he left, presents three formulations of the theory of propositional types, one of which is based upon equivalence. A proof of completeness is given. However, the systems differ from ours in various ways, principally in a rule of definition allowing the introduction of names for arbitrary elements of the hierarchy of propositional types.

References

[1] A. Church, *A formulation of the simple theory of types*, The Journal of Symbolic Logic, 5 (1940), pp. 56-68.

[2] — *Introduction to mathematical logic*, Vol. 1., Princeton University Press, Princeton (N. J.), 1956.

[3] A. Grzegorczyk, *The systems of Leśniewski in relation to contemporary logical research*, Studia Logica 3 (1955), pp. 77-95.

[4] L. Henkin, *Completeness in the theory of types*, The Journal of Symbolic Logic 15 (1950), pp. 81-91.

[5] A. Tarski, *Sur le terme primitif de la logistique*, Fundamenta Mathematicae 4 (1923), pp. 196-200.

Reçu par la Rédaction le 2. 7. 1962

A reduction of the axioms for the theory of propositional types

by

P. Andrews * (Princeton, N. J.)

Throughout this paper we shall follow the notation used by Henkin in his paper *A theory of propositional types* (this Volume, pp. 323-342), hereafter referred to as [H]. Reference numbers followed by 'H' refer to sections of that paper. (1)

Henkin's paper is of particular interest in that it takes symbols for the identity relation as the sole primitive constants. That there is ample historical precedent for special interest in such a system is attested by the following passage from Ramsey's article, *The Foundations of Mathematics*:

"The preceding and other considerations led Wittgenstein to the view that mathematics does not consist of tautologies, but of what he called 'equations', for which I should prefer to substitute 'identities'.... (It) is interesting to see whether a theory of mathematics could not be constructed with identities for its foundation. I have spent a lot of time developing such a theory, and found that it was faced with what seemed to me insuperable difficulties." (2)

The full beauty of Henkin's theory of propositional types can perhaps best be appreciated when the system of axioms in section 5.1H is simplified somewhat. Therefore let us replace this system of axioms by the following

AXIOMS.

(1) $(g_{00}T^n \wedge g_{00}F^n) \equiv \forall x_0(g_{00}x_0)$.

$(2^{\alpha 0})$ $(f_{\alpha 0} \equiv g_{\alpha 0}) \to (h_{0(\alpha 0)}f_{\alpha 0} \equiv h_{0(\alpha 0)}g_{\alpha 0})$.

---

* The author was supported by a National Science Foundation Cooperative Graduate Fellowship while working on this paper.

(1) I wish to express my appreciation to Professor Henkin for the privilege of reading his paper before publication, and for his many helpful suggestions, comments, and criticisms during the writing of the present paper.

(2) F. P. Ramsey, *The Foundations of Mathematics*, Proceedings of the London Mathematical Society, series 2, 25 (1926), p. 350.

$(3^{\alpha\beta})$ $\big(\forall x_\beta(f_{\alpha\beta}x_\beta \equiv g_{\alpha\beta}x_\beta)\big) \equiv (f_{\alpha\beta} \equiv g_{\alpha\beta})$.

(4) $\big((\lambda X_\beta B_\alpha) A_\beta\big) \equiv C_\alpha$, where C_α is obtained from B_α by replacing each free occurrence of X_β in B_α by an occurrence of A_β, providing no such occurrence of X_β is within a part of B_α which is a formula beginning '$(\lambda Y_\gamma$' where Y_γ is a variable free in A_β.

Note that our Axiom (4) is Axiom 7 of [H], while our Axioms (1), (2), and (3) are closely related to Axioms 4, 5.1, and 6, respectively, of [H].

We next show that each axiom of [H] can be derived from the above system of axioms with Rule R (5.2H) as the sole rule of inference.

(5) Theorem Schema. $\vdash A_\alpha \equiv A_\alpha$, *where* A_α *is any formula.*

Proof.

(5.1) $\vdash (\lambda x_0 A_\alpha) x_0 \equiv A_\alpha$ by Axiom (4).

(5.2) $\vdash (\lambda x_0 A_\alpha) x_0 \equiv A_\alpha$ by Axiom (4).

(5.3) $\vdash A_\alpha \equiv A_\alpha$ by Rule R, (5.1), (5.2).

This is Axiom Schema 1 of [H].

(6) Theorem. $\vdash T^n \wedge T^n$.

Proof.

(6.1) $\vdash \big(\lambda g_{00}(g_{00}T^n \wedge g_{00}F^n)\big)(\lambda y_0 T^n)$
$\equiv \big(\lambda g_{00}(g_{00}T^n \wedge g_{00}F^n)\big)(\lambda y_0 T^n)$ by (5).

