} = \{t \in T\colon \chi_A^{\dagger}(t) = 1\} = \overline{A^{\mathfrak{R}}} \subset \bar{A}\,,$$

$$(A \cup B)^{\mathfrak{R}} = A^{\mathfrak{R}} \cup B^{\mathfrak{R}}\,, \qquad (A \setminus B)^{\mathfrak{R}} \supset A^{\mathfrak{R}} \setminus B^{\mathfrak{R}}\,,$$

$$\inf_A x \leqslant \inf_{\mathfrak{R}\,A} x \leqslant \sup_{\mathfrak{R}\,A} x \leqslant \sup_A x = \sup_{\mathfrak{D}\,A} x \quad \text{for} \quad A \notin \mathfrak{R}\,,$$

$$x^{\downarrow}(t) \leqslant x^{\dagger}(t) = x^{\mathfrak{R}}(t) \leqslant x^{\mathfrak{D}}(t) = \max[x(t)\,,\ \overline{\lim_{u\to t}}\, x(u)]\,.$$

LEMMA 1. *If $x(t)$ is any bounded continuous functions on a dense subset A of T, then the function*

$$x_e(t) = \begin{cases} x(t) & \text{for} \quad t \in A\,, \\ \overline{\lim\limits_{u\to t}}\, x(u) & \text{for} \quad t \in T \setminus A\,, \end{cases}$$

is continuous at any point of T.

We omit the easy proof. A subset H of T will be called a *D-set in T* if $A = T\setminus H$ is dense in T and if there exists a real-valued bounded continuous function z on A such that z_e is discontinuous at every point of H.

1.2. The space $\mathcal{H}(T, \mathfrak{R})$. The family $\mathfrak{H}$ of all sets of the form $H = G \cup R$ where $G \in \mathfrak{G}$ and $R \in \mathfrak{R}$ satisfies the following conditions: (1) the empty set and whole space T belongs to $\mathfrak{H}$, (2) if $A \in \mathfrak{H}$ and $B \in \mathfrak{H}$, then $A \cap B \in \mathfrak{H}$, (3) if $A_n \in \mathfrak{H}$ $(n = 1, 2, \ldots)$, then $\bigcup A_n \in \mathfrak{H}$. Thus, the class Z of all real-valued functions $x(t)$ on T such that the set $\{t \in T\colon a < x(t) < b\}$ belongs to $\mathfrak{H}$ for every a and b is closed with respect to addition, multiplication, supremum, and infimum of a finite number of functions and is closed with respect to the uniform convergence (Hausdorff [12], pp. 232-270, Carathéodory [6], pp. 369-393). Its subclass $Z \cap m(T)$ is just the class $Y(\mathfrak{R})$ of bounded functions with $D(x) \in \mathfrak{R}$ (Alexiewicz [1]). Consequently, $Y(\mathfrak{R})$ is a closed linear subring with unit and a sublattice of $m(T)$. $\mathcal{H}(T, \mathfrak{R})$ will denote the *quotient space* $Y(\mathfrak{R})/\mathfrak{R}$ *of all functions of* $Y(\mathfrak{R})$ *considered up to sets of* $\mathfrak{R}$. In other words, we consider the set

$$I = \{x \in Y(\mathfrak{R})\colon x =_{\mathfrak{R}} 0 \text{ on } T\} = \{x \in Y(\mathfrak{R})\colon \sup_{\mathfrak{R}} |x| = 0\}\,,$$

and we consider the quotient space $Y(\mathfrak{R})/I$. Since I is a linear ideal in the ring and lattice sense, $\mathcal{H}(T, \mathfrak{R})$ is a linear lattice (cf. Birkhoff [4], p. 222) and a ring as well. At the same time, $\mathcal{H}(T, \mathfrak{R})$ is a normed space with

$$\|x\| = \sup_{\mathfrak{R}\,T} |x(t)|\,.$$

It is complete and the proof of completeness is analogous to that for the space of Riemann-integrable functions with the uniform convergence a.e. ([1]). Namely, if $\{x_n\}$ is a Cauchy sequence in $\mathcal{H}(T, \mathfrak{R})$, then the corresponding functions $x_n(t)$ of $Y(\mathfrak{R})$ converge uniformly on $T\setminus P$ with some $P \in \mathfrak{R}$, and $R_n = D(x_n) \in \mathfrak{R}$. Hence, $R = \bigcup R_n \cup P \in \mathfrak{R}$, $x(t) = \lim x_n(t)$ is continuous on $T\setminus R$ and its extension x_e (according to Lemma 1) belongs to $Y(\mathfrak{R})$ and determines the limit coset $x_e/\mathfrak{R}$ in $\mathcal{H}(T, \mathfrak{R})$.

1.3. Examples. 1. The space $\mathcal{H}(\mathcal{J}, \mathfrak{N})$, is the space of Riemann-integrable functions considered with identification of functions equal a.e. ([2]).

2. The space $\mathcal{H}(T, \mathfrak{D})$ is just the space $C^*(T)$; e.g. $\mathcal{H}(\mathcal{N}, \mathfrak{D}) = m(\mathcal{N})$.

