droite sont homéomorphes ([1], théorème 4, p. 359), l'homéomorphie pouvant être obtenue au moyen d'une fonction croissante.

La nécessité de la condition résulte de ce que tout ensemble de mesure complète vérifie cette condition et l'homéomorphie ne modifie ni le type, ni la puissance de l'ensemble. Nous omettons la démonstration détaillée, car elle ne présente pas de grandes difficultés (la démonstration omise est analogue à celle qui se trouve dans [1], au N° 276, p. 286 et N° 337, p. 357-359).

Travaux cités

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Reçu par la Rédaction le 7. 11. 1960

On the decomposition of 2-dimensional ANR-s into a Cartesian product

by

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1. The object of the note Notations. A space X is said to be *topologically first* if it contains at least two points and is homeomorphic with no Cartesian product of factors containing at least two points each.

It is known (see [1], [2], [5], [7], [8]) that the decomposition of a space into topologically first spaces is in general not unique. Nevertheless, in certain special cases this uniqueness holds. For instance, the decomposition of an arbitrary polytope into topologically first spaces of dimension $\leqslant 1$ is (disregarding the permutations of the factors and their homeomorphisms) unique (see [4]). However, the problem posed by K. Borsuk (see [3], p. 140), whether the decomposition of an arbitrary space X into prime factors of dimension $\leqslant 1$ is unique, remains still unsolved.

The purpose of this paper is to show that this problem has a positive solution provided that the space X is an ANR (i.e. a compact absolute neighbourhood retract) of dimension $\leqslant 2$. Namely, we prove the following

THEOREM. *If X is an* ANR *of dimension $\leqslant 2$ and if X is homeomorphic with the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ of some topologically first spaces, then the system $X_1, X_2, \ldots, X_n$ is (disregarding its permutations and homeomorphisms) uniquely determined by X.*

It is clear that the factors $X_1, X_2, \ldots, X_n$ must be ANR-sets of dimension $\leqslant 2$ and that $\dim X = \dim X_1 + \dim X_2 + \ldots + \dim X_n$.

The following notation will be used in the sequel:

$\operatorname{ord}_{x_0} X$ will denote the order of the point x_0 in the space X in the sense of Menger-Urysohn.

$X^{[n]}$ will denote the set of the points of X of order $\geqslant n$.

If Q is a disk (i.e. a topological image of the circle $|z| \leqslant 1$) and L an arc (i.e. a topological image of the interval $0 \leqslant x \leqslant 1$), then Q° and L° will denote the interiors of these sets and $Q^\cdot$ and $L^\cdot$ their boundaries.

If $A \subset X \times Y$, then A_X (respectively A_Y) will denote the projection of A into X (respectively into Y), i.e. the set of all points $x \in X$ (resp.

$y \in Y$), so that there exists a point $y \in Y$ (resp. $x \in X$) such that $(x, y) \in A$.

The set $(x_0) \times Y_0 \subset X \times Y$, where $x_0 \in X$, $Y_0 \subset Y$, will be denoted by (x_0, Y_0).

2. Elementary lemmas.

LEMMA 1. *Let Q be a disk, $D \subset Q$ a compactum such that every component of it is a dendrite and that D contains no continuum of convergence (containing more than one point), and z_0 a point of D.*

(i) *If $z_0 \in Q^\circ$ and D cuts every neighbourhood of z_0 in Q, then there exists an arc $L \subset D$ such that $z_0 \in L^\circ \subset Q^\circ$, $L^\cdot \subset Q^\cdot$.*

(ii) *If $z_0 \in Q^\cdot$ and there exist two arcs $L_1, L_2 \subset Q$ such that $z_0 = L_1 \cap L_2 = L_1^\cdot \cap L_2^\cdot$ and that D cuts Q between $L_1 - z_0$ and $L_2 - z_0$, then there exists an arc $L \subset D$ such that $z_0 \in L^\cdot \subset Q^\cdot$, $L^\circ \subset Q^\circ$.*

Proof of (i). Suppose that $z_0 \in Q^\circ$ and that D cuts every neighbourhood of z_0 in Q. Considering the fact that D contains no continuum of convergence (containing more than one point), one can easily show that the family Λ of all arcs contained in D with the end-points lying on $Q^\cdot$ is at most countable. Moreover, it is clear that, if it is countable, then the diameters of the arcs belonging to it converge to zero. This implies that the set C, being the union of $Q^\cdot$ and all arcs of the family Λ, is a locally connected continuum.