(6.2) $\vdash \big(\lambda g_{00}(\forall x_0(g_{00}x_0))\big)(\lambda y_0 T^n)$
$\equiv \big(\lambda g_{00}(g_{00}T^n \wedge g_{00}F^n)\big)(\lambda y_0 T^n)$ by Rule R, Axiom (1), (6.1).

(6.3) $\vdash \forall x_0\big((\lambda y_0 T^n) x_0\big) \equiv . (\lambda y_0 T^n) T^n \wedge (\lambda y_0 T^n) F^n$
by Rule R, Axiom (4), (6.2).

(6.4) $\vdash \forall x_0 T^n \equiv . T^n \wedge T^n$ by Rule R, Axiom (4), (6.3).

(6.5) $\vdash \forall x_0 T^n$ by (5), since by definition of $\forall$ (4.6H), $\forall x_0 T^n$ is $(\lambda x_0 T^n) \equiv (\lambda x_0 T^n)$.

(6.6) $\vdash T^n \wedge T^n$ by Rule R, (6.4), (6.5).

(7) Theorem Schema. $\vdash (A_0 \equiv T^n) \equiv A_0$, *where* A_0 *is any formula.*

Proof. Let X_0 be any variable not occurring in A_0.

(7.1) $\vdash \big(\lambda f_{00}(f_{00}T^n \equiv T^n)\big) \equiv \big(\lambda f_{00}(f_{00}T^n)\big)$
by Rule R, Axiom (4), (6), definition of $\wedge$ (4.4H).

(7.2) $\vdash \big(\lambda f_{00}(f_{00}T^n \equiv T^n)\big)(\lambda X_0 A_0) \equiv \big(\lambda f_{00}(f_{00}T^n \equiv T^n)\big)(\lambda X_0 A_0)$ by (5).

(7.3) $\vdash \big(\lambda f_{00}(f_{00}T^n \equiv T^n)\big)(\lambda X_0 A_0) \equiv \big(\lambda f_{00}(f_{00}T^n)\big)(\lambda X_0 A_0)$
by Rule R, (7.1), (7.2).

(7.4) $\vdash (A_0 \equiv T^n) \equiv A_0$ by Rule R, Axiom (4), (7.3), condition on X_0.

This is Axiom Schema 2 of [H].

Remark: It is now easy to check that sections 7.1H-7.8H apply to our system as well as to [H], so we are free to use the theorems and derived rules of inference in these sections.

(8) Theorem. $\vdash (T^n \wedge F^n) \equiv F^n$.

Proof.

(8.1) $\vdash \big((\lambda y_0 y_0) T^n \wedge (\lambda y_0 y_0) F^n\big) \equiv \forall x_0 (\lambda y_0 y_0) x_0$
by Rule Sub (7.6H), Axiom (1).

(8.2) $\vdash (T^n \wedge F^n) \equiv \forall x_0 x_0$ by Rule R, Axiom (4), (8.1).

(8.3) $\vdash (T^n \wedge F^n) \equiv F^n$ from (8.2) by definitions of $\forall$ (4.6H) and F^n (4.2H); indeed, $\forall x_0 x_0$ is $(\lambda x_0 x_0) \equiv (\lambda x_0 T^n)$, which is F^n.

We thus obtain Axiom 3 of [H].

(9) Lemma. *Let* X_β *be any variable of type* β, *let* A_0 *be any formula in which* X_β *does not occur, and let* B_0 *be the result of substituting* X_β *for* x_β *at all free occurrences of* x_β *in* A_0. *Then* $\vdash (\lambda x_\beta A_0) \equiv (\lambda X_\beta B_0)$.

Proof (essentially as in 7.21H).

(9.1) $\vdash \forall x_\beta . A_0 \equiv A_0$ by Rule G (7.4H), (5).

(9.2) $\vdash \forall x_\beta . (\lambda x_\beta A_0) x_\beta \equiv (\lambda X_\beta B_0) x_\beta$
by Rule R, E-Rules (7.2H), Axiom (4), (9.1).

(9.3) $\vdash \forall x_\beta\big((\lambda x_\beta A_0) x_\beta \equiv (\lambda X_\beta B_0) x_\beta\big) \equiv . (\lambda x_\beta A_0) \equiv (\lambda X_\beta B_0)$
by Rule Sub (7.6H), Axiom $(3^{0\beta})$.