More general, let A be a fixed dense subset of T. We shall show that $\mathcal{H}(T, \mathfrak{P}_{T\setminus A})$ may be identified with $C^*(T)$. Indeed, given any point $t \in T\setminus A$, there exists a direct set $\{t_\tau\}$ of elements of A which is convergent to t in the sense of Moore and Smith (cf. Kelley [16], p. 66), and there exists a multiplicative linear functional ξ_t over the space $C^*(T\setminus A)$ such that the value $\xi_t(x)$ is the limit of a subnet of $x(t_\tau)$ (depending on x) and $\underline{\lim}\, x(t_\tau) \leqslant \xi_t(x) \leqslant \overline{\lim}\, x(t_\tau)$ (Mazur [23], Sikorski [37], p. 118). Denoting

$$x_H(t) = \begin{cases} \xi_t(x) & \text{for} \quad t \in T\setminus A\,, \\ x(t) & \text{for} \quad t \in A \end{cases}$$

we extend any function $x \in C^*(A)$ to a function x_H defined on T. Similarly to Lemma 1, we have $D(x_H) \subset T\setminus A$, i.e. the extended function is continuous at each point of A.

Thus, we have proved that there exists a simultaneous extension $x \to x_H$ of all bounded continuous functions on A to bounded functions on T with $D(x_H) \in \mathfrak{P}_{T\setminus A}$. The operation $U(x) = x_H$ is a ring and lattice isomorphism from $C^*(A)$ into $Y(\mathfrak{P}_{T\setminus A})$ with preserved norm, and to every coset of $\mathcal{H}(T, \mathfrak{P}_{T\setminus A})$ there is a unique function x_H belonging to the coset.

([1]) The author is indebted to Professor W. Orlicz for suggestion of this proof. The original proof of completeness of $\mathcal{H}(T, \mathfrak{N})$, due to Orlicz [28], applies some results of Carathéodory [7]; the simplified Orlicz's proof has not been published.

([2]) This space was considered first by Carathéodory [7] and Orlicz [28]. For a modern theory of Riemann integration, see Haupt [11], Bauer [3] and Marcus [22].

3. The space $\mathcal{H}(T, \mathfrak{B})$, where T is of the second category at each of its points, is the space of the so-called point-wise discontinuous functions considered up to sets of the first category.

Now, let us consider any bounded function $x(t)$ satisfying the condition of Baire on T (this means that counter-images $x^{-1}(E)$ of intervals are of form $(G \cup Q) \setminus R$ with $G \in \mathfrak{G}$ and $Q, R \in \mathfrak{B}$). There exists a set $P \in \mathfrak{B}$ such that x is relatively continuous on $A = T \setminus P$ (cf. Kuratowski [20], p. 306). Thus $x|A \in C_B(A)$ and the function $x_0(t) = (x|A)_e$ is (accordingly to Lemma 1) continuous at any points of A, whence $D(x_0) \in \mathfrak{B}$ and x_0 is point-wise discontinuous on T. It is easily seen that the map $x/\mathfrak{B} \to x_0/\mathfrak{B}$ is a ring and lattice isomorphism and isometry from the space of bounded functions satisfying the condition of Baire, considered up to sets of the first category, onto the space $\mathcal{H}(T, \mathfrak{B})$.

1.4. Classes $\mathcal{H}(T, \mathfrak{R})$ with various $\mathfrak{R}$. Given T and two σ-ideals $\mathfrak{R}_1, \mathfrak{R}_2$ of boundary subsets of T, $\mathcal{H}(T, \mathfrak{R}_1) \subset \mathcal{H}(T, \mathfrak{R}_2)$ will mean that there is a natural one-one correspondence between the cosets of $\mathcal{H}(T, \mathfrak{R}_1)$ and cosets of $\mathcal{H}(T, \mathfrak{R}_2)$. More precisely, this will mean that the following two conditions are satisfied: (1) If $x \in Y(\mathfrak{R}_1)$, then there exists $y \in Y(\mathfrak{R}_2)$ such that $x =_{\mathfrak{R}_2} y$. (2) If $x, x' \in Y(\mathfrak{R}_1)$, if $y, y' \in Y(\mathfrak{R}_2)$ and if $x =_{\mathfrak{R}_1} y$ and $x' =_{\mathfrak{R}_1} y'$, then $x =_{\mathfrak{R}_2} x'$ is equivalent to $y =_{\mathfrak{R}_2} y'$.

$\mathcal{H}(T, \mathfrak{R}_1) = \mathcal{H}(T, \mathfrak{R}_2)$ will mean that $\mathcal{H}(T, \mathfrak{R}_1) \subset \mathcal{H}(T, \mathfrak{R}_2)$ and $\mathcal{H}(T, \mathfrak{R}_2) \subset \mathcal{H}(T, \mathfrak{R}_1)$.

