sionalen euklidischen Raum, während die hier ausführlich konstruierte Menge X^{m+1} in den $(2m+2)$-dimensionalen euklidischen Raum eingebettet ist. Wir haben die Menge X^{m+1} und nicht die Menge $\tilde{X}^{m+1}$ konstruiert, da die Konstruktion von $\tilde{X}^{m+1}$ komplizierter als die von X^{m+1} ist.

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Reçu par la Rédaction le 15. 3. 1962

Some mappings of ANR-sets

by

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Recently, in connection with a method of construction of ANR-sets (¹) having some paradoxical properties, K. Borsuk raised the following problem:

Given an ANR-*set* X. *Take a sequence of mutually disjoint* AR-*sets* $X_1, X_2, \ldots$ *in* X *which have diameters converging to zero. Suppose* ξ_i *is a mapping of* X_i *such that* $\xi_i(X_i)$ *is an* AR-*set for* $i = 1, 2, \ldots$ *Then the decomposition of* X *into the sets* $\xi_i^{-1}(y)$, *where* $i = 1, 2, \ldots$ *and* $y \in \xi_i(X_i)$, *and the points belonging to* $X - (X_1 \cup X_2 \cup \ldots)$, *is upper semicontinuous; thus it induces a mapping* ξ *of* X. *Is* $\xi(X)$ *an* ANR-*set?*

We shall solve the problem in the affirmative for the case when $\xi(X)$ has a finite dimension (see Corollary 7 below). This will be done by showing that $\xi(X)$ is LC^n, i.e. homotopically locally connected in dimensions up to n (see [3], p. 506), for $n = 0, 1, \ldots$ Therefore the condition that $\xi(X)$ is finitely dimensional plays an essential role in our result. It is now an open question if the above problem has an affirmative solution also for $\xi(X)$ with infinite dimension.

THEOREM. *Let* $n = 0, 1, \ldots$, *let* f *be a continuous mapping of an* ANR-*set* X *and let* $X_1, X_2, \ldots$ *be subsets of* X *such that*

(i) X_i *are* $(n-1)$-*connected* ANR-*sets* (²) *for* $i = 1, 2, \ldots$,

(ii) $f(X_i)$ *are* n-*connected* ANR-*sets for* $i = 1, 2, \ldots$,

(iii) f *is* 1-1 *on the set* $X_0 = X - (X_1 \cup X_2 \cup \ldots)$ *and* $f(X_i)$ *are mutually disjoint sets for* $i = 0, 1, \ldots$ *with diameters converging to zero as* i *tends to infinity.*

Then the image $f(X)$ *is* LC^n.

Proof. Put

$$X_\infty = X \times \{0, 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\}$$

(¹) By ANR-*set* (or AR-*set*) we understand a compact metric absolute neighbourhood (or absolute) retract.

(²) A set is said to be n-*connected* if it is homotopically connected in dimensions up to n, i.e. if all its l-dimensional homotopy groups vanish for $l = 0, 1, \ldots, n$. The (-1)-connectedness means that no condition is required.

and consider the decomposition of X_∞ into the sets $f^{-1}(y)\times\{0\}$, where $y\in f(X)$, the sets $f^{-1}(y)\times\{1/i\}$, where $y\in f(X)$ and $f^{-1}(y)\subset X_0\cup\ldots\cup X_i$, and the points belonging to $(X_{i+1}\cup X_{i+2}\cup\ldots)\times\{1/i\}$, for every $i=1,2,\ldots$ It is a decomposition of the whole space X_∞, since the sets $f(X_i)$ are mutually disjoint according to (iii), and is upper semicontinuous by the continuity of f. It thus induces a continuous mapping f_∞ of X_∞. Write

$$F_\infty = f_\infty(X_\infty)\,,$$
$$f_0 = f_\infty|X\times\{0\}\,,\qquad F_0 = f_\infty(X\times\{0\})\,,$$
$$f_i = f_\infty|X\times\{1/i\}\,,\qquad F_i = f_\infty(X\times\{1/i\})$$

for $i=1,2,\ldots$ Then the sets $F_0, F_1, \ldots$ are mutually disjoint and F_0 is homeomorphic to $f(X)$. So it is sufficient to show that F_0 is LC^n.

For every $i=1,2,\ldots$ we have $F_i=f_i(X\times\{1/i\})$, the mapping f_i is 1-1 on the set $[X-(X_1\cup\ldots\cup X_i)]\times\{1/i\}$ and the sets $X_1,\ldots,X_i$ are mutually disjoint, according to (iii). Thus their union $X_1\cup\ldots\cup X_i$ is an ANR-set, by (i). We also have $f_i\big((X_1\cup\ldots\cup X_i)\times\{1/i\}\big)=f_i(X_1\times\times\{1/i\})\cup\ldots\cup f_i(X_i\times\{1/i\})$ and the union on the right is that of mutually disjoint sets homeomorphic to the sets $f(X_1),\ldots,f(X_i)$, respectively. Thus it is also an ANR-set, by (ii). It follows from the Borsuk theorem (see [1], p. 250) that the compactum F_i is locally contractible for $i=1,2,\ldots$ Hence $F_1, F_2, \ldots$ are LC^n.

