Some mappings of ANR-sets

by
A. Lelek (Wrocław)

Recently, in connection with a method of construction of ANR-sets (1) having some paradoxical properties, K. Borsuk raised the following problem:

Given an ANR-set $X$. Take a sequence of mutually disjoint AR-sets $X_1, X_2, ...$ in $X$ which have diameters converging to zero. Suppose $\xi$ is a mapping of $X_i$ such that $\xi_i(x)$ is an AR-set for $i = 1, 2, ...$ Then the decomposition of $X$ into the sets $\xi^{-1}(y)$, where $i = 1, 2, ...$ and $y \in \xi_i(X_i)$, and the points belonging to $X - \bigcup X_i$, is upper semicontinuous; thus it induces a mapping $\xi$ of $X$. Is $\xi(X)$ an ANR-set?

We shall solve the problem in the affirmative for the case when $\xi(X)$ has a finite dimension (see Corollary 7 below). This will be done by showing that $\xi(X)$ is $\text{LC}^n$, i.e., homotopically locally connected in dimensions up to $n$ (see [3], p. 500), for $n = 0, 1, ...$ Therefore the condition that $\xi(X)$ is finitely dimensional plays an essential role in our result. It is now an open question if the above problem has an affirmative solution for $\xi(X)$ with infinite dimension.

**Theorem.** Let $n = 0, 1, ...$, let $f$ be a continuous mapping of an ANR-set $X$ and let $X_1, X_2, ...$ be subsets of $X$ such that

(i) $X_i$ are $(n-1)$-connected ANR-sets (2) for $i = 1, 2, ...$,

(ii) $f(X_i)$ are $n$-connected ANR-sets for $i = 1, 2, ...$,

(iii) $f$ is 1-1 on the set $X_0 = X - \bigcup X_i$ and $f(X_i)$ are mutually disjoint sets for $i = 0, 1, ...$ with diameters converging to zero as $i$ tends to infinity.

Then the image $f(X)$ is $\text{LC}^n$.

**Proof.** Put

$X_\infty = X \times \{0, 1, \frac{1}{2}, \frac{1}{4}, ...\}$

(1) By ANR-set (or AR-set) we understand a compact metric absolute neighbourhood (or absolute) retract.

(2) A set is said to be $n$-connected if it is homotopically connected in dimensions up to $n$, i.e. if all its $i$-dimensional homotopy groups vanish for $i = 0, 1, ..., n$. The $(-1)$-connectedness means that no condition is required.
and consider the decomposition of $\mathcal{X}_n$ into the sets $f^{-1}(y) \times \{0\}$, where $y \in f(X)$, the sets $f^{-1}(y) \times \{1\}$, where $y \in f(X)$ and $f^{-1}(y) \subset \mathcal{X}_n \cup \ldots \cup \mathcal{X}_i$, and the points belonging to $(\mathcal{X}_{i+1} \cup \mathcal{X}_{i+2} \cup \ldots) \times \{1\}$, for every $i = 1, 2, \ldots$. It is a decomposition of the whole space $\mathcal{X}_n$, since the sets $f(\mathcal{X}_i)$ are mutually disjoint according to (iii), and is upper semicontinuous by the continuity of $f$. It thus induces a continuous mapping $f_\infty$ of $\mathcal{X}_n$. Write

$$
\begin{align*}
F_\infty &= f_\infty(\mathcal{X}_n), \\
f_i &= f_\infty|\{X \times \{0\}\}, \\
f_{i-1} &= f_\infty|\{X \times \{1\}\}, \\
F_i &= f_\infty|\{X \times \{1\}\}
\end{align*}
$$

for $i = 1, 2, \ldots$ Then the sets $F_1, F_2, \ldots$ are mutually disjoint and $F_1$ is homeomorphic to $f(X)$. So it is sufficient to show that $F_\infty$ is LC$^\alpha$.