It is easy to see that $\mathfrak{R}_1 \subset \mathfrak{R}_2$ implies $\mathcal{H}(T, \mathfrak{R}_1) \subset \mathcal{H}(T, \mathfrak{R}_2)$, but $\mathcal{H}(T, \mathfrak{R})$ does not determine $\mathfrak{R}$ uniquely, i.e. $\mathcal{H}(T, \mathfrak{R}_1) = \mathcal{H}(T, \mathfrak{R}_2)$ need not imply $\mathfrak{R}_1 = \mathfrak{R}_2$. There exists, however, the least σ-ideal leading to a given class $\mathcal{H}(T, \mathfrak{R})$, namely the σ-ideal $\mathfrak{R}^D$ spanned upon the D-sets of $\mathfrak{R}$. Indeed, $\mathcal{H}(T, \mathfrak{R}^D) \subset \mathcal{H}(T, \mathfrak{R})$ is obvious; further, every $x \in Y(\mathfrak{R})$ is $\mathfrak{R}$-equivalent to the extension (according to Lemma 1) of the restricted function $x\|[T \setminus D(x)]$, and if $x, y \in Y(\mathfrak{R}^D)$ and $x =_{\mathfrak{R}} y$, then $\{t: x(t) \neq y(t)\} \subset D(x) \cup D(y) \in \mathfrak{R}^D$, whence $x =_{\mathfrak{R}^D} y$. Thus $\mathcal{H}(T, \mathfrak{R}^D) = \mathcal{H}(T, \mathfrak{R})$. To show that $\mathfrak{R}^D$ is the least σ-ideal of this property, let us suppose that $\mathcal{H}(T, \mathfrak{R}) = \mathcal{H}(T, \mathfrak{R}_1)$; we claim that all D-sets of $\mathfrak{R}$ belong to $\mathfrak{R}_1$. If it were not so, there would exist $z \in Y(\mathfrak{R}^D)$ such that $D(z) \notin \mathfrak{R}_1$ and no function $\mathfrak{R}$-equivalent to z could be continuous at any point of $D(x)$, so $z =_{\mathfrak{R}} y$ would imply $D(y) \supset D(z)$ and $y \notin Y(\mathfrak{R}_1)$.

If T is of the second category at any of its points (i.e. if $T^{\mathfrak{B}} = T$), then $\mathfrak{D}$ and $\mathfrak{B}$ are two extreme ideals in the family of all σ-ideals of boundary subsets of T in the sense that $\mathfrak{D} \subset \mathfrak{R}^D \subset \mathfrak{B}$ and

$$C^*(T) = \mathcal{H}(T, \mathfrak{D}) \subset \mathcal{H}(T, \mathfrak{R}) \subset \mathcal{H}(T, \mathfrak{B})$$

for any $\mathfrak{R}$ in the family.

2. Semicontinuous majorants

Let $\mathfrak{R}$ be a σ-ideal of subsets of T; in sections 2.1 and 2.2 we shall not assume $\mathfrak{R}$ to contain only boundary subsets.

2.1. Ideals with Banach localization property. We shall say that $\mathfrak{R}$ *has the Banach localization porperty with respect to a set* A ($A \subset T$) if conditions $B \subset A$ and $A \cap B^{\mathfrak{R}} = 0$ imply $B \in \mathfrak{R}$. The class of all such ideals will be denoted by $\boldsymbol{B}(A)$. If $A = T$, we shall shortly say that $\mathfrak{R}$ *has the Banach localization property.*

We shall say that an ideal $\mathfrak{R}$ *has the strong Banach localization property* if, for every family B_ϑ ($\vartheta \in \Theta$) of sets relatively open in their union $\bigcup B_\vartheta$, the conditions $B_\vartheta \in \mathfrak{R}$ ($\vartheta \in \Theta$) imply $\bigcup B_\vartheta \in \mathfrak{R}$. The class of all such ideals will be denoted by $\boldsymbol{B}_s$.

Former condition is essentially weaker than latter, i.e. $\boldsymbol{B}_s \subset \boldsymbol{B}(T)$ but $\boldsymbol{B}_s$ need not be equal to $\boldsymbol{B}(T)$. E.g. the ideal of all countable subsets of the set $T = \Gamma(\omega_1)$ of all ordinals less than or equal to the first uncountable ordinal ω_1 belongs to $\boldsymbol{B}(T) \setminus \boldsymbol{B}_s$. In this example one-point open sets (i.e. isolated points) belong to $\mathfrak{R}$ but it is easy to modify this example to have an ideal of boundary subsets with this property (e.g. considering $\Gamma(\omega_1) \times \mathcal{J}$), Further, if T is a Lindelöf space (cf. [16], p. 50) and $\mathfrak{R}$ is any σ-ideal, then $\mathfrak{R} \in \boldsymbol{B}(T)$. Condition $\mathfrak{R} \in \boldsymbol{B}_s$ is satisfied if and only if $\mathfrak{R}_A \in \boldsymbol{B}(A)$ for every $A \subset T$. Thus

$$\boldsymbol{B}_s = \bigcap_{A \subset T} \boldsymbol{B}(A).$$

If an ideal is invariant with respect to the operation $\mathcal{M}$ of Montgomery (cf. Montgomery [26], Kuratowski [19], p. 536, and [20] p. 268), then $\mathfrak{R} \in \boldsymbol{B}_s$. If T is metrizable and if $\mathfrak{R}$ is a σ-ideal [3] in $\boldsymbol{B}_s$, then $\mathfrak{R}$ is invariant with respect to $\mathcal{M}$.

