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Totality of uniform structures with linearly ordered base

by

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According to [2] a uniform structure is said to be *total* if it is the finest member of its p-equivalence class, i.e. if it is finer than any other uniform structure with the same uniform set-neighbourhoods. (The above form of the definition, specialized to proper uniform structures, is the third characterization of totality in Theorem 3 of [2]. For the notion of p-equivalence, cf. also [1], p. 97.)

Yu. M. Smirnov has proved that every metrizable uniform structure is total [5], p. 570 (cf. also [2] Theorem 4). In the present paper we prove that every uniform structure with a linearly ordered base is total. The method of proof is a transfinite extension of a technique which goes back to Efremovič (proof of Lemma 1 of [4], p. 190).

LEMMA 1. *Every well-ordered set B contains a cofinal, well ordered subset C such that every proper segment of C has strictly smaller power than C and every cofinal subset of C has the same power as C.*

Proof. Let χ be the smallest cardinal of any cofinal subset of B. If $\chi = 1$, then B contains a last element b, and we shall be through with $C = \{b\}$.

For $\chi > 1$ the set I of ordinals of cardinality strictly less than χ contains non-void proper segments $I_a = \{\beta \mid \beta \in I, \beta < a\}$, and I_a has cardinal strictly less than χ for every $a \in I$, whereas I itself has cardinal χ.

Let ψ be some one-one mapping of I into a cofinal subset of B. The set $\psi(I_a)$ can not be cofinal in B for any $a \in I$ since it has cardinal strictly less than χ. Let $\varphi(a)$ denote the least upper bound of $\psi(I_a)$ in B for every $a \in I$. Clearly φ is a non-decreasing mapping of I onto a cofinal subset $C = \varphi(I)$ of B. Moreover, C is well ordered (in the ordering induced from B), since it is the image of the well-ordered set I by the non-decreasing mapping φ. The cardinal of C cannot exceed χ since $C = \varphi(I)$; hence by the cofinal nature of C it must equal to χ.

Now we assume that D is some proper segment of C. It follows from the non-decreasing nature of φ that $F = \varphi^{-1}(D)$ is a proper segment of I. Hence the cardinal of F, and hence also of $D = \varphi(F)$, must be strictly less than χ. Thus we have proved that the cofinal well-ordered subset C

of B has the first of the two required properties. The second property follows from the fact that every cofinal subset of C also is cofinal in B, and hence its cardinal cannot be strictly less than χ.

LEMMA 2. *Let U and W be subsets of a Cartesian product $S\times S$, and assume that W is symmetric and contains the diagonal Δ, and that $W^4\subset U$. Moreover, let $(x_a, y_a)_{a\in A}$ be some generalized sequence with a lineraly ordered index set A, and assume that $(x_a, y_a)\notin U$ for $a\in A$. Then there exists a cofinal subset Γ of A such that $(x_\beta, y_\gamma)\notin W$ whenever β and γ both belong to Γ.*

Proof. By transfinite induction we may extract a cofinal, well-ordered subset B of A, and from B we may extract a cofinal well-ordered subset C with properties of the set C of Lemma 1. Since C is cofinal in A as well as in B, it will be sufficient to construct a cofinal subset Γ of C which has the desired property.

For every $a\in C$, we define

$$D_a=\{\beta|\ \beta\in C,(x_a, y_\beta)\in W\}\,,\qquad E_a=\{\beta|\ \beta\in C,(x_\beta, y_a)\in W\}\,.$$

For arbitrary elements β, γ of D_a we shall have $(x_\beta, y_\gamma)\notin W^2$; for otherwise the three relations:

$$(x_\beta, y_\gamma)\in W^2\,,\qquad (x_\gamma, y_a)\in W^{-1}\,,\qquad (x_a, y_\beta)\in W\,,$$

would imply $(x_\beta, y_\beta)\in W^4\subset U$, contrary to hypothesis.

The assumption $\Delta\subset W$ entails $W\subset W^2$, and so we have actually proved that $(x_\beta, y_\gamma)\notin W$ whenever $\beta,\gamma\in D_a$. Thus if D_a is cofinal in C for any a, say $a=a_0$, then we shall be through with $\Gamma=D_{a_0}$. Similarly we prove that if E_a is cofinal for any a, say $a=a_0$, then we may write $\Gamma=E_{a_0}$.

In the remaining case in which none of the sets D_a, E_a are cofinal in C, we define $\psi(a)$ to be the least upper bound of $D_a\cup E_a$ in C for every $a\in C$. We may assume that C has no last element a', for otherwise we should be through with $\Gamma=\{a'\}$. Now the sets

$$F_a=\{\beta|\ \beta\in C,\beta\leqslant a\}=\{\beta|\ \beta\in C,\beta<a+1\}$$

are proper segments of C for all $a\in C$. Hence they have strictly smaller power than C. The same statement holds for $\psi(F_a)$, and so $\psi(F_a)$ is non-cofinal in C for every $a\in C$. Let $\varphi(a)$ denote the least strict upper bound of $\varphi(F_a)$ and a in C for every $a\in C$. Clearly φ is a non-decreasing mapping of C into itself, and has the following two properties:

(1) $\varphi(a)\leqslant\beta\to(x_a, y_\beta)\notin W,\ (x_\beta, y_a)\notin W.$

(2) $a<\varphi(a)$ for every $a\in C$.