For every $i = 1, 2, \ldots$, we have $F_i = f_i(X \times \{1\})$, the mapping $f_i$ is 1-1 on the set $(X \times \{1\}) \cup \{x \in \mathcal{X}_i \times \{1\}\}$ and the sets $X _1, \ldots, X_i$ are mutually disjoint, according to (iii). Thus their union $X_1 \cup \ldots \cup X_i$ is an ANR-set, by (ii). We also have $f_i(X \times \{1\}) \cup \{x \in \mathcal{X}_i \times \{1\}\} = f_i(X \times \{1\}) \cup \{x \in \mathcal{X}_i \times \{1\}\}$ and the union on the right is that of mutually disjoint sets homeomorphic to the sets $f_i(X_1), \ldots, f_i(X_i)$, respectively. Thus it is also an ANR-set, by (ii). It follows from the Borsuk theorem (see [1], p. 205) that the compactness $F_1$ is locally contractible for $i = 1, 2, \ldots$. Hence $F_1, F_2, \ldots$ are LC$^\alpha$.

But from \(\lim_{i \to \infty} X_i \times \{1\} = X \times \{0\}\) we obtain

$$
\lim_{i \to \infty} F_i = F_\infty
$$

and all $F_1, F_2, \ldots$ are closed subsets of the compact set $F_\infty$. Consequently, by virtue of the Kurschewski theorem (see [2], p. 122, Theorem A), in order to show that $F_\infty$ is LC$^\alpha$ it is enough to prove that $F_1, F_2, \ldots$ are uniformly LC$^\alpha$ (see [2], p. 121, footnote).

Namely, let $p$ be an arbitrary point of $F_\infty$, and $U$ an arbitrary open set in $F_\infty$ to which $p$ belongs. We ought to find an open subset $V$ of $F_\infty$ such that $p \in V$ and every continuous mapping of the $i$-dimensional sphere $S_i (i \leq n)$ into $V \cap F_\infty$ is homotopic in $U \cap F_\infty$ with a constant mapping, i.e. $\forall n \in U \cap F_\infty$ for every $i = 1, 2, \ldots$, we have $S_i \to V \cap F_i$.

Let us first observe that since $p \in F_\infty$, a positive integer $i$ exists such that

$$
p \in f_i(X_1 \times \{1\}) \cup \ldots \cup f_i(X_i \times \{1\})\,.
$$

Next, we have $F_i \cap \text{ran}(A) = f_i(A \cap f_i(X_1 \times \{1\}))$ for every $A \subset \mathcal{X}_n$, whence

$$
F_i \cap \text{ran}(A) = f_i(A \cap (X \times \{1\}))
$$

Further, let $r_i: \mathcal{X}_n \to X \times \{0\}$ be a retraction defined by the formula $r_i(x) = (x, 0)$ for $i = 0, 1, \frac{1}{2}, \ldots$ and take another retraction $r$ so that the diagram

$$
\begin{array}{ccc}
\mathcal{X}_n & \xrightarrow{\gamma} & X \times \{0\} \\
\downarrow r & & \downarrow r_0 \\
\mathcal{X}_n & \to & F_0
\end{array}
$$

be commutative, namely by putting $r = f_0 r_0 f_0^{-1}$. This is possible because the inverses of points under $f_0$ are contained in those under $f_0^{-1}$.

Setting $r_i = r | F_i$ for $i = 1, 2, \ldots$, we get $r_i^{-1}(p) = F_i \cap r^{-1}(p) = F_i \cap \text{ran}(A)$, where $A = r_i^{-1}(r_i^{-1}(p))$, and $F_i \cap \text{ran}(p) = f_i^{-1}(\{0\}) \times \{0\}$, where $\psi \in f_i(X_1 \times \{1\})$, according to (2) and the definition of $f_i$. Hence

$$
r_i^{-1}(p) = f_i^{-1}(\{0\}) \times \{0\}
$$

by (3), and $f_i^{-1}(\{0\}) \subset X_i \subset \mathcal{X}_n$. It follows from the definition of $f_i$ that the set $r_i^{-1}(p)$ reduces to a single point, say $p'$.