2.2. Equivalences. The Banach localization properties are related to local $\mathfrak{R}$-essential properties of functions.

LEMMA 2. *The following conditions are equivalent and characterize the class* $\boldsymbol{B}(T)$:

(a) $A^{\mathfrak{R}} = 0$ *implies* $A \in \mathfrak{R}$ *for all* $A \subset T$.

(b) *Given any* $x \in m(T)$ *and any basis of neighbourhoods* U_ϑ ($\vartheta \in \Theta$) *in* T, *the identity* $\sup_{\mathfrak{R}\,T} x = \sup_\vartheta [\sup_{\mathfrak{R}\,U_\vartheta} x]$ *holds.*

(c) $\sup_{\mathfrak{R}\,T} x = \max [\sup_{t \in T \setminus T_{\mathfrak{R}}} x(t), \sup_{t \in T} \overline{\lim}_{\mathfrak{R}\, u \to t} x(u)]$ *holds for any* $x \in m(T)$.

(d) $\mathfrak{R}_F \in \boldsymbol{B}(F)$ *for every closed subset* F *of* T.

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(3) An ideal belonging to $\boldsymbol{B}_s$ need not be a σ-ideal, e.g. the ideal of no-where dense sets belongs to $\boldsymbol{B}_s$ ([20], p. 41).

Proof. (a)$\Rightarrow$(c): Given $x \in m(T)$ and $\varepsilon > 0$, let $B = M_{\alpha+\varepsilon}(x)$ where α is the right-hand side of the equality in (c). Then, for every $t \in T$, there exists a neighbourhood $U(t)$ of t such that $x \leqslant_{\mathfrak{R}} \alpha+\varepsilon$ on $U(t)$ which mean that $B \cap U(t) \in \mathfrak{R}$ for all $t \in T$. Hence, by (a), $B \in \mathfrak{R}$ and $\sup_{\mathfrak{R}} x \leqslant \alpha+\varepsilon$. Thus, we get $\sup_{\mathfrak{R}} x \leqslant \alpha$, the converse inequality being obvious.

(b)$\Rightarrow$(a): Let B be any subset of T such that $B^{\mathfrak{R}} = 0$ and let $x(t)$ be its characteristic function. There exist neighbourhoods $U(t)$ of the points $t \in T$ such that $U(t) \cap B \in \mathfrak{R}$. Applying (b) to the basis consisting of all open non-void subsets of the sets $U(t)$ we obtain $\sup_{\mathfrak{R}} x = 0$, whence $x =_{\mathfrak{R}} 0$ and $B \in \mathfrak{R}$.

Implications (c)$\Rightarrow$(b) and (a)$\Leftrightarrow$(d) are obvious.

LEMMA 3. *The following conditions are equivalent and characterize the class* $\boldsymbol{B}_s$:

(a) $A \cap A^{\mathfrak{R}} = 0$ *implies* $A \in \mathfrak{R}$ *for all* $A \subset T$,

(b) $\mathfrak{R}_G \in \boldsymbol{B}(G)$ *for all* $G \in \mathfrak{G}$.

(c) $A \setminus A^{\mathfrak{R}} \in \mathfrak{R}$ *for all* $A \subset T$.

(d) $x \leqslant_{\mathfrak{R}} x^{\uparrow}$ *for all* $x \in m(T)$.

Proof. (a)$\Leftrightarrow$(b): Obvious.

(a)$\Rightarrow$(c): Given $A \subset T$, let us choose neighbourhoods $G_u \in \mathfrak{G}(u)$ such that $A \cap G_u \in \mathfrak{R}$ and $G_u \cap A^{\mathfrak{R}} = 0$ for all $u \in A \setminus A^{\mathfrak{R}}$. Then $A \setminus A^{\mathfrak{R}} = \bigcup_{u \in A \setminus A^{\mathfrak{R}}} (A \setminus A^{\mathfrak{R}}) \cap G_u$ belongs to $\mathfrak{R}$ as well.

(c)$\Rightarrow$(a): We have $A = (A \setminus A^{\mathfrak{R}}) \cup (A \cap A^{\mathfrak{R}})$. If $A \setminus A^{\mathfrak{R}} \in \mathfrak{R}$, then $A \cap A^{\mathfrak{R}} = 0$ implies $A \in \mathfrak{R}$

(c)$\Rightarrow$(d): If $x(t) = \sum_{i=1}^{n} a_i \chi_{A_i}(t)$ with $A_i \cap A_j = 0$ for $i \neq j$, $\bigcup_{i=1}^{n} A_i = T$ and $a_1 > a_2 > \ldots > a_n$, then $\bigcup A_i^{\mathfrak{R}} = T^{\mathfrak{R}} = T$ and $x^{\uparrow}(t) = \max\{a_i\colon\ t \in A_i^{\mathfrak{R}}\} \geqslant_{\mathfrak{R}} x(t)$. If x is any bounded function, then it is the uniform limit of a sequence x_n of simple functions. Hence

$$\{t \in T\colon\ x(t) > x^{\uparrow}(t)\} \subset \bigcup_{n=1}^{\infty} \{t \in T\colon\ x_n(t) > x_n^{\uparrow}(t)\} \in \mathfrak{R}\,.$$

(d)$\Rightarrow$(c): If $A \subset T$, then $A \setminus A^{\mathfrak{R}} = \{t \in T\colon\ \chi_A(t) > \chi_A^{\uparrow}(t)\} \in \mathfrak{R}$.