Suppose that (i) does not hold. Then $z_0 \notin C$. Let Q_0 denote the component of $Q - C$ containing z_0. Since the continuum C is disconnected by no point, $\bar{Q}_0$ is a disk (see [6], p. 360, Theorem 4). Each component of $\bar{Q}_0 \cap D$ has at most one point in common with the boundary of $\bar{Q}_0$. On the contrary, the set D would contain an arc $L \subset Q_0$ whose end-points lie on $\mathrm{Fr}(Q_0) \subset C$. Then, by the definition of C, this arc would be extended to an arc with the end-points lying on $Q^\cdot$, contradicting the assumption that $Q_0 \subset Q - C$.

It follows that no component of $\bar{Q}_0 \cap D$ cuts $\bar{Q}_0$ and consequently the disk $\bar{Q}_0$ is a neighbourhood of z_0 which is not disconnected by D, contrary to our assumption.

Proof of (ii). Let $z_1 \in L_1 - z_0$, $z_2 \in L_2 - z_0$. By hypothesis there is a component D_0 of D which cuts Q between z_1 and z_2. Then $z_0 \in D_0 \cap Q^\cdot$. For otherwise $L_1 \cup L_2 \subset Q - D_0$ would be a continuum joining the points z_1 and z_2. Since D_0 is a dendrite cutting the disk Q, there exists another point $z_3 \in D_0 \cap Q^\cdot$. Hence, we may take for L the arc joining the points z_0 and z_3 in the dendrite $D_0 \subset D$.

In the sequel, we shall use the notion of local dendrite. This is a compactum each point of which has a neighbourhood which is a dendrite. The class of local dendrites coincides with the class of 1-dimensional ANR-s.

For given spaces X and Y let us put:

(1) $$(X, Y)_1 = [(X^{[1]} - X^{[2]}) \times Y] \cup [X \times (Y^{[1]} - Y^{[2]})],$$

(2) $$(X, Y)_2 = (X^{[2]} - X^{[3]}) \times (Y^{[2]} - Y^{[3]}),$$

(3) $$(X, Y)_3 = (X^{[3]} \times Y^{[2]}) \cup (X^{[2]} \times Y^{[3]}).$$

LEMMA 2. *Let Z be a Cartesian product of two local dendrites and let $z_0 \in Z$. Then:*

(i) *The point $z_0 = (x_0, y_0)$ belongs to the set $(X, Y)_1$ if and only if there exists no disk $Q \subset Z$ such that $z_0 \in Q^\circ$.*

(ii) *The point $z_0 = (x_0, y_0)$ belongs to the set $(X, Y)_3$ if and only if there exist three disks $Q_1, Q_2, Q_3 \subset Z$ disjoint except an arc $L \subset Q_1^\cdot \cap Q_2^\cdot \cap Q_3^\cdot$ containing z_0 in its interior.*

(iii) *The point $z_0 = (x_0, y_0)$ belongs to the set $(X, Y)_2$ if and only if it has neither of the properties mentioned in* (i) *and* (ii).

Proof of (i). First, let $z_0 = (x_0, y_0) \in (X, Y)_1$ and let $\mathrm{ord}_{x_0} X = 1$, $\mathrm{ord}_{y_0} Y \geqslant 1$. Suppose, contrary to (i), that $Q \subset Z$ is a disk such that $z_0 \in Q^\circ$. Then $\varrho(z_0, Q^\cdot) = \varepsilon > 0$. By our assumptions, there exist two dendrites $D' \subset X$ and $D'' \subset Y$ such that $x_0 \in \mathrm{Int}(D')$, $y_0 \in \mathrm{Int}(D'')$, $\mathrm{Fr}(D') = x_1$, $\mathrm{Fr}(D'') = \bigcup_{i=1}^{n} y_i$, $\delta(D' \times D'') < \varepsilon$. Let

$$D = \mathrm{Fr}(D' \times D'') = (x_1, D'') \cup \bigcup_{i=1}^{n} (D', y_i).$$

This formula implies that D is a dendrite. Observing the properties of the dendrites D' and D'' we see that $z_0 \in \mathrm{Int}(D' \times D'')$, $Q^\cdot \subset Z - (D' \times D'')$. Therefore, the dendrite $D = \mathrm{Fr}(D' \times D'')$ cuts the space Z between z_0 and $Q^\cdot$. It follows that the set $D \cap Q \subset Q^\circ$ cuts the disk Q between the point $z_0 \in Q^\circ$ and the boundary $Q^\cdot$. But it is false, because $D \cap Q \subset Q^\circ$ is a compactum whose components are dendrites.