Now, since $r$ is a retraction of $\mathcal{X}_n$ to $F_0$ and $p \in U \cap F_0$, there exist, by (1), a positive integer $k$ such that

$$
(F_0 \cup \bigcup_{m=0}^{k} F_i \cap \text{ran}^{-1}(p')) \subset U ,
$$

and an open subset $V'$ of $F_0$ such that $p \in V'$ and

$$
(F_0 \cup \bigcup_{m=0}^{k} F_i \cap \text{ran}^{-1}(V')) \subset U .
$$

Since $r_i^{-1}(V') = F_i \cap \text{ran}^{-1}(V')$, we get

$$
r_i^{-1}(V') \subset U \cap F_i
$$

for $m \geq k$. Moreover, $r_i^{-1}(V')$ is an open subset of $F_i$ and $(p') = r_i^{-1}(p) \subset r_i^{-1}(V')$. But $F_i$ is LC$^\alpha$. Consequently, an open subset $W'$ of $F_i$ exists such that $p \in W'$ and

$$
\gamma \approx W' \quad \text{for every} \quad \psi : S_i \to W',
$$

$l = 0, 1, \ldots, n$. We also can choose an open subset $V'$ of $F_0$ such that $p \in V'$ and $r_i^{-1}(V') \subset W'$.

The sets $f_i(X_1 \times \{0\})$ are homeomorphic to $f_i(X_1 \times \{0\})$ for $i = 1, 2, \ldots$, respectively, and, by (3), they are mutually disjoint and have diameters converging to zero as $i$ tends to infinity. By (ii), they are compact subsets of $F_i$. It follows that the decomposition of $F_i$ into the sets $f_i(X_1 \times \{0\})$, $\ldots$, $f_i(X_i \times \{0\})$
where \( i = 1, 2, \ldots \), and the remaining points, is upper semicontinuous, and therefore it induces a continuous mapping \( g_0 \) of \( F_1 \). Since no set \( f_i(X_{i+1} \times 0) \) meets \( f_i(X_{i-1} \times 0) \) according to (iii), we get
\[
\mathcal{G}^{-1}(p) = \mathcal{G}' \cap V',
\]
by (2). Hence putting
\[
G = F_0 - \mathcal{G}^{-1}(V')
\]
we obtain \( G \subset V' \) and \( p \in G \). Let \( G' \) be a component of \( G \) to which \( p \) belongs. Since the set \( F_1 \), being homeomorphic to \( f(X_i) \), is locally connected and \( G \) is an open subset of \( F_1 \), the set \( G' \) is open in \( F_1 \). Thus the set
\[
W = \mathcal{G}'^{-1}(G')
\]
is open in \( F_1 \) and
\[
W \subset \mathcal{G}'^{-1}(G') \subset V'.
\]
Moreover, the set \( \mathcal{G}^{-1}(F_1 - V') \) either does not intersect or contains the set \( f_i(X_{i+1} \times 0) \) for \( i = 1, 2, \ldots \). Consequently, the set \( G \) and its component \( G' \) do the same, since all sets \( f_i(X_{i+1} \times 0) \) are connected, by (ii). This yields
\[
G' = \mathcal{G}^{-1}(G'),
\]
and we see that the set \( W \) either does not intersect or contains the set
\[
r_{i+1} f_1(X_{i+1} \times 0)
\]
for \( i = 1, 2, \ldots \). On the other hand, we have
\[
r_{i+1} f_1(X_{i+1} \times 0) = f_i \circ r_i^{-1}(f_i(X_{i+1} \times 0)) = f_i \circ f_i^{-1}(B),
\]
where \( B = f_i^{-1}(f_i(X_{i+1} \times 0)) \), and the remaining points, is upper semicontinuous for \( m = 1, 2, \ldots \), and so it induces a continuous mapping \( g_m \) of \( F_1 \). By (8), the set \( W \) either does not intersect or contains \( \mathcal{G}^{-1}(y) \) for \( y \in \mathcal{G}^{-1}(F_1) \). Then the inverses of points under \( g_m \) \( W \) coincide with the ones under \( g_m \) for \( m = 1, 2, \ldots \). The decomposition of \( F_1 \) into the mutually disjoint sets \( f_1(X_{i+1} \times 1) \), where \( i = 1, 2, \ldots, m \), and the remaining points, is upper semicontinuous for \( m = 1, 2, \ldots \). It follows that the mapping \( g_m \) \( W \) is proper in the sense of Smale (see [4], p. 604), i.e., the inverses of compact sets under \( g_m \) \( W \) are compact, for \( m = 1, 2, \ldots \).