LEMMA 4. *Let* $\mathfrak{R} \in \boldsymbol{B}_s$. *Then* (4)

(i) $(A^{\mathfrak{R}})^{\mathfrak{R}} = A^{\mathfrak{R}}$ *for any* $A \subset T$,

(ii) $(x^{\uparrow})^{\uparrow} = x^{\uparrow}$ *for any* $x \in m(T)$.

Proof. Since $A^{\mathfrak{R}} \setminus (A^{\mathfrak{R}})^{\mathfrak{R}} \subset (A \setminus A^{\mathfrak{R}})^{\mathfrak{R}} = 0$, we get $A^{\mathfrak{R}} \subset (A^{\mathfrak{R}})^{\mathfrak{R}}$; the converse inclusion is always true. Equivalence (i)$\Leftrightarrow$(ii) can be proved analogously to (c)$\Leftrightarrow$(d) in Lemma 3.

(4) An example analogous to that considered in 2.1 shows that (i) does not imply $\mathfrak{R} \in \boldsymbol{B}_s$.

2.3. Criteria of belonging to $\mathcal{H}(T, \mathfrak{R})$. In this section we shall consider semicontinuous representations of cosets of $\mathcal{H}(T, \mathfrak{R})$.

THEOREM 1. *Let $\mathfrak{R}$ have the strong Banach localization property and let x be any bounded function on T. Then:*

(i) *The function $x^{\uparrow}$ is the least function upper semicontinuous and majorizing the function x $\mathfrak{R}$-almost everywhere, i.e. $x \leqslant_{\mathfrak{R}} x^{\uparrow}$ and if z is any upper semicontinuous function on T such that $x \leqslant_{\mathfrak{R}} z$, then $x^{\uparrow}(t) \leqslant z(t)$ for all $t \in T$.*

(ii) *There exists a function y defined on T and such that*

$$x =_{\mathfrak{R}} y \quad \textit{on} \quad T\,,$$

$$x^{\downarrow}(t) = y^{\downarrow}(t) \leqslant y(t) \leqslant y^{\uparrow}(t) = x^{\uparrow}(t) \quad \textit{for all} \quad t \in T\,,$$

$$x^{\uparrow}(t) = \max[y(t)\,,\ \varlimsup_{u \to t} y(u)]\,, \qquad x^{\downarrow}(t) = \min[y(t)\,,\ \varliminf_{u \to t} y(u)]\,.$$

In other words, $x^{\mathfrak{R}} = y^{\mathfrak{R}} = y^{\mathfrak{D}}$.

(iii) *In order that there exist a function $z \in m(T)$ such that $x =_{\mathfrak{R}} z$ on T and $D(z) \in \mathfrak{R}$, it is necessary and sufficient that $x^{\uparrow} =_{\mathfrak{R}} x^{\downarrow}$.*

Proof. (5) Upper semicontinuity of $x^{\uparrow}$ is obvious; $x \leqslant_{\mathfrak{R}} x^{\uparrow}$ has been proved in Lemma 3. So let us suppose that $x \leqslant_{\mathfrak{R}} z$ and that z is upper semicontinuous, i.e. $z = z^{\mathfrak{D}}$. Then $x^{\mathfrak{R}} \leqslant z^{\mathfrak{R}} \leqslant z^{\mathfrak{D}} = z$. Condition (ii) is satisfied by the function

$$y(t) = \begin{cases} x^{\uparrow}(t) & \text{if} \quad x(t) > x^{\uparrow}(t)\,, \\ x^{\downarrow}(t) & \text{if} \quad x(t) < x^{\downarrow}(t)\,, \\ x(t) & \text{elsewhere on } T\,. \end{cases}$$

and (ii) yields (iii).

3. Multiplicative linear functionals on $\mathcal{H}(T, \mathfrak{R})$

3.1. General remarks. Ω or $\Omega(T, \mathfrak{R})$ will denote the set of all non-trivial multiplicative linear functionals on $X = \mathcal{H}(T, \mathfrak{R})$, the trivial functional O being excluded. In other words, Ω is the set of all ring homomorphisms (or, equivalently, the set of all lattice homomorphisms) from X onto the set of reals. Ω is a compact Hausdorff space in the *-weak topology of functionals. X is an M-space with unit in the sense of Kakutani [15] and is equivalent (in the linear, metric and lattice sense)

(5) Similar conditions have been proved by Carathéodory [7] in case of Riemann-integrable functions; some conditions are related to results of Levi [21] who assumes, however, some countability axioms.