(9.4) $\vdash (\lambda x_\beta A_0) \equiv (\lambda X_\beta B_0)$ by Rule R, (9.3), (9.2).

(10) Theorem. $\vdash (g_{00}T^n \wedge g_{00}F^n) \equiv \big(\forall X_0(g_{00}X_0)\big)$.

Proof. This is obtained from Axiom (1) by the use of lemma (9), Rule R, and the definition of $\forall$ (4.6H).

This is Axiom Schema 4 of [H].

Remark. It is now easy to see that sections 7.9H-7.12H also apply to our system.

(11) Theorem Schema. $\vdash (A_0 \wedge A_0) \equiv A_0$, *where* A_0 *is any formula.*

Proof.

(11.1) $\vdash \big((\lambda x_0 x_0) \equiv (\lambda x_0 T^n)\big) \to . (\lambda y_{00} . y_{00} F^n)(\lambda x_0 x_0)$
$\equiv (\lambda y_{00} . y_{00} F^n)(\lambda x_0 T^n)$ by Rule Sub (7.6H), Axiom (2^{00}).

(11.2) $\vdash \big((\lambda x_0 x_0) \equiv (\lambda x_0 T^n)\big) \to . F^n \equiv T^n$ by Rule R, Axiom (4), (11.1)

(11.3) $\vdash F^n \to F^n$ by Rule R, (7), (11.2), definition of F^n (4.2H).

(11.4) $\vdash (F^n \wedge F^n) \equiv F^n$
by Rule R, Axiom (4), (11.3), definition of $\to$ (4.5H).

(11.5) $\vdash (x_0 \wedge x_0) \equiv x_0$ by Rule of Cases (7.9H), (7.7H), (11.4).

(11.6) $\vdash (A_0 \wedge A_0) \equiv A_0$ by Rule Sub (7.6H), (11.5).

(12) THEOREM. $\vdash (f_a \equiv g_a) \to . h_{0a} f_a \equiv h_{0a} g_a$, *where a is any type symbol.*

Proof.

(12.1) $\vdash ((\lambda z_0 f_a) \equiv (\lambda z_0 g_a)) \to . (\lambda t_{a0} . h_{0a}(t_{a0} w_0))(\lambda z_0 f_a)$
$\equiv (\lambda t_{a0} . h_{0a}(t_{a0} w_0))(\lambda z_0 g_a)$
by Rule Sub (7.6H), Axiom (2^{a0}).

(12.2) $\vdash ((\lambda z_0 f_a) \equiv (\lambda z_0 g_a)) \to . h_{0a} f_a \equiv h_{0a} g_a$
by Rule R, Axiom (4), (12.1).

(12.3) $\vdash \forall x_0 ((\lambda z_0 f_a) x_0 \equiv (\lambda z_0 g_a) x_0) \equiv . (\lambda z_0 f_a) \equiv (\lambda z_0 g_a)$
by Rule Sub (7.6H), Axiom (3^{a0}).

(12.4) $\vdash \forall x_0 (f_a \equiv g_a) \equiv . (\lambda z_0 f_a) \equiv (\lambda z_0 g_a)$
by Rule R, Axiom (4), (12.3).

(12.5) $\vdash ((\lambda t_0 (f_a \equiv g_a)) T^n \wedge (\lambda t_0 (f_a \equiv g_a)) F^n) \equiv \forall x_0 . (\lambda t_0 (f_a \equiv g_a)) x_0$
by Rule Sub (7.6H), Axiom (1).

(12.6) $\vdash ((f_a \equiv g_a) \wedge (f_a \equiv g_a)) \equiv \forall x_0 . f_a \equiv g_a$
by Rule R, Axiom (4), (12.5).

(12.7) $\vdash ((\lambda z_0 f_a) \equiv (\lambda z_0 g_a)) \equiv . f_a \equiv g_a$
by E-Rules (7.2H), (12.4), (12.6), (11).

(12.8) $\vdash (f_a \equiv g_a) \to . h_{0a} f_a \equiv h_{0a} g_a$
by Rule R, (12.7), (12.2).

Remark. Axiom (2^{00}) seems to be necessary for the proof of theorem (11), which is used in the proof of (12.7) above. However, it is clear that any finite number of instances of our Axiom Schema (2) other than (2^{00}) may be deleted from our list of axioms, and then derived by the method used in our proof of theorem (12). Indeed, certain infinite sets of instances of Axiom Schema (2) might be deleted from the list of axioms; for example, it would suffice to take only those instances of Axiom Schema (2) of the forms (2^{00}) or ($2^{(a0)0}$) as axioms.