Next, let $z_0 = (x_0, y_0) \notin (X, Y)_1$. Then $\mathrm{ord}_{x_0} X > 1$ and $\mathrm{ord}_{y_0} Y > 1$; therefore there exist two arcs $L' \subset X$ and $L'' \subset Y$ such that $x_0 \in L'^\circ$, $y_0 \in L''^\circ$. Now, putting $Q = L' \times L''$, we obtain the disk Q with the required property.

Proof of (ii) runs similarly to the preceding one. In fact, if $z_0 = (x_0, y_0) \in (X, Y)_3$, then either $\mathrm{ord}_{x_0} X \geqslant 3$ and $\mathrm{ord}_{y_0} Y \geqslant 2$ or $\mathrm{ord}_{x_0} X \geqslant 2$ and $\mathrm{ord}_{y_0} Y \geqslant 3$. In either case the space Z contains three disks with the required property. Conversely, if $z_0 = (x_0, y_0) \notin (X, Y)_3$ and there exist three disks $Q_1, Q_2, Q_3 \subset Z$ whose common part is an arc $L \subset Q_1^\cdot \cap Q_2^\cdot \cap Q_3^\cdot$ containing z_0 in its interior, then, in view of (i), $z_0 \notin (X, Y)_1$. It follows that $\mathrm{ord}_{x_0} X = 2$ and $\mathrm{ord}_{y_0} Y = 2$. Therefore, there exist two dendrites $D' \subset X$, $D'' \subset Y$ such that $x_0 \in \mathrm{Int}(D')$, $y_0 \in \mathrm{Int}(D'')$, $\mathrm{Fr}(D')$

$= x_1 \cup x_2$, $\mathrm{Fr}(D'') = y_1 \cup y_2$, $\delta(D'\times D'') < \varrho[z_0, (Q_1^\bullet \cup Q_2^\bullet \cup Q_3^\bullet) - L^\circ]$. Then the set $\mathrm{Fr}(D'\times D'') \cap (Q_1 \cup Q_2 \cup Q_3)$ contains at most one simple closed curve and cuts $Q_1 \cup Q_2 \cup Q_3$ between the point z_0 and the set $(Q_1^\bullet \cup Q_2^\bullet \cup Q_3^\bullet) - L^\circ$, which is impossible.

The truth of (iii) is an immediate consequence of the formula $(X, Y)_2 = X \times Y - (X, Y)_1 - (X, Y)_3$.

This lemma characterises the subsets $(X, Y)_\nu$, where $\nu = 1, 2, 3$, of the space Z, which is a 2-dimensional ANR in an invariant manner, i.e. independently of the fixed decomposition of Z into the Cartesian product $X \times Y$. Hence, in the sequel we shall use the following notation:

(4) $$(X, Y)_\nu = Z_\nu \quad \text{for} \quad \nu = 1, 2, 3\,.$$

3. Some families of sets R and S. In Sections 3-6 we shall consider a fixed connected 2-dimensional ANR of the form $Z = X \times Y$, where X and Y are curves. It is clear that X and Y must be connected 1-dimensional ANR-sets.

In this Section we define two families R, S of subsets of the space Z. In Sections 4-6 we shall give the proof of identity of these families (see (7)). Our Theorem, whose proof we shall give in Section 8, is an easy consequence of this fact.

Definition of the family R. The set $M \subset Z$ belongs to the family R if and only if it has either the form (x_0, Y) or the form (X, y_0), where $x_0 \in X^{[3]}$, $y_0 \in Y^{[3]}$.

For each closed subset N of the space Z, let us denote by $R\|N$ the family of all sets of the form $M \cap N$, where $M \in R$. Let us notice that:

(1R) *Each two sets $M_1, M_2 \in R$ having at least two different common points are identical.*

(2R) *If $N = \bar{N} \subset Z_3$, then the family $R\|N$ constitutes a countable covering of N by closed sets.*

(1R) requires no proof. (2R) follows from the formula $Z_3 = (X, Y)_3 \subset (X^{[3]} \times Y) \cup (X \times Y^{[3]})$ (see (3) and (4)) with regard to the countability of $X^{[3]}$ and $Y^{[3]}$, which are the sets of the points of ramification of the local dendrites X and Y.

The above definition of the family R depends explicitly on the decomposition of the space Z into the Cartesian product $X \times Y$. Conversely, the definition of the family S has an invariant character, i.e. depends only on some topological properties of the space Z. However, we shall show that these families coincide.

Definition of the family S. The set A belongs to the family S is and only if:

(1A) *A is a connected local dendrite.*

(2A) $A^{[1]} - A^{[2]} = A \cap Z_1$.

(3A) $A^{[2]} = A \cap Z_3$.

(4A) *There is a continuum $F \subset Z$ such that the following conditions are satisfied:*

(i) $A \subset \mathrm{Int}(F)$.