The converse theorem is true in the following sense: If T is completely regular for each $x \in m(T)$ and there exists the least upper semi-continuous function y such that $y \geqslant_{\mathfrak{R}} x$, then $\mathfrak{R} \in B_s$. Indeed, suppose that $A \notin \mathfrak{R}$ and $A \cap A^{\mathfrak{R}} = 0$. Consider $x = \chi_A$. If $y \geqslant_{\mathfrak{R}} x$, then y must not be identically 0, whence we can construct an upper semi-continuous y_1 such that $0 \leqslant y_1 \leqslant y$, $y_1 \neq y$ and $y_1 =_{\mathfrak{R}} y$, whence $y_1 \geqslant_{\mathfrak{R}} x$.

to the space $C(\Omega)$ (Kakutani [15], M. Krein and S. Krein [18]). At the same time, this isomorphism given by the formula

$$x(\xi) = \xi(x) \quad \text{where} \quad x \in X\,,\ \xi \in \Omega\,,\ x \in C(\Omega)$$

is a ring isomorphism from X onto $C(\Omega)$.

Now, let us examine the examples of 1.3.

1. The set $\Omega(\mathcal{J}, \mathfrak{N})$ corresponding to the class of Riemann-integrable functions is 0-dimensional (= totally disconnected, Ω being compact) and is the Stone space corresponding to the Boolean algebra $\mathfrak{A}/\mathfrak{N}$ of Jordan-measurable subsets of $\mathcal{J}$ considered up to sets of measure 0. This is an immediate consequence of the following theorem (cf. Marcus [22]): given any Riemann-integrable function x on $\mathcal{J}$ and $\varepsilon > 0$, there exists a simple Riemann-integrable function z (i.e. a linear combination of a finite number of characteristic functions of Jordan-measurable subsets of $\mathcal{J}$ such that $|x(t)-z(t)| < \varepsilon$ for all $t \in \mathcal{J}$. Next, a set is Jordan-measurable if and only if the measure of its boundary is zero; hence the sets A, $\bar{A}$ and $\mathrm{Int}(A)$ are equal modulo $\mathfrak{N}$ for any $A \in \mathfrak{A}$. Thus, every coset in $\mathfrak{A}/\mathfrak{N}$ contains a closed set and an open set as well.

$\mathfrak{A}/\mathfrak{N}$ is not σ-complete. Indeed, $I_k^{(n)}$ $(k = 1, \ldots, 2^{n-1}, n = 1, 2, \ldots)$, being the intervals of the usual partition leading to a set $C = \mathcal{J} \setminus \bigcup \bigcup I_k^{(n)}$ of Cantor type and of positive measure, the family $\{I_k^{(2n)}\}$ has no l.u.b.

The space $m = m(\mathcal{N})$ of bounded sequences can be embedded into $\mathcal{H}(\mathcal{J}, \mathfrak{N})$ as a subring with unit; consequently, by the Stone continuous image theorem ([39], p. 475), there exists a continuous map from $\Omega(\mathcal{J}, \mathfrak{N})$ onto $\beta(\mathcal{N})$. Hence $\Omega(\mathcal{J}, \mathfrak{N})$ is of power $2^{\mathfrak{c}}$.

2. If T is completely regular, the set $\Omega(T, \mathfrak{O})$ may be identified with the Stone-Čech compactification $\beta(T)$ and $\Omega(T, \mathfrak{P}_A) \underset{\text{top}}{=} \beta(T \setminus A)$.

3. By the consideration of 1.3, the set $\Omega(T, \mathfrak{B})$ is the Stone space of the Boolean algebra of the Borel sets in T considered up to sets of the first category; this algebra being complete, $\Omega(T, \mathfrak{B})$ is extremely disconnected (cf. Stone [38], Sikorski [35] and [36]).

Finally, let us remark that if T is a metric space and $\mathfrak{N} \neq \mathfrak{O}$, then the power of $\Omega(T, \mathfrak{N})$ is at least $2^{\mathfrak{c}}$. Indeed, if t is a point of $T_{\mathfrak{N}}$, it is not isolated in T, whence there exists a continuous map from $\Omega(T, \mathfrak{N})$ onto $\beta(T \setminus t)$ (by Lemma 1) and $\beta(T \setminus t)$ contains topologically the set $\beta\mathcal{N} \setminus \mathcal{N}$ of power $2^{\mathfrak{c}}$.

3.2. Separability of Ω. $\Omega(T, \mathfrak{N})$ satisfies the second axiom of countability (i.e. it is metrisable) if and only if $\mathfrak{N} = \mathfrak{O}$ and T is metrisable and compact (if that is the case, $\Omega \underset{\text{top}}{=} T$).

Now, in order that there exist a countable set dense in Ω it is necessary and sufficient that there exist a continuous map from $\beta\mathcal{N}$ onto Ω

(sufficiency is trivial, necessity is a consequence of the Stone continuous image theorem). Let T be a completely regular.

Theorem 2. *The density character (i.e. the least cardinal number of a dense subset) of $\Omega(T, \mathfrak{N})$ is equal to that of βT.*

In particular, in the sets $\Omega(\mathcal{J}, \mathfrak{N})$ and $\Omega(\mathcal{J}, \mathfrak{B})$ there exist countable dense subsets, whence there exist strictly positive measures on these spaces (cf. also Tarski [41], p. 229 and Heider [13], p. 218).

