A characterization of the exponential and logarithmic functions by functional equations

by

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As is well known (cf. e.g. [1]), the functions e^{cx} and $\log cx$ can be characterized by means of the functional equations in two variables:

$$\varphi(x+y) = \varphi(x)\varphi(y)$$

and

$$\varphi(xy) = \varphi(x)+\varphi(y),$$

respectively. The purpose of the present note is to characterize these functions with the aid of the following functional equations in a single variable:

$$\varphi(2x) = [\varphi(x)]^2 \tag{1}$$

and

$$\varphi(x^2) = 2\varphi(x), \tag{2}$$

and some additional conditions.

At first let us notice that a function continuous at the point $x = 0$, satisfying equation (1) and the condition

$$\varphi(0) = 1, \tag{3}$$

must be strictly positive. In fact, it is evident from (1) that $\varphi(x) \geqslant 0$. If there were an x_0 such that $\varphi(x_0) = 0$; then, according to (1), we would have

$$\varphi\left(\frac{x_0}{2^n}\right) = 0, \quad n = 1, 2, \ldots,$$

which, on account of the continuity of $\varphi(x)$ at zero, would imply $\varphi(0) = 0$ and contradict condition (3).

Thus $\varphi(x) > 0$, and we may take the logarithms of both sides of equation (1). Setting $\varphi^*(x) = \log\varphi(x)$ we see that the function $\varphi^*(x)$ satisfies the functional equation

$$\varphi(2x) = 2\varphi(x). \tag{4}$$

Equations (2) and (4) are particular cases of the Schröder equation,

$$\varphi[f(x)] = s\varphi(x)\,, \quad s \neq 0\,, \tag{5}$$

($f(x)$ and s given, $\varphi(x)$ to be found). We now turn to the study of this equation and prove two theorems which, when specialized [1] to equations (4) and (2), yield the desired characterization of the exponential and logarithmic functions.

THEOREM 1. *If the function $f(x)$ is of class C^1 in an interval $\langle a, b)$, $f(x) \neq x$, $f'(x) > 0$ in (a, b), $f(a) = a$, and $f'(a) = s$, then for any number $a \neq 0$ there exists at most one function $\varphi(x)$ of class C^1 in $\langle a, b)$ satisfying equation* (5) *and the conditions*

$$\varphi(a) = 0\,, \quad \varphi'(a) = \alpha\,. \tag{6}$$

Similarly, if the function $f(x)$ is of class C^1 in an interval $(a, b\rangle$, $f(x) \neq x$, $f'(x) > 0$ in (a, b), $f(b) = b$, and $f'(b) = s$, then for any number $\alpha \neq 0$ there exists at most one function $\varphi(x)$ of class C^1 in $(a, b\rangle$ satisfying equation (5) *and the conditions*

$$\varphi(b) = 0\,, \quad \varphi'(b) = \alpha\,. \tag{7}$$

Proof. We shall prove only the first part of the theorem; the proof of the second part is quite analogous.

Let $\varphi_1(x)$ and $\varphi_2(x)$ be two functions of class C^1 in $\langle a, b)$, satisfying equation (5) and condition (6). Then their derivatives satisfy the equation

$$\varphi'[f(x)] = \frac{s}{f'(x)}\varphi'(x)\,. \tag{8}$$

From the assumption $\alpha \neq 0$, conditions (6) and the continuity of the functions $\varphi_i'(x)$ $(i = 1, 2)$ it follows (by an argument similar to that used at the beginning of this paper) that $\varphi_i'(x) \neq 0$ in $\langle a, b)$. Consequently the function $\psi(x) \overset{\text{def}}{=} \varphi_1'(x)/\varphi_2'(x)$ is continuous in $\langle a, b)$ and satisfies the functional equation

$$\psi[f(x)] = \psi(x)\,.$$

It follows (cf. [3]) that $\psi(x) \equiv \text{const}$, and thus, taking into account conditions (6), we obtain $\varphi_1(x) \equiv \varphi_2(x)$, which was to be proved.