(13) THEOREM. $\vdash (x_\beta \equiv y_\beta) \to . (f_{a\beta} \equiv g_{a\beta}) \to (f_{a\beta} x_\beta \equiv g_{a\beta} y_\beta)$.

Proof.

(13.1) $\vdash (x_\beta \equiv y_\beta) \to . (\lambda z_\beta ((f_{a\beta} \equiv g_{a\beta}) \to (f_{a\beta} z_\beta \equiv g_{a\beta} y_\beta))) x_\beta$
$\equiv (\lambda z_\beta ((f_{a\beta} \equiv g_{a\beta}) \to (f_{a\beta} z_\beta \equiv g_{a\beta} y_\beta))) y_\beta$
by Rule Sub (7.6H), (12).

(13.2) $\vdash (x_\beta \equiv y_\beta) \to . ((f_{a\beta} \equiv g_{a\beta}) \to (f_{a\beta} x_\beta \equiv g_{a\beta} y_\beta))$
$\equiv ((f_{a\beta} \equiv g_{a\beta}) \to (f_{a\beta} y_\beta \equiv g_{a\beta} y_\beta))$
by Rule R, Axiom (4), (13.1).

(13.3) $\vdash (f_{a\beta} \equiv g_{a\beta}) \to . (f_{a\beta} y_\beta \equiv g_{a\beta} y_\beta) \equiv (g_{a\beta} y_\beta \equiv g_{a\beta} y_\beta)$
by Rule Sub (7.6H), (12), Rule R, Axiom (4).

(13.4) $\vdash (f_{a\beta} \equiv g_{a\beta}) \to (f_{a\beta} y_\beta \equiv g_{a\beta} y_\beta)$
from (13.3) by (5), Rule T (7.3H), Rule R, (7).

(13.5) $\vdash (x_\beta \equiv y_\beta) \to . (f_{a\beta} \equiv g_{a\beta}) \to (f_{a\beta} x_\beta \equiv g_{a\beta} y_\beta)$
by Rule T (7.3H), (13.4), Rule R, (7), (13.2).

This is Axiom 5 of [H].

Remark. It is now easy to see that sections 7.13H-7.20H also apply to our system.

(14) THEOREM. $\vdash (\forall X_\beta (f_{a\beta} X_\beta \equiv g_{a\beta} X_\beta)) \to (f_{a\beta} \equiv g_{a\beta})$.

Proof.

(14.1) $\vdash (\forall X_\beta (f_{a\beta} X_\beta \equiv g_{a\beta} X_\beta)) \equiv (f_{a\beta} \equiv g_{a\beta})$ from Axiom ($3^{a\beta}$) by lemma (9), Rule R, and the definition of $\forall$ (4.6H).

(14.2) $\vdash ((\forall X_\beta (f_{a\beta} X_\beta \equiv g_{a\beta} X_\beta)) \equiv (f_{a\beta} \equiv g_{a\beta}))$
$\to . (\forall X_\beta (f_{a\beta} X_\beta \equiv g_{a\beta} X_\beta)) \to (f_{a\beta} \equiv g_{a\beta})$
by 7.20H, since this is a tautological formula.

(14.3) $\vdash (\forall X_\beta (f_{a\beta} X_\beta \equiv g_{a\beta} X_\beta)) \to (f_{a\beta} \equiv g_{a\beta})$
by Rule MP (7.12H), (14.1), (14.2).

This is Axiom Schema 6 of [H]. We have thus completed the task of showing that each axiom of [H] can be derived from our axioms.

We remark that the entire theory of propositional types can be developed from our axioms without making any use of the definition of $\wedge$ (4.4H), and so any definition could be used which did not render the system inconsistent. (For example, we could, if we wished, define $\wedge$ as $\lambda x_0 . \lambda y_0 . (\lambda g_{000} (g_{000} x_0 y_0)) \equiv (\lambda g_{000} (g_{000} T^n T^n))$.) To see this, we remark that the only place the definition of $\wedge$ is used in [H] is in the proof of 7.7H $\vdash (T^n \wedge T^n) \equiv T^n$, and the only place the definition of $\wedge$ is used in the present paper is in the proof of theorem (7): $\vdash (A_0 \equiv T^n) \equiv A_0$. To prove these theorems without using the definition of $\wedge$ one may proceed along the following lines:

First show that by using Rule R, 7.21H, and Axiom (4) one can put any formula into λ-normal form, that is a form in which it contains no well-formed parts of the form $((\lambda X_\beta B_a) A_\beta)$. Next show that if $\vdash A_0$, then the λ-normal form of A_0 has the form $B_a \equiv C_a$. One then sees that Rule Sub can be proved essentially as in 7.4H-7.6H without using Rule T. One substitutes $(\lambda y_a y_a)$ for f_{aa} and g_{aa} in Axiom (3^{aa}) and uses theorem (5) to prove $\vdash \forall x_a (x_a \equiv x_a)$, from which one proceeds as in 7.5H to prove $\vdash (B_a \equiv B_a) \equiv T^n$. Hence one easily proves Rule T.