(ii) *A is an irreductible cut of F, i.e. A is the common boundary in F for every component of $F - A$.*

(iii) *If z_0 is an arbitrary point of A and G an arbitrary component of $F - A$, then there exists no disk $Q \subset \bar{G}$ such that $z_0 \in Q^\circ$.*

(iv) *If L is an arbitrary arc contained in A and G an arbitrary component of $F - A$, then there exists a disk $Q \subset \bar{G}$ such that $L = Q \cap A \subset Q^\bullet$.*

(v) *If z_0 is an arbitrary point of A, G is an arbitrary component of $F - A$, $L_1, L_2 \subset A$ are arcs such that $z_0 \in L_1^\circ \cap L_2^\circ$ and $Q_1, Q_2 \subset \bar{G}$ are disks such that $L_1 \subset Q_1$, $L_2 \subset Q_2$, then there is an arc $L_0 \subset Q_1 \cap Q_2$ such that $z_0 \in L_0^\bullet$, $L_0 - z_0 \subset Q_1^\circ \cap Q_2^\circ$.*

The following property is a consequence of the above properties of A:

(5A) *If $Q \subset Z$ is a disk, then for each point $z \in Q^\circ \cap A$ there exists an arc $L \subset Q \cap A$ such that $z \in L^\circ \subset Q^\circ$. Moreover, if the disk Q is sufficiently small, then it is possible to find an arc L satisfying the additional condition $L^\bullet \subset Q^\bullet$.*

In fact, let $Q \subset Z$ be a given disk and $z \in Q^\circ \cap A$ a given point. Let F be a continuum satisfying conditions (i)-(v) mentioned in (4A). Replacing, if necessary, the disk Q by a smaller one (containing z in its interior), we can assume that $Q \subset F$ and that the set $Q \cap A$ contains no simple closed curve. Then, (4A) (iii) implies that the set $Q \cap A$ cuts every neighbourhood of the point z in Q. Hence, the assumptions of Lemma 1 (i) are satisfied by the disk Q, by the compactum $D = Q \cap A$ and by the point z. Hence, there is an arc $L \subset D = Q \cap A$ with the required property.

4. The inclusion $R \subset S$. Let M be a given set of the family R. Without loss of generality we can assume that $M = (x_0, Y)$, where $x_0 \in X^{[3]}$. To prove that $M \in S$ it is necessary to show (1M)-(4M).

The properties (1M)-(3M) follow at once from the form of the set M and from the definitions of $Z_1 = (X, Y)_1$ and $Z_3 = (X, Y)_3$ (see (1) and (3)). To prove (4M) let us put $F = D' \times Y$, where D' is a dendrite constituting a neighbourhood of the point x_0 (with respect to X). Of course, conditions (i) and (ii) are then satisfied.

Let $z_0 = (x_0, y_0)$ be a given point of $M = (x_0, Y)$ and G a given component of $F - M = (D' - x_0) \times Y$. Then $G = G' \times Y$, where G' is a component of $D'-x_0$. Therefore G' is a dendrite such that $x_0 \in \overline{G'}^{[1]} - \overline{G'}^{[2]}$. This means that $z_0 = (x_0, y_0) \in (\overline{G'}, Y)_1$, so that in virtue of Lemma 2 (i) condition (iii) is satisfied.

Now, let $G = G' \times Y$ be again a component of $F-M$ and $L \subset M = (x_0, Y)$ a given arc. Then $L = (x_0, L'')$, where $L'' = L_Y \subset Y$ is an arc. Let us denote by L' any arc contained in $\overline{G'}$ such that $x_0 \in L'^{\bullet}$. Now, putting $Q = L' \times L''$, we obtain a disk Q with the property required by (iv).

Passing to (v), let us consider a point $z_0 = (x_0, y_0) \in M$, a component $G = G' \times Y$ of $F-M$, two arcs $L_1 = (x_0, L_1'')$, $L_2 = (x_0, L_2'') \subset M$ and two disks $Q_1, Q_2 \subset \overline{G}$ such that $z_0 \in L_1^\circ \cap L_2^\circ$, $L_1 \subset Q_1$, $L_2 \subset Q_2$. We can assume that the sets $L_1, L_2, Q_1, Q_2 \subset \overline{G} = \overline{G'} \times Y$ lie in a neighbourhood of z_0 (with respect to $\overline{G}$) constituted by the Cartesian product $\overline{G'} \times G''$, where G'' is a dendrite constituting a neighbourhood of y_0 (with respect to Y).