Remark. If $s \neq 1$, then every solution of equation (5) in $\langle a, b)$ must fulfil the condition $\varphi(a) = 0$ (this follows immediately from (5) on setting $x = a$). If, however, $s = 1$, then a solution of equation (5) in $\langle a, b)$ may assume an arbitrary value at $x = a$. And it is evident that, in this case, the theorem remains valid when condition (6) is replaced

[1] For equation (4) corresponding theorems have been proved by Oeconomou [6], but, as far as we know, these results have not been used to characterize the exponential function.

by the condition $\varphi(a) = \varphi_0$, where φ_0 is an arbitrary constant (and, similarly, in (7) $\varphi(b) = \varphi_0$). To see this, it is sufficient to apply the previous argument to the function $\varphi(x) - \varphi(a)$. On the other hand, the restriction $\alpha \neq 0$ is essential. We shall show, however, that if there *exists* a solution $\varphi_0(x)$ of equation (5), fulfilling the condition $\varphi_0'(a) = \alpha \neq 0$, and if $\varphi(x)$ is a solution of (5) satisfying the condition $\varphi(a) = \varphi'(a) = 0$, then $\varphi(x)$ is identically zero.

In fact, let $\varphi(x)$ be a solution of equation (5) (of class C^1) such that $\varphi(a) = \varphi'(a) = 0$. Then the function $\varphi_0(x) - \varphi(x)$ also satisfies equation (5) and $\varphi_0'(a) - \varphi'(a) = \alpha$. But on account of what has just been proved $\varphi_0(x) - \varphi(x) \equiv \varphi_0(x)$, i.e. $\varphi(x) \equiv 0$.

It can happen, however, that all the solutions of equation (5) which are of class C^1 in (a, b) (under our conditions there are infinitely many such solutions; cf. [2]) fulfil the condition

$$\lim_{x\to a+0} \varphi'(x) = 0\,.$$

In particular, this is the case if

$$\lim_{n\to\infty} \frac{s^n}{\prod\limits_{\nu=0}^{n} f'[f^\nu(x)]} = 0$$

($f^\nu(x)$ denotes the νth iterate of the function $f(x)$).

The condition $f'(a) = s$, occurring in the hypotheses of the theorem, may seem restrictive. But it is evident from relation (8) that this condition is necessary for the existence of a solution $\varphi(x)$ of (5) such that $\varphi'(a) \neq 0$.

The requirement for $\varphi(x)$ to be of class C^1 is necessary to ensure uniqueness. As has been proved in [3], under the conditions of theorem 1 equation (5) has an infinity of solutions continuous in $\langle a, b)$; in fact every function $\varphi_0(x)$ continuous in an interval $\langle x_0, f(x_0)\rangle$ $\big(x_0 \in (a, b)\big)$ and fulfilling the condition $\varphi_0[f(x_0)] = s\varphi_0(x_0)$ can be uniquely extended to a solution of (5) continuous in $\langle a, b)$ (cf. [3]). The requirement of class C^1 for solutions, however, can be replaced by the condition of convexity.

THEOREM 2. *Under the conditions of theorem* 1 *equation* (5) *has at most a one-parameter family of convex solutions* [2] *in* (a, b).

[2] Or concave. If $\varphi(x)$ is a convex solution of (5), then $-\varphi(x)$ is a concave solution of (5), and conversely.

We say that the function $\varphi(x)$ is convex if it satisfies the inequality

$$\varphi(\lambda x + (1-\lambda)y) \leqslant \lambda\varphi(x) + (1-\lambda)\varphi(y)\,, \quad \lambda \in (0, 1)\,, \quad x, y \in (a, b)\,.$$

This definition implies the continuity of a convex function.

Proof. Let us assume that $f(x)$ is of class C^1 in $\langle a, b)$, $f(a) = a$, $f'(a) = s$; the proof in the other case is quite similar.