For some fixed element a_0 of C we shall construct a suitable transfinite continuation Γ of the sequence:

$$a_0,\ \varphi(a_0)\,,\ \varphi\big(\varphi(a_0)\big)\,,\ \ldots$$

More specifically, let Γ be the intersection of all subsets G of C with the following properties:

(3) $a_0\in G$.

(4) $a\in G\to\varphi(a)\in G$.

(5) Every least upper bound (in C) of a subset of G again belongs to G.

There certainly exist sets with the properties (3)-(5), e.g. the set of all successors of a_0, and their intersection Γ is non-void since $a_0\in\Gamma$. It is also easily verified that Γ itself has the properties (3)-(5), and so it is the smallest set with these properties. By a standard, although not quite trivial, argument (cf. e.g. [3], p. 5), we can prove that the minimality of Γ together with the properties (2)-(5) yield the following additional property of Γ:

(6) $\gamma,\delta\in\Gamma$ and $\gamma<\delta\Rightarrow\varphi(\gamma)\leqslant\delta$.

The set Γ must be cofinal in C, for otherwise it would have a least upper bound γ' which should belong to Γ by (5) thus giving $\gamma'<\varphi(\gamma')\in\Gamma$ by (2) and (4) contrary to the definition of γ'.

Now, let γ and δ be arbitrary elements of Γ. If $\gamma=\delta$, we have $(x_\gamma, y_\delta)\notin W^4$, and hence also $(x_\gamma, y_\delta)\notin W$. If $\gamma<\delta$, we shall have $\varphi(\gamma)\leqslant\delta$ by (6), and hence we obtain $(x_\gamma, y_\delta)\notin W$ by virtue of (1). Similarly $\delta<\gamma$ implies $(x_\gamma, y_\delta)\notin W$, and so the proof is accomplished.

THEOREM. *Every uniform structure with a linearly ordered base is total.*

Proof. Let $\mathcal{U}$ be a uniform structure on a set S which admits a base of *entourages* V_a, $a\in A$, where A is some linearly ordered set and $V_a\supset V^\beta$ whenever $a<\beta$. As usual we shall apply the notation $E\Subset F$ to denote that a subset F of S is a uniform neighbourhood of another subset E. (Cf. e.g. [1], p. 97.)

Let $\mathcal{U}'$ be some other uniform structure of S belonging to the same p-equivalence class as $\mathcal{U}$. In other words, the relation $E\Subset F$ has the same meaning relatively to $\mathcal{U}$ and $\mathcal{U}'$.

If $\mathcal{U}'$ were not coarser than $\mathcal{U}$, there would exist an *entourage* U of $\mathcal{U}'$ not containing any V_a, $a\in A$. In that case we might assign (by the axiom of choice) to every $a\in A$ a couple (x_a, y_a) such that $(x_a, y_a)\in V_a$, $(x_a, y_a)\notin U$.

Let W be a symmetric *entourage* of $\mathcal{U}'$ such that $W^4\subset U$. By Lemma 2, there exists a cofinal subset Γ of A such that $(x_\gamma, y_\delta)\notin W$ whenever γ and δ both belong to Γ. Defining $X=\{x_\gamma|\ \gamma\in\Gamma\}$, $Y=\{y_\gamma|\ \gamma\in\Gamma\}$, we obtain $W(X)\cap Y=\emptyset$, or equivalently $W(X)\subset\complement Y$ which means that $X\Subset\complement Y$.

On the other hand $(x_\gamma, y_\gamma)\in V_\gamma$; hence $V_\gamma(X)\cap Y\neq\emptyset$, or equivalently $V(X)\not\subset\complement Y$, for $\gamma\in\Gamma$. Since Γ is cofinal in A, $\{V_\gamma\}_{\gamma\in\Gamma}$ must be a base of $\mathcal{U}$, and hence, the relation just proved means that $X\not\Subset\complement Y$. This contradiction establishes the proof.

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О рангах систем множеств и размерности пространств

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Настоящая работа посвящена в основном исследованию понятия ранга системы множеств и связи этого понятия с размерностью пространства. Наряду с понятием ранга системы множеств в смысле Нагата оказывается полезным рассматривать ранги несколько по иному определенные, в частности так, как это было сделано мной ранее в [2].

В § 1 приводятся наиболее общие результаты, из которых важнейший — характеристику размерности произвольного топологического пространства — дает теорема 1.4.

Достоинства метрических пространств и бикомпактов позволяют доказать для них более сильные результаты, собранные в § 2.

Наконец случай слабо-счетномерных и счетномерных пространств разобран отдельно в § 3. Развитая там теория позволяет доказать инвариантность класса произвольных слабо-счетномерных пространств и метрических счетномерных пространств при открытых, непрерывных, конечнократных отображениях.

Замечу, что в этой работе все покрытия предполагаются открытыми, а размерность пространства есть всюду размерность, определенная с помощью покрытий. Под *слабо-счетномерными* пространствами понимаются представимые в виде суммы счетного множества своих замкнутых конечномерных подпространств, а под *счетномерными* — представимые в виде суммы счетного числа своих нульмерных подмножеств. Наконец, k у нас — всегда некоторое целое положительное число.

Приведем основные определения.

Нагата (¹) называет два множества *зависимыми*, если одно из них содержится в другом. Система множеств называется *зависимой*, если она содержит зависимые множества, в противном случае она называется независимой.

(¹) Результаты Нагата, относящиеся к рангам систем множеств, которые я отмечаю в этой работе, мне известны только в виде формулировок из устных источников. Доказательства их, повидимому, еще не опубликованы.