Proof. Necessity is an immediate consequence of the existence of a continuous map from $\Omega(T, \mathfrak{N})$ onto βT. To prove sufficiency, let us consider the ideal $\mathfrak{N}_\beta = \mathfrak{N} \vee \mathfrak{P}_{\beta T \setminus T}$ of all sets of the form $R \cup A$ where $R \in \mathfrak{N}$ and $A \subset \beta T \setminus T$. By Lemma 1, the spaces $\mathcal{H}(T, \mathfrak{N})$ and $\mathcal{H}(\beta T, \mathfrak{N}_\beta)$ are equivalent. Consequently, we may assume for simplicity that a set $\{t_a\}$ is dense in T. Let ξ_a be any generalized limit at t_a, i.e. a functional of $\Omega(T, \mathfrak{N})$ such that

$$\underset{u \to t_a}{\underline{\lim}_{\mathfrak{N}}}\, x \leqslant \xi_a(x) \leqslant \underset{u \to t_a}{\overline{\lim}_{\mathfrak{N}}}\, x$$

for all $x \in \mathcal{H}(T, \mathfrak{N})$ (cf. [31], [32] and [33]). We shall prove that the sequence $\{\xi_a\}$ is dense in Ω. Since $\mathcal{H}(T, \mathfrak{N}) \equiv C(\Omega)$, we have to prove that $\|x\| = \sup_{\mathfrak{N}}|x(t)| = \sup_a |\xi_a(x)|$ for any $x \in Y(\mathfrak{N})$. Let us choose $\varepsilon > 0$ and $z \in Y(\mathfrak{N})$ arbitrarily and then, successively, a point $u \in T \setminus D(z)$ such that $|z(u)| > \|z\| - \varepsilon$, a neighbourhood G of u such that $t \in G$ implies $|z(t) - z(u)| < \varepsilon$ and, finally, a point $t_a \in G$. Then

$$|\xi_a(z)| = \xi_a(|z|) \geqslant \underset{t \to t_a}{\underline{\lim}_{\mathfrak{N}}} |z(t)| \geqslant \inf_{G}{}_{\mathfrak{N}} |z| \geqslant |z(u)| - \varepsilon > \|z\| - 2\varepsilon\,.$$

Since $|\xi_a(z)| \leqslant \|z\|$ and since ε may be arbitrarily small, the theorem has been proved.

Now, we can deduce a variant of a theorem of Szpilrajn [40] (cf. also [14], p. 490; [34]).

Theorem 3. *Let T be a completely regular space with a countable dense subset and such that $\beta T \setminus T$ is of the first category in βT. There exists no finite atomless Borel measure μ on T such that a set $A \subset T$ is of the first category if and only if $\mu(A) = 0$.*

Proof. Let us assume, *a contrario*, a non-trivial non-negative atomless regular Borel measure μ onto T vanish exactly on the sets of the first category. We extend μ to βT by the formula $\mu(E) = \mu(E \cap T)$, i.e. $\mu(B) = 0$ for $B \subset \beta T \setminus T$. Obviously, the extended measure is still an atomless Borel measure and vanishes just on the sets of the first category in βT, since E is of the first category in βT if and only if $E \cap T$ is of the first category in T (cf. [20], p. 50). Consequently, the spaces $\mathcal{H}(\beta T, \mathfrak{B})$ and $L_\infty(\beta T, \mu)$ coincide, though no countable set is dense in the Stone space of the Boolean algebra of the μ-measurable sets ([33], Chapt. VII).

The assumption of existence of a countable dense subset in T (or, more general, in βT) as well as the assumption that μ is atomless are essential. Indeed, in the Stone space of the algebra of Lebesgue measurable subsets of $\mathcal{I}$ considered modulo $\mathfrak{N}$ and in $\beta\mathcal{N}$ there exist measures such that a set is of the first category if and only if its measure is zero (cf. Stone [38], Mibu [24], Sikorski [35], p. 257 and [36], Heider [13], p. 221).

Corollary. *The spaces $\mathcal{H}(\mathcal{I}, \mathfrak{B})$ and $m = m(\mathcal{N})$ are isomorphic as Banach spaces, i.e. there exists a linear bicontinuous map from one space onto the other.*

Proof. This is a consequence of a theorem of Pełczyński [29], since either space is isomorphic to a subspace of the other ($\mathcal{H}(\mathcal{I}, \mathfrak{B})$ is isometric to a subspace of m by Theorem 2), and either space has the extension property (cf. [8], p. 94).

3.3. Canonical irreducible maps. A. M. Gleason ([10], see also [30]) has shown the following theorem: For each compact Hausdorff space S there exists a unique extremally disconnected compact Hausdorff space G_S and a (unique up to homeomorphism) continuous irreducible map g_S from G_S onto S (irreducibility means that $F = \overline{F} \subset G_S$ and $F \neq G_S$ imply $g_S(F) \neq S$). G_S will be called the *minimal Stonian resolution of* S and g_S will be called the *canonical irreducible map* onto S.

