be left, presents three formulations of the theory of propositional types, one of which is based upon equivalence. A proof of completeness is given. However, the systems differ from ours in various ways, principally in a rule of definition allowing the introduction of names for arbitrary elements of the hierarchy of propositional types.

References


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A reduction of the axioms for the theory of propositional types

by

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Throughout this paper we shall follow the notation used by Henkin in his paper A theory of propositional types (this Volume, pp. 323-342), hereafter referred to as [H]. Reference numbers followed by 'H' refer to sections of that paper. (*)

Henkin's paper is of particular interest in that it takes symbols for the identity relation as the sole primitive constants. That there is ample historical precedent for special interest in such a system is attested by the following passage from Ramsey's article, The Foundations of Mathematics:

"The preceding and other considerations led Wittgenstein to the view that mathematics does not consist of tautologies, but of what he called 'equations', for which I should prefer to substitute 'identities'. (...) is interesting to see whether a theory of mathematics could not be constructed with identities for its foundation. I have spent a lot of time developing such a theory, and found that it was faced with what seemed to me insuperable difficulties." (†)

The full beauty of Henkin's theory of propositional types can perhaps best be appreciated when the system of axioms in section 5.1 H is simplified somewhat. Therefore let us replace this system of axioms by the following

AXIOMS.

(1) \((\varphi \land \varphi') \land \varphi' \land \varphi') = \varphi \land \varphi'

(2) \((\varphi \equiv \varphi') \land \varphi \equiv \varphi'

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Remark: It is now easy to check that sections 7.1H-7.8H apply to our system as well as to [H], so we are free to use the theorems and derived rules of inference in these sections.

(8) Theorem. \( \vdash (T^n \land F^n) = F^n \).

Proof.

(8.1) \( \vdash (\exists y_0 x_n) T^n \land (\exists y_0 x_n) F^n = \forall y_0 (\exists y_0 x_n) x_0 \) by Rule Sub (7.6H), Axiom 1.

(8.2) \( \vdash (T^n \land F^n) = \forall y_0 x_0 \) by Rule R, Axiom 4, (8.1).

(8.3) \( \vdash (T^n \land F^n) = F^n \) from (8.2) by definitions of \( V \) (4.6H) and \( \exists y_0 \) (4.5H); indeed, \( \forall y_0 x_0 = (\exists y_0) T^n \) which is \( F^n \).

We thus obtain Axiom 3 of [H].

(9) Lemma. Let \( x_3 \) be any variable of type \( \beta \), let \( A_4 \) be any formula in which \( x_3 \) does not occur, and let \( B_5 \) be the result of substituting \( x_3 \) for \( x_2 \) at all free occurrences of \( x_2 \) in \( A_4 \). Then \( \vdash (\exists x_2 A_4) = (\exists x_2 B_5) \).

Proof (essentially as in 7.21H).

(9.1) \( \vdash \forall x_2 A_4 = A_4 \) by Rule G (7.4H), (5).

(9.2) \( \vdash \forall x_2 (\exists x_2 A_4) x_2 = (\exists x_2 B_5) x_2 \) by Rule R, E-Rules (7.2H), Axiom 4, (9.1).

(9.3) \( \vdash \forall x_2 [ (\exists x_2 A_4) x_2 = (\exists x_2 B_5) x_2] = (\exists x_2 A_4) = (\exists x_2 B_5) \) by Rule Sub (7.6H), Axiom 3.(6).

(9.4) \( \vdash (\exists x_2 A_4) = (\exists x_2 B_5) \) by Rule R, (9.3), (9.2).

(10) Theorem. \( \vdash \forall x_2 (T^n \land F^n) = \forall x_2 (\exists y_0 x_n) \).

Proof. This is obtained from Axiom 1 by the use of lemma (9), Rule B, and the definition of \( V \) (4.6H).

This is Axiom Schema 4 of [H].

Remark. It is now easy to see that sections 7.9H-7.12H also apply to our system.

(11) Theorem Schema. \( \vdash (A_4 \land A_4) = A_4 \), where \( A_4 \) is any formula.

Proof.

(11.1) \( \vdash (\exists x_2 x_0) = (\exists x_2 T^n) \rightarrow (\exists x_2 x_0) (\exists x_2 x_0) \) by Rule Sub (7.6H), Axiom 2.(4).

(11.2) \( \vdash \forall x_2 (\exists x_2 T^n) \rightarrow F^n = F^n \) by Rule B, Axiom 4, (11.1)

(11.3) \( \vdash F^n \rightarrow F^n \) by Rule R, (11.2), (7.1), definition of \( F^n \) (4.2H).

(11.4) \( \vdash F^n \rightarrow F^n \) by Rule R, Axiom 4, (11.3), definition of \( \rightarrow (4.5H) \).

(11.5) \( \vdash (x_2 \land x_3) = x_2 \) by Rule of Cases (7.9H), (7.12H).

(11.6) \( \vdash (\exists x_2 A_4) = A_4 \) by Rule Sub (7.6H), (11.5).
(12) **Theorem.** \( \vdash (g = g) \rightarrow h_0a = h_0g, \) where \( a \) is any type symbol.

**Proof.**

(12.1) \( \vdash [(\lambda a_1)g, a_1 = (\lambda a_2, h_0a_1)\lambda a_2] = (\lambda a_2, h_0a_2) \lambda a_2 \)

by Rule Sub (7.6H), Axiom (2\textsuperscript{nd}).