Let \( s_m(x, 1) = (x, 1(t - m)) \) for \( x \in X \) and \( m = 1, 2, \ldots \). We get
\[
r_{i+1} s_m(x, 1) = r_{i+1} s_m(x, 1),
\]
and can define a continuous mapping \( h_m : F_i \to F_{i+m} \) so that the diagram
\[
\begin{array}{ccc}
X \times 1 & \overset{r_i}{\to} & X \times 1 \\
\downarrow_{f_i} & & \downarrow_{f_i} \\
F_i & \overset{r_i}{\to} & F_{i+m}
\end{array}
\]
is commutative, namely by putting \( h_m = f_{i+m} \circ s_m \circ f_i^{-1} \) for \( m = 1, 2, \ldots \). The definition is right since the inverses of points under \( f_{i+m} \) lie in those under \( f_i \). Let us observe that the inverses of points under \( h_m \) are single points or subsets of \( f_i(X_{i+1} \times 1) \), where \( i = 1, 2, \ldots, m \) (compare the definition of \( h_0 \) and \( f_1 \), p. 226). It follows that the inverses of points under \( h_m \) \( W \) are contained in those under \( g_m \) \( W \). The formula
\[
w_m = (g_m \circ g) \circ (h_m \circ W) = (g_m \circ W)
\]
defines thus a continuous mapping such that the diagram
\[
\begin{array}{ccc}
W \circ & \overset{r_i}{\to} & W \circ \\
\downarrow_{h_i} & & \downarrow_{h_i} \\
h_m(W) & \overset{r_i}{\to} & g_m(W)
\end{array}
\]
is commutative. Moreover, \( g_m \) \( W \) being a proper mapping, \( u_m \) is the same, according to the continuity of \( h_m \) \( W \), for \( m = 1, 2, \ldots \).

Since \( f_i = f_i^{-1} \) on \( F_1 \), we get
\[
r_{i+1} s_m(x) = f_{i+m} s_m f_i(x) = f_{i+m} s_m f_i(x) = r_{i+m}(x)
\]
for \( x \in F_1 \), whence
\[
r_i = r_{i+m} s_m
\]
for \( m = 1, 2, \ldots \).

We now define the desired open subset \( V \) of \( F_m \) (see p. 226) by setting
\[
V = r^{-1}(G') = (F_1 \cup \ldots \cup F_{b+1-1})
\]
and verify its properties as follows.

The point \( p \) belongs to \( F_1 \), that is \( p = r(p) \), and so \( p \in V \) \( r^{-1}(p) \cap r^{-1}(G') \).

The set \( F_1 \), being disjoint with any set \( F_i \) for \( i = 1, 2, \ldots, j + k - 1 \), we conclude that \( p \in V \). Furthermore, \( V \cap F_{b+1-1} = r^{-1}(G') \) for \( m \geq k \), whence
\[
V \cap F_{b+1-1} = h_m^{-1} r^{-1}(G') = h_m^{-1} r^{-1}(G') = h_m(W)
\]
for \( m \geq k \), according to (9).