Since $z_0 \in M \cap Q_1$ and $Q_1 \subset \overline{G}$, in virtue of (iii) we have $z_0 \in Q_1^{\bullet}$. Because $z_0 \in L_1^\circ$ and $L_1 \subset \overline{G'} \times G''$, we see that $y_0 \in L_1''^\circ$ and $L_1'' \subset G''$. Therefore the point y_0 cuts the dendrite G'' between the two components of $L_1''-y_0$. Consequently, the dendrite $(\overline{G'}, y_0)$ cuts $\overline{G'} \times G''$ between the two components of $L_1 - z_0 = (x_0, L_1''-y_0)$. By this, since $L_1 \subset Q_1 \subset \overline{G'} \times G''$, the compactum $D_1 = (\overline{G'}, y_0) \cap Q_1$ cuts the disk Q_1 between the two components of $L_1 - z_0$. Hence, we may apply Lemma 1 (ii) to the disk Q_1, the compactum D_1 and the point z_0. Then we find an arc $L_{1,0} \subset D_1 = (\overline{G'}, y_0) \cap Q_1$ such that $z_0 \in L_{1,0}^{\bullet}$, $L_{1,0} - z_0 \subset Q_1^\circ$.

By an analogous proceeding for the disk Q_2 and the arc L_2, we obtain an arc $L_{2,0} \subset (\overline{G'}, y_0) \cap Q_2$ such that $z_0 \in L_{2,0}^{\bullet}$, $L_{2,0} - z_0 \subset Q_2^\circ$. Then we have $L_{1,0} = (L_{1,0}', y_0)$, $L_{2,0} = (L_{2,0}', y_0)$, $L_{1,0}', L_{2,0}' \subset \overline{G'}$, $x_0 \in L_{1,0}'^{\bullet} \cap L_{2,0}'^{\bullet}$. Consequently (since $\overline{G'}$ is a dendrite such that $\mathrm{ord}_{x_0} \overline{G'} = 1$) there exists an arc $L_0' \subset L_{1,0}' \cap L_{2,0}'$ such that $x_0 \in L'^{\bullet}$. Now, putting $L_0 = (L_0', y_0)$, we obtain an arc L_0 with the properties required by (v). Thus, the proof of the inclusion

$$R \subset S \tag{5}$$

is complete.

5. Some properties of the sets of the family S. Throughout this Section A will be a fixed set of the family S. We shall give here the definition of turning and Lemmas 3 and 4 related to this notion. Corollary 1, arising from these Lemmas, will be used in Section 6 to prove the inclusion $S \subset R$.

DEFINITION OF TURNING. The point $z_0 = (x_0, y_0)$ belonging to an arc $I \subset A$ will be called a *turning of this arc* if there exist two arcs $J, K \subset I$ such that $z_0 = J \cap K = J^{\bullet} \cap K^{\bullet}$, $J_Y = (y_0)$ and $K_X = (x_0)$.

LEMMA 3. *Each arc $I \subset A$ such that neither of the projections I_X, I_Y reduces to a single point has a turning.*

Proof. Let $I \subset A$ be a given arc with the above-mentioned property. Since $I \cap (A - A^{[2]}) \subset I^{\bullet}$, we can assume, reducing this arc if necessary, that $I \subset A^{[2]}$.

Since, in virtue of (3A), $A^{[2]} \subset Z_3$, we see in view of (2R), that the family $R \| I$ covers the arc I. Let K denote the subset of the arc I consisting of those points which lie in the interior (with respect to I) of any set of the family $R\|I$. By (1R), the interiors of two different sets of the family $R\|I$ are disjoint. Hence, each component of K is a component of the interior of exactly one set of the family $R\|I$. By hypothesis the arc I is not contained in any set of the family R. Therefore $I - K \neq 0$.

Consider the set $N = I - K$. Since $N = \overline{N}$ and $N \subset Z_3$, in virtue of (2R) we infer that the family $R\|N$ constitutes a countable covering of N by closed sets. Hence, by Baire's Theorem, there exist a point $z_0 = (x_0, y_0) \in N - I^{\bullet}$, an arc $I_0 \subset I$ and a set $M_0 \in R$ such that $z_0 \in I_0^\circ$, $I_0 \cap N \subset M_0$. Let, for instance, $M_0 = (x_0, Y)$. We are going to show that the point z_0 is a turning of the arc I.

