The power of the continuum and some propositions of plane geometry

by

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Introduction. In 1951, Sierpiński [5] announced the equivalence for $n = 1, 2$ of the hypothesis

(H_n) $\quad 2^{\aleph_0} \leqslant \aleph_n$

to the following proposition of elementary geometry:

(P_n) *Euclidean $(n+2)$-dimensional space can be decomposed into $n+2$ sets E_i $(i = 1, 2, \ldots, n+2)$ such that each line parallel to the coordinate axis OX_i intersects E_i $(i = 1, 2, \ldots, n+2)$ in only a finite number of points.*

The proof for $n = 1$ was published by Sierpiński ([6]; [7]; [8], p. 397), and a proof for every positive integer n by Kuratowski [3]. The object of the present paper is to establish the equivalence of the hypothesis (H_n) to a corresponding proposition of elementary plane geometry:

(Q_n) *The Euclidean plane can be decomposed into $n+2$ sets E_i $(i = 1, 2, \ldots, n+2)$ such that, for some $n+2$ directions θ_i in the plane, each line in the direction θ_i intersects E_i $(i = 1, 2, \ldots, n+2)$ in only a finite number of points.*

Bagemihl proved [1] that $(\mathrm{Q}_1) \Rightarrow (\mathrm{H}_1)$, and I have recently [2] proved that $(\mathrm{H}_1) \Rightarrow (\mathrm{Q}_1)$. Thus proposition (Q_1), the question of whose validity was raised by Sierpiński ([6]; [8], p. 399), is already known to be equivalent to (H_1), in other words to the continuum hypothesis.

Proof that $(\mathrm{Q}_n) \Rightarrow (\mathrm{H}_n)$**.** Suppose if possible that (Q_n) is true and (H_n) false; then

(1) $$1 < \aleph_0 < \aleph_1 < \ldots < \aleph_{n+1} \leqslant 2^{\aleph_0}.$$

Using transfinite induction and the axiom of choice, it is easy to prove the following:

If C is any plane set of power $\mathfrak{a} < 2^{\aleph_0}$, θ any direction in the plane and $\mathfrak{b}$ any cardinal number satisfying $\mathfrak{a} < \mathfrak{b} \leqslant 2^{\aleph_0}$, then we can construct $\mathfrak{b}$ disjoint sets, each of which is congruent to C by a translation in the direction θ. If C^* denotes the union of these $\mathfrak{b}$ disjoint sets, it follows that C^* is of power $\mathfrak{b}$, but lies on $\mathfrak{a}$ straight lines in the direction θ.

Using these results and (1), it is clear how to construct by induction a sequence of $n+3$ plane sets $C_0, C_1, \ldots, C_{n+2}$, where C_0 *consists of a single point*, and for $i = 1, 2, \ldots, n+2$ *the set* C_i *is the union of* $\aleph_{i-1}$ *disjoint sets each a translation of the set* C_{i-1}, *but lies on* $\aleph_{i-2}$ *straight lines (one straight line, if* $i = 1$) *in the direction* θ_i.

The set C_{n+2} lies on $\aleph_n$ straight lines in the direction θ_{n+2}, and each of these lines intersects the set E_{n+2}, of proposition (Q_n), in only a finite number of points; consequently, C_{n+2} intersects E_{n+2} in at most $\aleph_n$ points. However, C_{n+2} is the union of $\aleph_{n+1}$ disjoint sets each congruent to C_{n+1}, and therefore at least one is free of points of E_{n+2}; let C'_{n+1} denote one such set.

The set C'_{n+1} (being a translation of C_{n+1}) lies on $\aleph_{n-1}$ straight lines in the direction θ_{n+1}, and each of these intersects the set E_{n+1} in only a finite number of points; consequently, C'_{n+1} intersects E_{n+1} in at most $\aleph_{n-1}$ points. However, C'_{n+1} is the union of $\aleph_n$ disjoint sets each congruent to C_n, and therefore at least one is free of points of E_{n+1}; let C'_n denote one such set.

Continuing like this, we obtain a decreasing sequence of sets

$$C'_{n+1} \supseteq C'_n \supseteq \ldots \supseteq C'_0,$$

where C'_{i-1} is congruent to C_{i-1} and is free of points of E_i $(i = 1, 2, \ldots, n+2)$. It follows that the set C'_0 consists of a single point which does not belong to any of the sets E_i, and this is a contradiction.

It may be observed that the above proof is rather similar to Mazurkiewicz's proof [4] that the plane is not the union of a finite number of curves.

Proof that $(\mathrm{H}_n) \Rightarrow (\mathrm{Q}_n)$. We shall prove that the hypothesis (H_n) implies the following proposition, more general than (Q_n):

Given any $n+2$ *directions* θ_i $(i = 1, 2, \ldots, n+2)$ *in the plane, no two of which are parallel, the plane can be decomposed into* $n+2$ *sets* E_i *such that each line in the direction* θ_i *intersects* E_i $(i = 1, 2, \ldots, n+2)$ *in only a finite number of points.*

The proof is based on the same idea as [2].

We shall call a line *special* if it is in one of the directions θ_i, and we shall call a set N of special lines a *network* if whenever two of the special lines through a point p belong to N so do all the special lines through p.

LEMMA 1. *If* M *is any infinite set of special lines, then the smallest network* N *containing* M *is a set of the same power as* M.

Proof. Let M have power $\boldsymbol{m} \geqslant \aleph_0$. The power of the set of points of intersection of lines of M does not exceed that of the Cartesian square of M, which is $\boldsymbol{m}^2 = \boldsymbol{m}$. Hence the set $f(M)$, defined to consist of M together with all special lines through these points of intersection, also has power $\boldsymbol{m}$. Clearly $N = M \cup f(M) \cup f(f(M)) \cup \ldots$ and has power $\geqslant \boldsymbol{m}$ and $\leqslant \aleph_0 \cdot \boldsymbol{m} = \boldsymbol{m}$.