Now, let us point out that our theorem trivially holds for \( n = 0 \). We can thus assume that \( n > 0 \).

Then the superposition \( g_i r_i \) is a monotone mapping. Indeed, for \( x \in g_i F_i \), the inverse \( g_i^{-1}(x) \) is either \( x^2 \) a single point \( y \) belonging to \( f_0(X_{i+1} \times 0) \), where \( i = 0, \ldots, j \), or \( 2^2 \) a set \( f_0(X_{i+1} \times 0) \), where \( i = 1, 2, \ldots \).

If \( 1^2 \), we have
\[
g_i r_i^{-1}(x) = r_i^{-1} g_i^{-1}(x) = r_i^{-1}(y)
\]
and confirm, in the same way as previously for \( p \) (see p. 227) instead of \( y \), that the set \( r_i^{-1}(y) \) reduces to a single point. If \( 2^2 \), we obtain
\[
g_i r_i^{-1}(x) = r_i^{-1}(f_0(X_{i+1} \times 0))
\]
whence by (8) the inverse of \( z \) under \( g_r f_i \) is the set \( f_j(X_{i+1} \times 1(j)) \), homeomorphic to \( X_{i+1} \) by the definition of \( f_j \). It is therefore connected according to (i) and the inequality \( n > 0 \).

But since (7) implies

\[ W = r_{i+1}^* G' = (g_r f_i)^{-1} g_{i+1}^* (G') \],

the connectedness of \( G' \) yields that of \( W \).

Let \( m > k \) be an integer. Then the set \( h_m(W) \) is open in \( F_{i+m} \) according to (10). The compacta \( F_i \) and \( F_{i+m} \) being \( LC^\infty \), their open subsets \( W \) and \( h_m(W) \), respectively, are \( LC^\infty \) too. They are also locally compact and 0-connected, since \( W \) is connected.

Furthermore, the inverse of any point under \( g_m(W) \) is that under \( g_m \), i.e., it is a single point or one of the sets \( f_j(X_{i+1} \times 1(j)) \), where \( i = 1, \ldots, m \).

As we have just asserted, all these sets are homeomorphic to \( X_{i+1} \), respectively. Hence the inverses of points under \( g_m(W) \) are \( (m-1) \)-connected ANR-sets by (i). Consequently, the inverse of any point under \( w_m \) is a single point or one of the sets

\[ h_m(f_j(X_{i+1} \times 1(j))) = f_{j+m} w_m f_j^{-1} (f_j(X_{i+1} \times 1(j)) \) = \[ f_{j+m} (X_{i+1} \times 1(j+m)) \],

where \( m = 1, \ldots, m \). These sets are homeomorphic to \( f_j(X_{i+1}) \), respectively, as we have already seen (see [4], p. 226), and so the inverses of points under \( w_m \) are \( m \)-connected ANR-sets by (ii).

The mappings \( g_m(W) \) and \( w_m \) being proper, we thus conclude from the Snake theorem (see [3], p. 604) that the homomorphism

\[ (g_m(W))_x \rightarrow (n_m(W))_x \]

is a homomorphism for every \( i = 0, \ldots, n \). Hence in view of the equality

\[ (g_m(W))_x = (w_m(W))_x \]

\( (h_m(W))_x \) is an epimorphism in all dimensions \( i = 0, \ldots, n \).

Now, every mapping \( \psi : S_i \rightarrow F_i \), where \( i = 1, 2, \ldots \) and \( i = 0, \ldots, n \), must satisfy \( f_{i+k} \leq k \) because \( V \cap F_i = 0 \) for \( i < k \). So \( f_{i+k} \leq m \), where \( m > k \). Then \( V \cap F_{i+m} = h_m(W) \), by (10). Since \( (h_m(W))_x \) is an epimorphism, there exists a mapping \( \psi : S_i \rightarrow W \) such that \( \psi \equiv h_m \) in \( h_m(W) \). It follows from (5) and (6) that \( \psi \equiv 0 \) in \( r_{i+1}^* (V') \), whence

\[ \psi \equiv h_m \psi \equiv 0 \text{ in } h_m r_{i+1}^* (V') = r_{i+1}^* h_m (V') \subset U \cap F_{i+1} = U \cap F_i \]

by (4) and (9). This completes the proof of the theorem.