Let T be a compact Hausdorff space. Then all compact sets $\Omega(T, \mathfrak{R})$ have the same minimal Stonian resolution. Specifically, $\Omega(T, \mathfrak{B})$ is the G_S for each $S = \Omega(T, \mathfrak{R})$ and g_S is just the natural map of $\Omega(T, \mathfrak{B})$ onto $\Omega(T, \mathfrak{R})$, i.e. if ξ is a multiplicative linear functional on $\mathcal{H}(T, \mathfrak{B})$, then $g_S(\xi)$ is the restriction of ξ to the subring $\mathcal{H}(T, \mathfrak{R})$. Consequently if $\mathfrak{R}_1 \subset \mathfrak{R}_2$, then the canonical map onto $\Omega(T, \mathfrak{R}_1)$ is the composition of that of $\Omega(T, \mathfrak{R}_2)$ and of the natural map of $\Omega(T, \mathfrak{R}_2)$ onto $\Omega(T, \mathfrak{R}_1)$.

The proof of irreducibility of the natural map from $\Omega(T, \mathfrak{B})$ onto T is easy and the fact that G_T is just the Stone space of the Boolean algebra $\mathfrak{A}/\mathfrak{B}$ of all Borel sets (or of Baire sets) on T modulo sets of the first category is implicitly contained in [38], [35] and [10]. Gleason considers the algebra of all closed domains (i.e. closures of open sets) which is isomorphic to $\mathfrak{A}/\mathfrak{B}$; indeed, in any coset A of there is a unique closed domain F ($F \subset T$), namely F is the set of points of T at which the elements of the coset are of the second category (cf. [20], p. 52). If A_0 is the closed-open subset of $G_T = \Omega(T, \mathfrak{B})$ corresponding to A in the Stone isomorphism, then F is just the image $g_T(A_0)$ of A_0. At the same time, $\mathcal{H}(T, \mathfrak{B})$ is the normal completion of $C(T)$ in the sense of Dilworth [9] and any coset of $\mathcal{H}(T, \mathfrak{B})$ contains exactly one "normal" function, i.e. an upper semi-continuous function x such that $x(t) = \overline{\lim\limits_{u\to t}}\, \underline{\lim\limits_{v\to u}}\, x(v)$. Note that χ_F is such a function if and only if F is a closed domain.

If T is a non-compact completely regular space, then $\mathcal{H}(T, \mathfrak{R}) = \mathcal{H}(\beta T, \mathfrak{R}_\beta)$ (cf. the proof of Theorem 2), whence the minimal Stonian resolution of $\Omega(T, \mathfrak{R})$ is $\Omega(\beta T, \mathfrak{B}_{\beta T})$. Note that T may be of the first category and cosets of $\mathcal{H}(\beta T, \mathfrak{B}_{\beta T})$ may have no meaning on T, so every space $\mathcal{H}(T, \mathfrak{R})$ corresponds to a space $\mathcal{H}(\beta T, \mathfrak{S})$ but not conversely.

If T is compact and $\Omega(T, \mathfrak{R})$ is extremally disconnected, then $\mathcal{H}(T, \mathfrak{R}) = \mathcal{H}(T, \mathfrak{B})$ by uniqueness of the canonical map. In particular, if T is compact and extremally disconnected, then all spaces $\mathcal{H}(T, \mathfrak{R})$ coincide. Another example is the space X of all bounded real-valued functions on $\mathcal{I}$ such that left-hand side and right-hand side limits exist at each point of $\mathcal{I}$; if $\mathfrak{R}$ is the ideal of all countable subsets of $\mathcal{I}$, then $C(\mathcal{I}) \subset X/\mathfrak{R} \subset \mathcal{H}(\mathcal{I}, \mathfrak{R})$ (natural embeddings). $X/\mathfrak{R}$ is equivalent to $C(S)$ where S is the lexicographical product of $\mathcal{I}$ and of a two-element set (with the order topology) and $G_S = G_{\mathcal{I}}$.

Proposition. *Given a compact Hausdorff space T, the following conditions are equivalent:*

(i) *The set T_0 of isolated points of T is dense in T.*

(ii) *G_T is homeomorphic to a space βN_α where N_α is a discrete space.*

(iii) *The set of isolated points of $\Omega(T, \mathfrak{R})$ is dense in $\Omega(T, \mathfrak{R})$ for some (or for each) $\mathfrak{R}$.*

Proof. (i)$\Rightarrow$(ii): If $T \setminus T_0$ is boundary, it is nowhere dense because it is closed. Hence $\mathfrak{B}_T = \mathfrak{B}_{T\setminus T_0}$ and every bounded function on T_0 corresponds to exactly one coset of $\mathcal{H}(T, \mathfrak{B}_T)$. Thus, $G_T = \beta(T_0)$ and g_T maps points of T_0 onto themselves. Implications (ii)$\Rightarrow$(iii)$\Rightarrow$(i) are trivial.

Further, for any continuous map h of $\beta N_\alpha \setminus N_\alpha$ onto a Hausdorff space S there is a unique compact T such that $G_T = \beta N_\alpha$, $T = S \cup N_\alpha$, $g_T(t) = t$ for $t \in N_\alpha$ and $g_T(t) = h(t)$ for $t \in \beta N_\alpha \setminus N_\alpha$.

Let us note that if S is compact and zero-dimensional, then G_S is the Stone space of the minimal extension (in the sense of MacNeille) of the algebra of open-closed subsets of S (cf. [36], p. 119).

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