But from $\operatorname*{Lim}_{i\to\infty} X\times\{1/i\}=X\times\{0\}$ we obtain

(1) $$\operatorname*{Lim}_{i\to\infty} F_i = F_0$$

and all $F_0, F_1, \ldots$ are closed subsets of the compactum F_∞. Consequently, by virtue of the Kuratowski theorem (see [2], p. 122, Theorem A_1), in order to show that F_0 is LC^n it is enough to prove that $F_1, F_2, \ldots$ are *uniformly* LC^n (see [2], p. 121, footnote). We are going to do it.

Namely, let p be an arbitrary point of F_0, and U—an arbitrary open set in F_∞ to which p belongs. We ought to find an open subset V of F_∞ such that $p\in V$ and every continuous mapping of the l-dimensional sphere S_l $(l\leqslant n)$ into $V\cap F_i$ is homotopic in $U\cap F_i$ with a constant mapping, i.e. $\varphi\simeq 0$ in $U\cap F_i$ for every $\varphi\colon S_l\to V\cap F_i$, $i=1,2,\ldots$ and $l=0,1,\ldots,n$.

Let us first observe that since $p\in F_0$, a positive integer j exists such that

(2) $$p\in f_0(X_{j-1}\times\{0\})\,.$$

Next, we have $F_j\cap f_\infty(A)=f_\infty\big(A\cap f_\infty^{-1}(F_j)\big)=f_\infty\big(A\cap(X\times 1/j)\big)$ for every $A\subset X_\infty$, whence

(3) $$F_j\cap f_\infty(A)=f_j\big(A\cap(X\times 1/j)\big)\,.$$

Further, let $r_0\colon X_\infty\to X\times\{0\}$ be a retraction defined by the formula $r_0(x,t)=(x,0)$ for $t=0,1,\frac{1}{2},\ldots$ and take another retraction r so that the diagram

$$\begin{array}{ccc} X_\infty & \xrightarrow{r_0} & X\times\{0\} \\ {\scriptstyle f_\infty}\downarrow & & \downarrow{\scriptstyle f_0} \\ F_\infty & \xrightarrow[r]{} & F_0 \end{array}$$

be commutative, namely by putting $r=f_0r_0f_\infty^{-1}$. This is possible because the inverses of points under f_∞ are contained in those under f_0r_0.

Setting
$$r_i = r\,|\,F_i$$
for $i=1,2,\ldots,$ we get $r_j^{-1}(p)=F_j\cap r^{-1}(p)=F_j\cap f_\infty(A)$, where $A=r_0^{-1}f_0^{-1}(p)$, and $f_0^{-1}(p)=f^{-1}(\widetilde{p})\times\{0\}$, where $\widetilde{p}\in f(X_{j-1})$, according to (2) and the definition of f_0. Hence

$$r_j^{-1}(p)=f_j\big(f^{-1}(\widetilde{p})\times 1/j\big)$$

by (3), and $f^{-1}(\widetilde{p})\subset X_{j-1}$. It follows from the definition of f_j that the set $r_j^{-1}(p)$ reduces to a single point, say p'.

Now, since r is a retraction of F_∞ to F_0 and $p\in U\cap F_0$, there exist, by (1), a positive integer k such that

$$\Big(F_0\cup\bigcup_{m=k}^{\infty}F_{j+m}\Big)\cap r^{-1}(p)\subset U\,,$$

and an open subset V' of F_0 such that $p\in V'$ and

$$\Big(F_0\cup\bigcup_{m=k}^{\infty}F_{j+m}\Big)\cap r^{-1}(V')\subset U\,.$$

Since $r_{j+m}^{-1}(V')=F_{j+m}\cap r^{-1}(V')$, we get

(4) $$r_{j+m}^{-1}(V')\subset U\cap F_{j+m}$$

for $m\geqslant k$. Moreover, $r_j^{-1}(V')$ is an open subset of F_j and $\{p'\}=r_j^{-1}(p)\subset r_j^{-1}(V')$. But F_j is LC^n. Consequently, an open subset W' of F_j exists such that $p'\in W'$ and

(5) $$\psi\simeq 0 \text{ in } r_j^{-1}(V') \quad\text{for every}\quad \psi\colon S_l\to W'\,,$$

$l=0,1,\ldots,n$. We also can choose an open subset V'' of F_0 such that $p\in V''$ and $r_j^{-1}(V'')\subset W'$.