One applies Rule T to our theorem (6) to obtain $\vdash (T^n \wedge T^n) \equiv T^n$, and combines this with theorem (8) via Rule of Cases and Rule Sub to obtain $\vdash (T^n \wedge A_0) \equiv A_0$. One readily proves $\vdash (T^n \equiv T^n) \equiv T^n$ by theorem (5) and Rule T, so to prove $\vdash (A_0 \equiv T^n) \equiv A_0$ by Rule of Cases and Rule Sub it suffices to prove $\vdash (F^n \equiv T^n) \equiv F^n$. This is done as follows:

Use Axiom (1) to prove $\vdash((T^n \equiv T^n) \wedge (F^n \equiv T^n)) \equiv \forall x_0(x_0 \equiv T^n)$, and hence $\vdash(F^n \equiv T^n) \equiv \forall x_0(x_0 \equiv T^n)$. Then substitute $(\lambda x_0 x_0)$ for f_{00} and $(\lambda x_0 T^n)$ for g_{00} in Axiom (3^{00}) and use the definition of F^n to obtain $\vdash \forall x_0(x_0 \equiv T^n) \equiv F^n$.

Henkin remarks at the end of [H] that when one passes from the theory of propositional types to the full theory of finite types, it becomes necessary to add a constant $\iota_{1(01)}$ to denote a descriptor function, and an appropriate axiom involving this constant. We note that for this axiom it suffices to take the simple formula

$$\iota_{1(01)}(\lambda x_1(x_1 \equiv y_1)) \equiv y_1 ,$$

from which the formula

$$(\exists! x_1)(f_{01} x_0) \rightarrow f_{01}(\iota_{1(01)} f_{01})$$

can be derived without difficulty.

PRINCETON, N. J.

Reçu par la Rédaction le 20. 8. 1962

О диадических пространствах

В. Пономарев (Москва)

1. Эта заметка примыкает к работе [1].

Определение. Вполне регулярное пространство X назовем (*неприводимо*) *диадическим*, если у него существует расширение $\overline{X}$, являющееся (неприводимо) диадическим бикомпактом (¹).

Заметим прежде всего, что всякое диадическое пространство X удовлетворяет отрицательной аксиоме счетности (т. е. в X не существует несчетной дизъюнктной системы открытых множеств). Действительно, если $\overline{X}$ есть расширение пространства X, являющееся диадическим бикомпактом, а $\{U_a\}$ дизъюнктная несчетная система открытых множеств пространства X, то рассматривая систему $\{OU_a\}$ открытых в $\overline{X}$ множеств, высекающую из X данную систему $\{U_a\}$, получим также несчетную дизъюнктную систему открытых в $\overline{X}$ множеств, чего в диадическом бикомпакте быть не может (Теорема Э. Марчевского [2]).

Теорема 1. *Диадическое паракомпактное пространство финально-компактно* (²).

Доказательство. Достаточно доказать, что всякое покрытие γ нормального пространства X, удовлетворяющего отрицательной аксиоме счетности, содержит счетное покрытие того-же пространства.

Предполагаем, что элементы покрытия γ занумерованы порядковыми числами; т. е. что $\gamma = \{\Gamma_a\}$, где

$$a = 1, 2, 3, \ldots, < \omega_\tau .$$

Так как γ — покрытие парокомпактного пространства X, то существует такое

(¹) Как известно, бикомпакт $\overline{X}$ веса τ называется (неприводимо) *диадическим*, если он является образом обобщенного канторова дисконтинуума D^τ (т. е. топологического произведения τ пространств, каждое из которых состоит из двух изолированных точек) при некотором (неприводимо) непрерывном отображении $f\colon D^\tau \rightarrow \overline{X}$. Непрерывное отображение f пространства R на пространство R' называется *неприводимым*, если для всякого замкнутого подмножества $A \neq R$ пространства R имеем $fA \neq R'$.

(²) Как известно, пространство X называется *финально-компактным* (лии *линделёфовым*), если из всякого его открытого покрытия можно выделить счетное множество элементов, также образующих покрытие пространства X.