LEMMA 2. *If* m *is a non-negative integer, then every network of power* $\aleph_m$ *can be ordered by a relation* $\prec$ *with the following property:*

(Π_m) *For any element* l *there exist only a finite number of systems of* $m+1$ *elements* $l_1, \ldots, l_{m+1}$ *such that* $l, l_1, \ldots, l_{m+1}$ *are concurrent and*

$$l_{m+1} \prec l_m \prec \ldots \prec l_1 \prec l.$$

Proof. We shall use induction on m. (Of course the result holds vacuously unless $\aleph_m \leqslant 2^{\aleph_0}$.)

If N is a network of power $\aleph_0$, then N can be ordered by some relation $\prec$ as an infinite sequence

$$k_1 \prec k_2 \prec \ldots,$$

and for any element $l = k_i$ of N there exist only a finite number of elements $l_1 \in N$ for which $l_1 \prec l$, namely the elements $k_1, \ldots, k_{i-1}$. This establishes Lemma 2 for $m = 0$.

Now suppose that it is true for some integer $m \geqslant 0$, and let N be any network of power $\aleph_{m+1}$. Then there exists a transfinite sequence

$$k_1, k_2, \ldots, k_\alpha, \ldots \quad (1 \leqslant \alpha < \omega_{m+1})$$

of type ω_{m+1}, where ω_{m+1} is the least ordinal of power $\aleph_{m+1}$, composed of all the elements of N. For each ordinal α, $\omega_m \leqslant \alpha < \omega_{m+1}$, denote by $N(\alpha)$ the smallest network containing all the elements $k_{\alpha'}$ for $1 \leqslant \alpha' \leqslant \alpha$. Then $N(\alpha)$ is of power $\aleph_m$, and consequently can be ordered by a relation $\prec_\alpha$ with the property (Π_m).

Given any element $k \in N$, let us denote by $a(k)$ the least ordinal α $(\omega_m \leqslant \alpha < \omega_{m+1})$ for which $k \in N(\alpha)$. Given any two elements g, h of N, let us write $g \prec h$ if either $a(g) < a(h)$, or $a(g) = a(h) = \alpha$, say, and $g \prec_\alpha h$. It is easy to see that this relation $\prec$ provides an ordering of N; we shall prove that it has the property (Π_{m+1}).

In fact, let $l \in N$ and let $l_1, \ldots, l_{m+2}$ be a system of $m+2$ elements of N such that $l, l_1, \ldots, l_{m+2}$ are concurrent and

$$l_{m+2} \prec l_{m+1} \prec \ldots \prec l_1 \prec l. \tag{2}$$

Then we have

$$a(l_{m+2}) \leqslant a(l_{m+1}) \leqslant \ldots \leqslant a(l_1) \leqslant a(l). \tag{3}$$

From the first inequality in (3) it follows that $N(a(l_{m+2})) \subseteq N(a(l_{m+1}))$, and consequently l_{m+2} belongs to $N(a(l_{m+1}))$ as well as l_{m+1}. Since $N(a(l_{m+1}))$ is a network, it therefore contains all the special lines through the point of intersection of l_{m+1} and l_{m+2}, and in particular $l \in N(a(l_{m+1}))$ and $a(l) \leqslant a(l_{m+1})$. From (3), we deduce that

$$a(l_{m+1}) = \ldots = a(l_1) = a(l) .$$

Thus, setting $a = a(l)$, all the lines $l, l_1, \ldots, l_{m+1}$ belong to $N(a)$, and (in view of (2))

$$l_{m+1} \prec_a \ldots \prec_a l_1 \prec_a l .$$

Since the relation $\prec_a$ ordering $N(a)$ has the property (Π_m), there exist for each l only a finite number of such systems $l_1, \ldots, l_{m+1}$ (with $l, l_1, \ldots, l_{m+1}$ concurrent). Finally, for each such system there exist only a finite number of special lines l_{m+2} through their point of intersection, and our result follows for $m+1$.

We now proceed to the construction of the sets E_i. The set of all special lines in the plane is a network, and the hypothesis (H_n) implies that it is of power $\leqslant \aleph_n$; it follows that it is of power $\aleph_m$ for some positive integer $m \leqslant n$, and therefore can be ordered by a relation $\prec$ with the property (Π_m). It is easy to see that such a relation possesses also the property (Π_n). If p is any point of the plane, we assign p to the set E_i if

$$p(\theta_s) \prec p(\theta_i) \quad (s = 1, \ldots, n+2;\ s \neq i) , \tag{4}$$

where $p(\theta)$ denotes the line through p in the direction θ. It is obvious that every point of the plane is thereby assigned to one of the sets E_i.

Our proof will be complete when we have shown that each line l in the direction θ_i intersects E_i in only a finite number of points $(i = 1, 2, \ldots, n+2)$. Let p be any point in the intersection $l \cap E_i$; then $l = p(\theta_i)$, and (4) holds. Hence if the $n+1$ lines $p(\theta_s)$ $(s = 1, \ldots, n+2;$ $s \neq i)$ are denoted by $l_1, \ldots, l_{n+1}$ in a suitable order, then $l, l_1, \ldots, l_{n+1}$ are concurrent and

$$l_{n+1} \prec \ldots \prec l_1 \prec l .$$

Since there exist only a finite number of such systems $l_1, \ldots, l_{n+1}$, there are only a finite number of points in the set $l \cap E_i$, as required.

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