The sets $f_0(X_{j+i}\times 0)$ are homeomorphic to $f(X_{j+i})$ for $i=1,2,\ldots,$ respectively, and, by (iii), they are mutually disjoint and have diameters converging to zero as i tends to infinity. By (ii), they are compact subsets of F_0. It follows that the decomposition of F_0 into the sets $f_0(X_{j+i}\times 0)$, where $i = 1, 2, \ldots$, and the remaining points, is upper semicontinuous, and therefore it induces a continuous mapping g_0 of F_0. Since no set $f_0(X_{j+i} \times 0)$ meets $f_0(X_{j-1} \times 0)$ according to (iii), we get

$$g_0^{-1} g_0(p) = \{p\} \subset V'',$$

by (2). Hence putting

$$G = F_0 - g_0^{-1} g_0(F_0 - V''),$$

we obtain $G \subset V''$ and $p \in G$. Let G' be a component of G to which p belongs. Since the set F_0, being homeomorphic to $f(X)$, is locally connected and G is an open subset of F_0, the set G' is open in F_0. Thus the set

$$W = r_j^{-1}(G')$$

is open in F_j and

$$W \subset r_j^{-1}(G) \subset r_j^{-1}(V'') \subset W'. \tag{6}$$

Moreover, the set $g_0^{-1} g_0(F_0 - V'')$ either does not intersect or contains the set $f_0(X_{j+i} \times 0)$ for $i = 1, 2, \ldots$ Consequently, the set G and its component G' do the same, since all sets $f_0(X_{j+i} \times 0)$ are connected, by (ii). This yields

$$G' = g_0^{-1} g_0(G'), \tag{7}$$

and we see that the set W either does not intersect or contains the set $r_j^{-1} f_0(X_{j+i} \times 0)$ for $i = 1, 2, \ldots$ On the other hand, we have

$$r_j^{-1} f_0(X_{j+i} \times 0) = F_j \cap r^{-1} f_0(X_{j+i} \times 0) = F_j \cap f_\infty(B),$$

where $B = r_0^{-1} f_0^{-1} f_0(X_{j+i} \times 0) = r_0^{-1}(X_{j+i} \times 0)$ by (iii). Hence (3) applied to $A = B$ gives

$$r_j^{-1} f_0(X_{j+i} \times 0) = f_j(X_{j+i} \times 1/j) \tag{8}$$

for $i = 1, 2, \ldots$

The decomposition of F_j into the mutually disjoint sets $f_j(X_{j+i} \times 1/j)$, where $i = 1, \ldots, m$, and the remaining points, is upper semicontinuous for $m = 1, 2, \ldots$, and so it induces a continuous mapping g_m of F_j. By (8), the set W either does not intersect or contains $g_m^{-1}(y)$ for $y \in g_m(F_j)$. Then the inverses of points under $g_m|W$ coincide with the ones under g_m for $m = 1, 2, \ldots$ It follows that the mapping $g_m|W$ is *proper* in the sense of Smale (see [4], p. 604), i.e. the inverses of compact sets under $g_m|W$ are compact, for $m = 1, 2, \ldots$

Let $s_m(x, 1/j) = (x, 1/(j+m))$ for $x \in X$ and $m = 1, 2, \ldots$ We get $r_0 s_m(x, 1/j) = r_0(x, 1/j)$ and can define a continuous mapping $h_m\colon F_j \to F_{j+m}$ so that the diagram

$$\begin{array}{ccc} X \times 1/j & \xrightarrow{s_m} & X \times 1/(j+m) \\ {\scriptstyle f_j}\downarrow & & \downarrow{\scriptstyle f_{j+m}} \\ F_j & \xrightarrow[h_m]{} & F_{j+m} \end{array}$$

is commutative, namely by putting $h_m = f_{j+m} s_m f_j^{-1}$ for $m = 1, 2, \ldots$ The definition is right since the inverses of points under f_j lie in those under $f_{j+m} s_m$. Let us observe that the inverses of points under h_m are single points or subsets of $f_j(X_{j+i} \times 1/j)$, where $i = 1, \ldots, m$ (compare the definition of f_∞ and f_i, p. 226). It follows that the inverses of points under $h_m|W$ are contained in those under $g_m|W$. The formula

$$u_m = (g_m|W)(h_m|W)^{-1}$$

defines thus a continuous mapping such that the diagram

$$\begin{array}{ccc} & W & \\ {\scriptstyle h_m|W}\swarrow & & \searrow{\scriptstyle g_m|W} \\ h_m(W) & \xrightarrow[u_m]{} & g_m(W) \end{array}$$

is commutative. Moreover, $g_m|W$ being a proper mapping, u_m is the same, according to the continuity of $h_m|W$, for $m = 1, 2, \ldots$

Since $f_\infty^{-1} = f_i^{-1}$ on F_i, we get

$$r h_m(x) = f_0 r_0 f_{j+m}^{-1} f_{j+m} s_m f_j^{-1}(x) = f_0 r_0 f_j^{-1}(x) = r(x)$$

for $x \in F_j$, whence

$$r_j = r_{j+m} h_m \tag{9}$$

for $m = 1, 2, \ldots$

We now define the desired open subset V of F_∞ (see p. 226) by setting

$$V = r^{-1}(G') - (F_1 \cup \ldots \cup F_{j+k-1})$$

and verify its properties as follows.

