Continua meeting an orbit at a point

by

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Anderson and Hunter have proved the following result [1]:

THEOREM (Anderson and Hunter). *Let X be a connected locally compact space, G a compact group acting on X and $p \in X$. Suppose further that*

(i) *the isotropy group G_p at p is trivial;*

(ii) *X is 1-semi-locally connected at $G(p) \neq X$;*

(iii) *G is separable metric.*

Then there is a non-degenerate continuum $M \subset X$ such that

$$M \cap G(p) = \{p\}.$$

It is the purpose of this note to prove this theorem without hypotheses (i), (ii), and (iii). The proof here is somewhat different from that of Anderson and Hunter, and at the same time also shorter and simpler (1).

A special case of this theorem was proved by the author and A. L. Hudson in showing that a compact connected homogeneous semigroup of finite dimension and with identity is a group [3]. The proof there used much more powerful machinery, and so this represents also a more elementary proof of that fact.

The corollary at the end of the paper illustrates the type of applications of the theorem possible in topological semigroups.

LEMMA 1. *Let X be a compact space, M a continuum, and ν: $X \to M$ an open mapping of X onto M. Then there is a continuum $M' \subset X$ such that $\nu(M') = M$.*

This result is well known, and the proof is a standard tower argument. Hence we omit the proof.

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(1) Anderson and Hunter have informed the author that they have also been able to remove condition (iii) and perhaps also condition (i) but not (ii). The author has not been informed of their methods, however.

LEMMA 2. *Let X be a compact space p a point of X and G a compact transformation group acting on X. Let $K_1 \subset K \subset G$ be compact normal subgroups of G such that K/K_1 is a Lie group. Denote*

$$\pi\colon X \to X/K,$$
$$\pi_1\colon X \to X/K_1,$$
$$\nu\colon X/K_1 \to X/K$$

the natural projections. Let M be a compact set in X, and $q \in X$ such that

(i) $K(M) = M$,

(ii) $K(M) \cap G(p) \subset K(p)$, $p \in M$,

(iii) $q \notin G(p)$ *and* $G(q) \cap M \neq \emptyset$,

(iv) M/K *is connected.*

Then there is a compact set $M_1 \subset M$ such that (i)-(iv) *are satisfied with M replaced by M_1 and K replaced by K_1.*

Proof [2]. Now K/K_1 acts on M/K_1 as a compact Lie group of transformations, and hence there is a neighborhood U of $\pi(p)$ in M/K and a slice T through $\pi_1(p)$ such that $\nu(T) = U^-$ [2]. Let W be a neighborhood of $\pi(p)$ in M/K such that $W^- \subset U$. Then, if $M_0 = \nu^{-1}(M/K - W^-) \cup T$, $\nu|M_0$ is an open mapping of M_0 onto M/K. By Lemma 1, there is a continuum $M' \subset M_0$ such that $\nu(M') = M$. Let $M_1 = \pi_1^{-1}(M')$. Then M_1 is clearly the desired set.

THEOREM [3]. *Let X be a locally compact connected space and G a compact transformation group acting on X. Then at any $p \in X$, there exists a non-degenerate continuum M such that $M \cap G(p) = p$, or $G(p) = X$.*

Proof. Let V be any open set with non-empty boundary and compact closure about p and such that $G(V) = V$. If $G(p) \neq X$, such a set exists. Let P be a component of V^- containing p, and $q \in (V^- - V) \cap P$. Since X is connected, such a q exists. Let B be the collection of all pairs (M, K) where M is a compact set, K is a compact normal subgroup of G, and K and M satisfy (i)-(iv) of Lemma 2. Then $\big(G(P), G\big) \in B$ so that B is non-empty. We order these pairs by containment: i.e., $(M_1, K_1) < (M, K)$ if $M_1 \subset M$ and $K_1 \subset K$. Clearly B is inductive, and hence there is a least element (M, K). If $K \neq 1$, there is a normal subgroup K_1 of G such that $K_1 \subset K$ and K/K_1 is a non-degenerate Lie group. By Lemma 2, we can find M_1 such that $(M_1, K_1) < (M, K)$. Thus, the theorem is proved.

The following generalizes the remaining two results of [1].

[2] We use the following notation: A^- denotes the closure of the set A; $A - B$ denotes the complement of B in A; $\nu|M_0$ denotes the function ν restricted to M_0.

[3] One may notice from the proof that the assumption that X is connected may be replaced by the assumption that X/G is connected.

COROLLARY. *Let S be a locally compact connected semigroup, $p \in S$, and H any compact subgroup of S. If H is not the minimal ideal of S, then there is a non-degenerate continuum A such that $A \cap Hp = ep$ or $A \cap pH = pe$, where $e^2 = e \in H$.*

Proof. Now e is the identity of H, then H acts on eS as a transformation group. Hence, if H is not the minimal ideal, in which case $eS \supset Hp$, the theorem applies.

References

[1] L. W. Anderson and R. P. Hunter, *Small continua at certain orbits*, to appear.

[2] A. Borel, *Seminar on transformation groups*, Annals of Mathematics Studies, study 46, Princeton Univ. Press, Princeton, N. J.

[3] Anne Lester Hudson and P. S. Mostert, *Finite dimensional homogeneous clans are groups*, to appear in Annals of Math.

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