Suppose that $K_0 \subset I_0^\circ$ is a component of K. Then there exists a set $M \in R$ such that $\overline{K}_0 \subset M$. Since $\overline{K}_0 - K_0 \subset I_0 \cap N \subset M_0$, applying (1$R$) we obtain $M = M_0$. Hence $K_0 \subset M_0$. It follows that every point of the arc $I_0 = (I_0 \cap N) \cup (I_0 \cap K)$, except possibly those which belong to a component of K containing a point of $I_0^{\bullet}$, lies in M_0. Hence, the arc I_0 is covered by at most three sets of the family R. Since the point $z_0 \in I_0^\circ \cap N \cap M_0$ does not belong to the interior of $M_0 = (x_0, Y)$, there is another set $M_1 \in R$ such that $z_0 = (x_0, y_0) \in M_1$. Therefore $M_1 = (X, y_0)$ and the point z_0 is a turning of the arc I_0, as well of the arc $I \supset I_0$.

LEMMA 4. *No arc $I \subset A$ may have a turning.*

Proof. Suppose on the contrary that a point $z_0 = (x_0, y_0) \in A$ is a turning of some arc contained in A. Hence, there exist two arcs $J_1 = (J_1', y_0)$, $K_1 = (x_0, K_1'')$ such that $z_0 = J_1 \cap K_1 = J_1^{\bullet} \cap K_1^{\bullet}$. Let F be a fixed continuum satisfying conditions (i)-(v) mentioned in (4A). We can assume, reducing these arcs if necessary, that $J_1, K_1 \subset D' \times D'' \subset F$, where $D' \subset X$, $D'' \subset Y$ are some dendrites such that $x_0 \in \mathrm{Int}(D')$, $y_0 \in \mathrm{Int}(D'')$. Moreover, we can assume that the set $(D' \times D'') \cap A$ contains no simple closed curve.

Since, in virtue of (3A), $z_0 \in A^{[2]} \subset Z_3 = (X, Y)_3$, we can assume that $\mathrm{ord}_{x_0} D' \geqslant 2$ and $\mathrm{ord}_{y_0} D'' \geqslant 3$. Hence, there are an arc $J_2' \subset D'$ such that $J_1' \cap J_2' = x_0 = J_1'^{\bullet} \cap J_2'^{\bullet}$ and two arcs K_2'', $K_3'' \subset D''$ such that $K_1'' \cap K_2'' = K_1'' \cap K_3'' = K_2'' \cap K_3'' = y_0 = K_1''^{\bullet} \cap K_2''^{\bullet} \cap K_3''^{\bullet}$. Let

$$Q_1 = J_1' \times K_1'', \quad Q_2 = (J_1' \cup J_2') \times (K_2'' \cap K_3'').$$

We see that $Q_1, Q_2 \subset F$ are disks such that $Q_1 \cap Q_2 = (J_1', y_0) = J_1 \subset A$, $J_1 \cup K_1 \subset Q_1^{\bullet} \cap A$, $z_0 \in Q_1^{\bullet} \cap Q_2^\circ \cap A$.

Let S_1' denote the closure of the component of $D'-x_0$ containing $J_1'-x_0$ and T_1'' the closure of the component of $D''-y_0$ containing $K_1''-y_0$. Let $S_1=(S_1', x_0)$, $T_1=(y_0, T_1'')$. Then, the dendrite $S_1\cup T_1$ cuts $D'\times D''$ between the interior and the complement (with respect to $D'\times D''$) of $S_1'\times T_1''$. Since $Q_1^\circ=J_1'^\circ\times K_1''^\circ\subset(S_1'-x_0)\times(T_1''-y_0)=\operatorname{Int}(S_1'\times T_1'')$ and $Q_2\cap(S_1'\times T_1'')=[(J_1'\cup J_2')\cap S_1']\times[(K_2''\cup K_3'')\cap T_1'']=(J_1', y_0)=Q_2\cap Q_1$, which implies that $Q_2-Q_1\subset(D'\times D'')-(S_1'\times T_1'')$, the dendrite $S_1\cup T_1$ cuts $D'\times D''$ also between Q_1° and Q_2-Q_1.

In order to obtain a contradiction we shall construct the disk Q and the arc L such that $L\subset Q\subset D'\times D''$, $z_0\in L^\circ\cap Q^\bullet$, that the components of $L-z_0$ will lie in Q_1° and Q_2-Q_1 respectively and that $Q\cap A\subset Q^\bullet$.