If $s = 1$, then all solutions of equation (5) must be constant on the set of points of the form $f^n(x_0)$ (for any $x_0 \in (a, b)$), and consequently, in this case, the only convex solutions of (5) are the constant ones. Thus in the sequel we assume that $s \neq 1$. We may also leave out of the considerations the trivial solution $\varphi(x) \equiv 0$.

Let $\varphi(x)$ be a convex solution of (5) in (a, b). Thus $\varphi(x)$ is differentiable almost everywhere in (a, b). Moreover, let us notice that, according to (5), if $\varphi(x)$ is differentiable at the point x_0, then it is also differentiable at the point $f(x_0)$. Hence it follows that the function $\varphi'(x)$ satisfies equation (8) for all x for which it is defined.

For every $x \in (a, b)$, the limit

$$\psi(x) \stackrel{\text{def}}{=} \lim_{\xi \to x-0} \varphi'(\xi)$$

exists; $\psi(x)$ is a monotonic function in (a, b) and satisfies equation (8) in (a, b). Consequently $\psi(x)$ has a constant sign in an interval $(a, a_0) \subset (a, b)$, and has no zeros in (a, a_1). (Otherwise $\psi(x)$ would have to be identically zero in a neighbourhood of a, and then, being a solution of (8), would vanish identically in (a, b). Then $\psi(x)$ would be constant, which implies that either $s = 1$, or $\varphi(x) \equiv 0$. However, both these cases have been excluded.)

The function $\mu(x) \stackrel{\text{def}}{=} |\psi(x)|$ satisfies the functional equation

$$\mu[f(x)] = \frac{s}{f'(x)}\mu(x). \tag{9}$$

Thus the function $\lambda(x) \stackrel{\text{def}}{=} \log\mu(x)$ is monotonic in (a, a_1) and satisfies the equation

$$\lambda[f(x)] - \lambda(x) = \log\frac{s}{f'(x)}, \qquad x \in (a, a_1). \tag{10}$$

Now, $\lim_{x \to a+0} \log \frac{s}{f'(x)} = 0$, and thus, as we have shown ([3]) in [4] (cf. also [5]), equation (10) has at most a one-parameter (with an additive constant) family of solutions monotonic in (a, a_1). It follows (in view of the fact that each solution of equation (8) is uniquely determined by its values in the interval (a, a_1)) that there can exist at most a one-parameter (with a multiplicative constant) family of monotonic functions $\psi(x)$ satisfying equation (8). Thus $\varphi'(x)$ is determined up to a multiplicative constant, except on at most a set of measure zero; and the function $\varphi(x)$—which, being convex, is absolutely continuous—is determined up to two constants (multiplicative and additive) in the whole of (a, b). But since $s \neq 1$, the additive constant can be determined from the fact that $\varphi(x)$ satisfies (5). This completes the proof.

([3]) In [4] and [5] the results concern the case $f(b) = b$, but they are also valid (and the proofs are quite analogous) in the case where $f(a) = a$.

Corollary. *Under the conditions of Theorem 2, every convex solution of equation* (5) *is of class* C^1.

Proof. The function $\psi^*(x) \stackrel{\text{def}}{=} \lim_{\xi \to x+0} \varphi'(\xi)$ is also a monotonic solution of equation (8) and equals $\psi(x)$ almost everywhere in (a, b). Since the monotonic solution of (8) is unique up to an additive constant, $\psi^*(x) \equiv \psi(x)$, which means that $\varphi'(x)$ is continuous in (a, b).

From theorems 1 and 2 and from the equivalence of equations (1) and (4), proved at the beginning of the paper, the following characterizations of the exponential and logarithmic functions result immediately:

Theorem 3. *The function* $\varphi(x) = e^x$ *is the unique function which is of class* C^1 *in* $\langle 0, \infty)$, *satisfies equation* (1) *and fulfils the conditions*

$$\varphi(0) = \varphi'(0) = 1.$$

Similarly, the function $\varphi(x) = \ln x$ *is the unique function which is of class* C^1 *in* $\langle 1, \infty)$, *satisfies equation* (2) *and fulfils the condition*

$$\varphi'(1) = 1.$$

Remark. In the above theorem the intervals $\langle 0, \infty)$ and $\langle 1, \infty)$ may be replaced by the intervals $(-\infty, 0\rangle$ and $(0, 1\rangle$, respectively.