Remarks. In the case where all \( f(X_i) \) are single points \( i = 1, 2, \ldots \) the theorem is a simple consequence of the Snake theorem (see [4], p. 604). Easy examples show that each of conditions (i)-(iii) is necessary. Analyzing our proof one sees that condition (a) the diameters of \( f(X_i) \) converge to zero when \( i \to \infty \), given by (iii), has been used only when we assert that the decomposition of \( f(X_i) \) into the sets \( f(X_i) \), where \( i \geq 1 \), and the remaining points, is upper seminoncilluous for every \( i = 1, 2, \ldots \) (cf. p. 227). The last statement is, however, equivalent to (a).

The following example shows that condition (a) cannot be replaced by a weaker one, namely by

(b) there is a homomorphism \( k \) of the union \( f(X_i) \cup f(X_0) \ldots \) such that the diameters of \( k(f) \) converge to zero when \( i \to \infty \).

Indeed, let \( I \) be the unit square on the plane and \( X_i \) an arc in \( I \) composed of two straight segments, one of which joins the points \( (1/2m, 1) \) and \( (1/2, 1) \) and another — the points \( (1/2n, 1) \) and \( (1/2, 1) \) for \( i = 1, 2, \ldots \). Consider the identification of points belonging to the same \( X_i \) and having the same ordinate. This determines a mapping \( f \) of \( I \) such that all \( X_i \) and \( f(X_i) \) are AR-sets (i.e., \( i = 1, 2, \ldots, i = 1 \); outside of \( X_i \cup X_0 \ldots \) and the image \( f(I) \) is not \( LC^\infty \). Here (b) holds, but (a) does not.

The preceding theorem states a local regularity of images of compacta under those mappings of special kind that are investigated here. This allows us to deduce corollaries, some of which say that also a total regularity of spaces is preserved.

COROLLARY 1. Let \( X_i \), \( f \), and \( n \) satisfy the hypotheses of the theorem (see p. 225). If \( d_{\min}(f(X_i)) \leq n \), then \( f(X_i) \) is an AR-set.

For if an \( n \)-dimensional compact metric space is \( LC^\infty \), it is an AR-set (see [3], p. 289).

COROLLARY 2. Let \( X_i \), \( f \), and \( n \) satisfy the hypotheses of the theorem. If \( x \in X_i \), then \( f \) induces an epimorphism

\[ \lambda_x : n(f(X_i), x) \rightarrow n(f(X_i), f(x)) \]

of homotopy groups for \( i = 0, 1, \ldots, n \).

In fact, according to (iii), the decomposition of \( f(X_i) \) into the sets \( f(X_i) \), where \( i = 1, 2, \ldots \), and the points belonging to \( f(X_0) \) is upper seminoncilluous. It thus induces a continuous mapping \( g(f(X_i)) \) such that the inverses of points under this decomposition, and so they are \( n \)-connected ANR-sets by (ii). Consequently, the inverse of a point under the superposition \( g(f) \) is a single point or one of the sets \( X_0 \cup X_0 \ldots \) by (iii). It is therefore an \( (n-1) \)-connected ANR-set by (1).

Since both compacta \( X_i \) and \( f(X_i) \) are \( LC^\infty \) by our theorem, it follows from the Snake theorem that

\[ g_x = g(f(X_i), f(x)) \rightarrow g(f(X_i), f(x)) \]

is a homomorphism (see [4], p. 607, Theorem 8) and

\[ (g_x)_x = n(X_i, x) \rightarrow n(f(X_i), f(x)) \]

by (4) and (9). This completes the proof of the theorem.
is an epimorphism (see [4], p. 608-609, Theorems 9 and 11) for \( l = 0, \ldots, n \). Hence \( f_* \) is an epimorphism, as \( (g/x)_* = g_* f_* \).