Then, the assumptions of Lemma 1 (ii) will be satisfied by the disk Q, the compactum $D=(S_1\cup T_1)\cap Q$ and the point z_0, because the set D will cut the disk $Q\subset D'\times D''$ between the two components of $L-z_0$. Therefore, we shall find an arc $L_1\subset Q\cap(S_1\cup T_1)$ such that $z_0\in L_1^\bullet$, $L_1^\circ\subset Q^\circ$. This arc must be contained in the closure of one component of $(S_1\cup T_1)-z_0$: for instance let $L_1\subset T_1$. Then, by the definition of T_1, we see that L_1, $K_1\subset T_1$, $z_0\in L_1^\bullet\cap K_1^\bullet$ and $\operatorname{ord}_{z_0}T_1=1$, which implies that $L_1^\circ\cap K_1^\circ\neq 0$. Since $L_1^\circ\subset Q^\circ$, $K_1\subset A$, it follows that $Q^\circ\cap A\neq 0$. However, the disk Q, which we want to construct, will satisfy the condition $Q\cap A\subset Q^\bullet$, which is a contradiction.

Now we pass to the construction of the disk Q and the arc L. Let us consider the disk Q_1 and let us recall that $z_0\in J_1\cup K_1\subset Q_1^\bullet\cap A$. According to (5$A$), each point $z\in Q_1^\circ\cap A$ lies on an arc of A joining two points of $Q_1^\bullet$. Hence, considering a component of $Q_1-A-Q_1^\bullet$ whose closure contains z_0, we find a disk $Q_{1,0}\subset Q_1$ and an arc $L_{1,0}\subset Q_{1,0}^\bullet$ such that $L_{1,0}=Q_{1,0}\cap A$, $z_0\in L_{1,0}^\circ$. Then $Q_{1,0}-L_{1,0}\subset F-A$, because $Q_{1,0}\subset Q_1\subset F$. Let G_0 denote the component of $F-A$ containing $Q_{1,0}-L_{1,0}$. Then we have $Q_{1,0}\subset\overline{G}_0$.

Since $z_0\in Q_2^\circ\cap A$, in virtue of (5A) there exists an arc $L_{2,0}\subset Q_2\cap A$ such that $z_0\in L_{2,0}^\circ\subset Q_2^\circ$. Further, by (4$A$) (iv), there is a disk $Q\subset\overline{G}_0$ such that $L_{2,0}=Q\cap A\subset Q^\bullet$. Since the set $D'\times D''$ is a neighbourhood of z_0, we can assume that $Q\subset D'\times D''$. Thus, we have obtained the desired disk Q.

Now, applying (4A) (v) to the point $z_0\in A$, the component G_0 of $F-A$, the arcs $L_{1,0}, L_{2,0}\subset A$ and the disks $Q_{1,0}$ and Q, we find an arc $L_0\subset Q_{1,0}\cap Q$ such that $z_0\in L_0^\bullet$, $L_0-z_0\subset Q_{1,0}^\circ\cap Q^\circ\subset Q_1^\circ\cap Q^\circ$. Finally, denoting by L the union of L_0 and that component of $L_{2,0}-z_0$ which lies in Q_2-Q_1, we obtain the desired arc L. Thus the proof is complete.

Lemmas 3 and 4 immediately imply the following

COROLLARY 1. *For each arc $I\subset A$ one of the projections I_X, I_Y reduces to a single point.*

6. The inclusion $S\subset R$. Let A be any set of the family S. It will be proved that A has either the form (x_0, Y) or the form (X, y_0), where $x_0\in X^{[3]}$, $y_0\in Y^{[3]}$. Let $I_0\subset A$ be a fixed arc. By Corollary 1, one of the projections I_{0_X}, I_{0_Y} reduces to a single point. For instance, let $I_{0_X}=(x_0)$. Let us choose a point $z_0=(x_0, y_0)\in I_0^\circ$ such that $\operatorname{ord}_{y_0}Y=2$. Then we have $\operatorname{ord}_{x_0}X\geqslant 3$, because $z_0\in A^{[2]}\subset Z_3=(X, Y)_3$ in virtue of (3A). We will prove that $A=(x_0, Y)$.

Let $z=(x, y)\in A$. Choose two distinct points $z_1=(x_0, y_1)$, $z_2=(x_0, y_2)\in I_0$. Denote by I_1 and I_2 some arcs joining in A the point z with the points z_1 and z_2 respectively. If $x\neq x_0$, then neither of the projections I_{1_X}, I_{2_X} can reduce to a single point. Therefore, we would have $I_{1_Y}=(y)=(y_1)$, $I_{2_Y}=(y)=(y_2)$, so that $y_1=y_2$, contrary to the assumption that $z_1\neq z_2$. Hence $x=x_0$, which proves that $A\subset(x_0, Y)$.