Theorem 4. *The function* $\varphi(x) = e^x$ *is the only function which is logarithmicaly convex in* $(0, \infty)$, *satisfies equation* (1) *and fulfils the condition* $\varphi(1) = e$.

Similarly, the function $\varphi(x) = \ln x$ *is the only function which is concave in* $(1, \infty)$, *satisfies equation* (2) *and fulfils the condition* $\varphi(e) = 1$.

Remark. In the above theorem the interval $(0, \infty)$ with the condition $\varphi(1) = e$, and the interval $(1, \infty)$ with the condition $\varphi(e) = 1$, may be replaced by the interval $(-\infty, 0)$ with the condition $\varphi(-1) = e^{-1}$, and the interval $(0, 1)$ with the condition $\varphi(e^{-1}) = -1$, respectively.

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A note on v^*-algebras

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A v^*-algebra is an abstract algebra $\mathfrak{A} = (\mathcal{A}, F)$ satisfying the following conditions:

(i) *If $a \in \mathcal{A}$ and a is not an algebraic constant, then the set $\{a\}$ is a set of independent elements.*

(ii) *If $\{a_1, \ldots, a_n\}$ is a set of independent elements and $\{a_1, \ldots, a_{n+1}\}$ is not a set of independent elements, then a_{n+1} belongs to the subalgebra generated by $\{a_1, \ldots, a_n\}$.* (Independence is to be understood in the sense of E. Marczewski. See [1], [2].)

Some properties of v^*-algebras have been developed in [3]. Here we shall prove a strengthening of theorem II of [3].

Let $\mathfrak{A}$ be any algebra. By $A^{(n)}$ we denote the set of all algebraic functions of n variables, and by $A^{(n,k)}$ we denote the set of all functions of $A^{(n)}$ depending on at most k variables.

THEOREM. *If $\mathfrak{A} = (\mathcal{A}, F)$ is an n-dimensional v^*-algebra, and $A^{(3)} = A^{(3,1)}$, then there exist a group G of transformations of the set $\mathcal{A}$ and a subset $\mathcal{A}_0 \subset \mathcal{A}$ containing all fixed points of the transformations from G such that $G(\mathcal{A}_0) \subset \mathcal{A}_0$, and moreover every algebraic function of n variables is of the form:*

$$f(x_1, \ldots, x_n) = g(x_i) \quad \text{for} \quad g \in G \text{ and } 1 \leqslant i \leqslant n,$$

or

$$f(x_1, \ldots, x_n) = a \quad \text{for} \quad a \in \mathcal{A}_0.$$

In view of theorem II of [3] it suffices to prove that $A^{(n)} = A^{(n,1)}$, whence the theorem results at once from the following

LEMMA. *If $\mathfrak{A}$ is an v^*-algebra, $k \geqslant 3$, $A^{(k)} = A^{(k,1)}$, and $\dim \mathfrak{A} \geqslant k+1$, then $A^{(k+1)} = A^{(k+1,1)}$.*

Proof. Suppose that the set $A^{(k+1)} \setminus A^{(k+1,k)}$ is non-void, and let $f \in A^{(k+1)} \setminus A^{(k+1,k)}$. Hence the set $\{f(x_1, \ldots, x_{k+1}), x_2, \ldots, x_{k+1}\}$ in the algebra $A^{(k+1)}$ is independent, and thus this set generates the whole algebra $A^{(k+1)}$. There exists an $F \in A^{(k+1)}$ such that

$$x_1 = F\big(f(x_1, \ldots, x_{k+1}), x_2, \ldots, x_{k+1}\big).$$