**Corollary 3.** Let \( X, f, u \) satisfy the hypotheses of the theorem (see p. 226). If \( X \) is \( u \)-connected, then \( f(X) \) is also \( u \)-connected.

This instantly follows from Corollary 2.

**Corollary 4.** Let \( X, f, u \) satisfy the hypotheses of the theorem. If \( X \) is \( u \)-connected and \( \operatorname{dim}(X) \leq u \), then \( f(X) \) is an AR-set.

For \( f(X) \) is an \( u \)-connected ANR-set by Corollaries 1 and 3 (see [3], p. 289).

**Corollary 5.** Let \( X, f, u \) satisfy the hypotheses of the theorem with conditions (i) and (ii) replaced by condition (iv) \( X \) and \( f(X) \) are \( u \)-connected ANR-sets for \( u = 1, 2, \ldots \).

If \( x \in X \), then \( f \) induces an isomorphism

\[ f_* : \pi_0(X, x) \cong \pi_0(f(X), f(x)) \]

of homotopy groups for \( l = 0, 1, \ldots, u \).

Indeed, taking the mapping \( g \), as previously after Corollary 2, we see that the inverses of points under \( g \) as well as under \( g/f \) are \( u \)-connected ANR-sets by (iv). It follows from the same Smale theorems that both \( g_* \) and \( (g/f)_* \) are isomorphisms, whence \( f_* = (g/f)_* g_* \) is an isomorphism.

**Corollary 6.** Let \( X, f, u \) satisfy the hypotheses of the theorem with conditions (i) and (ii) replaced by condition (iv). If \( \operatorname{dim}(X) \leq u \) and \( \operatorname{dim}(f(X)) \leq u \), then \( f \) is a homotopy equivalence, and so \( X \) and \( f(X) \) are of the same homotopy type.

In fact, the compacta \( X \) and \( f(X) \) are both ANR-sets by Corollary 1. Then \( X \) has a finite number of components \( C_1, \ldots, C_k \) each of which is a \( 0 \)-connected ANR-set with dimension less than or equal to \( u \). By (iv), all sets \( C_i \) are connected \( (i = 1, 2, \ldots) \), whence the sets \( f(C_i) \), \( f(C_k) \) are mutually disjoint according to (iii); thus they are components of \( f(X) \), i.e., \( 0 \)-connected ANR-sets with dimensions less than or equal to \( u \).

It follows that the hypotheses of the theorem (see p. 226) are satisfied for \( C_1 \) and \( f(C_i) \) instead of \( X \) and \( f \) \((j = 1, \ldots, k)\), respectively, and with conditions (i) and (ii) replaced by condition (iv). In view of the well-known fact that every \( m \)-dimensional ANR-set is dominated by an \( m \)-dimensional finite simplicial complex, we conclude from Corollary 5 and from the Whitehead theorem (see [5], p. 1133) that each \( f|C_i \) is a homotopy equivalence for \( j = 1, \ldots, k \). Hence \( f \) is also a homotopy equivalence.

**Corollary 7.** Let \( f \) be a continuous mapping of a space \( X \) and let \( X_1, X_2, \ldots \) be subsets of \( X \) satisfying condition (iii) and

(v) \( X \) and \( f(X) \) are AR-sets for \( i = 1, 2, \ldots \)

If the image \( f(X) \) has a finite dimension and \( X \) is an ANR-set (or AR-set), then \( f(X) \) is also an ANR-set (or AR-set). Further, if \( X \) is an ANR-set and both \( X \) and \( f(X) \) have finite dimensions, then they are of the same homotopy type.

Since (v) implies (iv) for every \( n = 0, 1, \ldots \), Corollary 7 directly follows from Corollaries 1, 4, and 6.

**References**


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