By (4A), the set A cuts some continuum F, which is its neighbourhood in the space $Z=X\times Y$. However, no proper subset of (x_0, Y) has this property. Hence, $A=(x_0, Y)\in R$, which proves the inclusion

(6) $$S\subset R.$$

From (5) and (6) we immediately infer that

(7) *The families R and S coincide.*

7. Proof of the Theorem in the case where the space X is connected. Let X be a connected 2-dimensional ANR. Suppose that there exist two pairs of curves X', Y' and X'', Y'' and two homeomorphisms $h'\colon X'\times Y'\to X$, $h''\colon X''\times Y''\to X$ such that $h'(X'\times Y')=X=h''(X''\times Y'')$. Let $Z'=X'\times Y'$, $Z''=X''\times Y''$, let Z_ν', Z_ν'', where $\nu=1,2,3$, be defined by formula (4), and let $h=h''^{-1}h'$.

First, let $Z_3'=0$. Then $X'^{[3]}=0=Y'^{[3]}$. Hence the space X is homeomorphic with the square $I^1\times I^1$, the cylinder $I^1\times S^1$, or the torus $S^1\times S^1$. These cases are topologically different, since in the first case the set Z_1' (and also $Z_1''=h(Z_1')$) is connected, in the second case it is not connected, and in the third case it is empty.

Next, let $Z_3'\neq 0$. Then either only one of the sets $X'^{[3]}$, $Y'^{[3]}$ is non-empty or both these sets are non-empty. In the first case the set Z_3' is not connected, in the second case it is connected.

Let $X'^{[3]}\neq 0$, $Y'^{[3]}=0$. Since Z_3'' is homeomorphic to Z_3' and therefore has the same properties, we may assume that $X''^{[3]}\neq 0$, $Y''^{[3]}=0$. Then, by the Theorem of Borsuk (see [3], p. 156), we have $X'\underset{\text{top}}{=}X''$ and $Y'\underset{\text{top}}{=}Y''$.

Finally, let $X'^{[3]}\neq 0\neq Y'^{[3]}$ and therefore also $X''^{[3]}\neq 0\neq Y''^{[3]}$. Let R', S' denote the families of subsets of the space Z' defined as in No. 3, and let R'', S'' denote the analogous families for the space Z''. Then, the homeomorphism h transforms the family S' on S'' by their topological character and, with regard to the identities $R' = S'$ and $R'' = S''$ arising from (7), it transforms the family R' on R''. Hence, if $y' \in Y'^{[3]}$ then $h(X', y')$ is equal to (X'', y'') or (x'', Y''). Suppose for instance that the first case holds. Then, $X' \underset{\text{top}}{=} X''$. If $X' \underset{\text{top}}{\neq} Y'$ then $h(x', Y') = (x'', Y'')$, so that $Y' \underset{\text{top}}{=} Y''$. On the other hand, if $X' \underset{\text{top}}{=} Y'$ then also (by the topological identity of the families $S' = R'$ and $S'' = R''$) $X'' \underset{\text{top}}{=} Y''$; therefore again $Y' \underset{\text{top}}{=} Y''$. Hence, in all cases the pairs X', Y' and X'', Y'' are topologically the same, which completes the proof of our Theorem in this case.

8. Proof of the Theorem in the general case. This proof runs similarly to the one given by K. Borsuk (see [4], p. 148). Namely, we associate with the topological type of any space X a certain polynomial $P(X)$ called the *characteristic polynomial of* X. This polynomial is the sum of the polynomial $P(C)$ corresponding to every component of X (see ibidem).

Let X be a given ANR of dimension $\leqslant 2$, which is homeomorphic with the Cartesian product $X_1 \times X_2 \times \ldots \times X_n$ of some topologically first spaces. Then $P(X) = P(X_1) \cdot P(X_2) \cdot \ldots \cdot P(X_n)$. Let C be a component of X. By definition we put $P(C) = 1$ if C is 0-dimensional (hence C contains only one point), $P(C) = x_i$ if C is 1-dimensional, $P(C) = x_j \cdot x_k$ if C is a Cartesian product of two 1-dimensional spaces, and $P(C) = z_i$ if C is a topologically first 2-dimensional set. By the preceding Section, each $P(C)$ is uniquely determined by C. We have

$$P(X) = \sum_{i,j} a_{i,j} x_i^{\varepsilon_i} \cdot x_j^{\varepsilon_j} + \sum_i b_i \cdot z_i ,$$

where $a_{i,j}, b_i$ are non-negative integers and ε_i is equal to 0 or 1.

It is easily seen that the space X is decomposable into a Cartesian product of some topologically first spaces if and only if the polynomial $P(X)$ is decomposable into a product of some indecomposable linear factors. Since this last decomposition is unique, the decomposition $X = X_1 \times X_2 \times \ldots \times X_n$ is also unique. Thus our Theorem is completely proved.

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Reçu par la Rédaction le 22. 6. 1961