(12.2) \( \vdash [\lambda a_1f_1(g, a_1) = (\lambda a_2, h_0a_2) \rightarrow h_0a_1 = h_0g] \)

by Rule R, Axiom (4), (12.1).

(12.3) \( \vdash [\lambda a_1f_1(g, a_1) = (\lambda a_2, h_0a_2) \rightarrow h_0a_1 = h_0g] \)

by Rule Sub (7.6H), Axiom (3\textsuperscript{rd}).

(12.4) \( \vdash [\lambda a_1f_1(g, a_1) = (\lambda a_2, h_0a_2) \rightarrow h_0a_1 = h_0g] \)

by Rule R, Axiom (4), (12.3).

(12.5) \( \vdash [(\lambda a_1f_1(g, a_1)) T \rightarrow (\lambda a_2, h_0a_2) F] \rightarrow [\lambda a_1f_1(g, a_1) F] \rightarrow \lambda a_2, h_0a_2 \]

by Rule Sub (7.6H), Axiom (1).

(12.6) \( \vdash [(\lambda a_1f_1(g, a_1)) \rightarrow h_0a_1 = h_0g] \)

by Rule R, Axiom (4), (12.5).

(12.7) \( \vdash [(\lambda a_1f_1(g, a_1)) \rightarrow h_0a_1 = h_0g] \)

by Rule E-Rules (7.2H), (12.4), (12.6), (11).

(12.8) \( \vdash (g = g) \rightarrow h_0a = h_0g \)

by Rule R, (12.7), (12.2).

**Remark.** Axiom (2\textsuperscript{nd}) seems to be necessary for the proof of theorem (11), which is used in the proof of (12.7) above. However, it is clear that any finite number of instances of our Axiom Schema (2) other than (2\textsuperscript{nd}) may be deleted from our list of axioms, and then derived by the method used in our proof of theorem (12). Indeed, certain finite sets of instances of Axiom Schema (2) might be deleted from the list of axioms; for example, it would suffice to take only those instances of Axiom Schema (2) of the forms (2\textsuperscript{nd}) or (2\textsuperscript{nd})* as axioms.

(13) **Theorem.** \( \vdash (a_y = g) \rightarrow (f_a \rightarrow (f_a \rightarrow g = g_a y_2)). \)

**Proof.**

(13.1) \( \vdash (a_y = g) \rightarrow (f_a \rightarrow (f_a \rightarrow g = g_a y_2)) \)

by Rule Sub (7.6H), (12).

(13.2) \( \vdash (a_y = g) \rightarrow (f_a \rightarrow (f_a \rightarrow (f_a \rightarrow g = g_a y_2))) \)

by Rule R, Axiom (4), (13.1).

(13.3) \( \vdash (a_y = g) \rightarrow (f_a \rightarrow (f_a \rightarrow g = g_a y_2)) \)

by Rule Sub (7.6H), (12), Rule R, Axiom (4).
Use Axiom (1) to prove \( \vdash (T^a = T^a) \land (T^a = T^a) \) \( \forall x (x = y) \), and hence \( \vdash (T^a = T^a) \) \( \forall x (x = y) \). Then substitute \( (\lambda x y_x) \) for \( f_{1a} \) and \( (\lambda x y_x) \) for \( f_{1a} \) in Axiom (3) to obtain \( \vdash \forall x (x = y) \) \( \forall x (x = y) \).

Henkin remarks at the end of [H] that when one passes from the theory of propositional types to the full theory of finite types, it becomes necessary to add a constant \( t \) to denote a descriptor function, and an appropriate axiom involving this constant. We note that for this axiom it suffices to take the simple formula

\[
t_{\text{ideal}}(\lambda x x_x = y) = y,
\]

from which the formula

\[
(\exists x_1) (f_{1a} x_1) \rightarrow f_{1a} (t_{\text{ideal}} x_1)
\]

can be derived without difficulty.

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О диадических пространствах

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1. Эта заметка примыкает к работе [1].

Определение. Вполне регулярное пространство $X$ называется (неприводимым) диадическим, если у него существует расширение $X$, являющееся (неприводимым) диадическим бикомпактом [2].

Заметим прежде всего, что всякое диадическое пространство $X$ удовлетворяет отрицательной аксиоме счетности (т.е. в $X$ не существует недийдиальной системы открытых множеств). Действительно, если $X$ есть расширение пространства $Y$, являющееся диадическим бикомпактом, а $(U_{a})$ диадическая несчетная система открытых множеств пространства $X$, то рассмотревая систему $(OU_{a})$ открытых в $X$ множеств, включающую из $X$ двумерную систему $(U_{a})$, получим такую несчетную диадическую систему открытых в $X$ множеств, что в диадическом бикомпакте быть не может (Теорема Э. Марчевского [21]).

Теорема 1. Диадическое паракомпактное пространство финально-компактно [2].

Доказательство. Достаточно доказать, что всякое покрытие у нормального пространства $X$, удовлетворяющего отрицательной аксиоме счетности, содержит счетное покрытие того же пространства.

Предполагаем, что элементы покрытия у заключены в порядке числами, т.е. что $y = (I_{a})$, где

$$
\alpha = 1, 2, 3, ..., \omega_1.
$$

Так как $y$ — покрытие паракомпактного пространства $X$, то существует такое