The point p belongs to F_0, that is $p = r(p)$, and so $p \in r^{-1}(p) \subset r^{-1}(G')$. The set F_0 being disjoint with any set F_i for $i = 1, \ldots, j+k-1$, we conclude that $p \in V$. Furthermore, $V \cap F_{j+m} = r_{j+m}^{-1}(G')$ for $m \geqslant k$, whence

$$V \cap F_{j+m} = h_m h_m^{-1} r_{j+m}^{-1}(G') = h_m r_j^{-1}(G') = h_m(W) \tag{10}$$

for $m \geqslant k$, according to (9).

Now, let us point out that our theorem trivially holds for $n = 0$. We can thus assume that $n > 0$.

Then the superposition $g_0 r_j$ is a monotone mapping. Indeed, for $z \in g_0(F_0)$, the inverse $g_0^{-1}(z)$ is either 1° a single point y belonging to $f_0(X_i \times 0)$, where $i = 0, \ldots, j$, or 2° a set $f_0(X_{j+i} \times 0)$, where $i = 1, 2, \ldots$ If 1°, we have

$$(g_0 r_j)^{-1}(z) = r_j^{-1} g_0^{-1}(z) = r_j^{-1}(y)$$

and confirm, in the same way as previously for p (see p. 227) instead of y, that the set $r_j^{-1}(y)$ reduces to a single point. If 2°, we obtain

$$(g_0 r_j)^{-1}(z) = r_j^{-1} f_0(X_{j+i} \times 0),$$

whence by (8) the inverse of z under $g_0 r_j$ is the set $f_j(X_{j+i}\times 1/j)$, homeomorphic to X_{j+i} by the definition of f_j. It is therefore connected according to (i) and the inequality $n > 0$.

But since (7) implies

$$W = r_j^{-1}(G') = (g_0 r_j)^{-1} g_0(G'),$$

the connectedness of G' yields that of W.

Let $m \geqslant k$ be an integer. Then the set $h_m(W)$ is open in F_{j+m} according to (10). The compacta F_j and F_{j+m} being LC^n, their open subsets W and $h_m(W)$, respectively, are LC^n too. They are also locally compact and 0-connected, since W is connected.

Furthermore, the inverse of any point under $g_m|W$ is that under g_m, i.e. it is a single point or one of the sets $f_j(X_{j+i}\times 1/j)$, where $i = 1, \ldots, m$. As we have just asserted, all these sets are homeomorphic to X_{j+i}, respectively. Hence the inverses of points under $g_m|W$ are $(n-1)$-connected ANR-sets by (i). Consequently, the inverse of any point under u_m is a single point or one of the sets

$$h_m f_j(X_{j+i}\times 1/j) = f_{j+m} s_m f_j^{-1} f_j(X_{j+i}\times 1/j) = f_{j+m}\big(X_{j+i}\times 1/(j+m)\big),$$

where $i = 1, \ldots, m$. These sets are homeomorphic to $f(X_{j+i})$, respectively, as we have already seen (compare p. 226), and so the inverses of points under u_m are n-connected ANR-sets by (ii).

The mappings $g_m|W$ and u_m being proper, we thus conclude from the Smale theorem (see [4], p. 604) that the homomorphism

$$(g_m|W)_*\colon \pi_l(W)\to\pi_l[g_m(W)]$$

of homotopy groups induced by $g_m|W$ is an epimorphism and that

$$u_{m*}\colon \pi_l[h_m(W)]\to\pi_l[g_m(W)]$$

is a monomorphism for every $l = 0, \ldots, n$. Hence in view of the equality

$$(g_m|W)_* = u_{m*}(h_m|W)_*$$

$(h_m|W)_*$ is an epimorphism in all dimensions $l = 0, \ldots, n$.

Now, every mapping $\varphi\colon S_l\to V\cap F_i$, where $i = 1, 2, \ldots$ and $l = 0, \ldots, n$, must satisfy $j+k\leqslant i$ because $V\cap F_i = 0$ for $i<j+k$. So $i = j+m$, where $m\geqslant k$. Then $V\cap F_i = h_m(W)$ by (10). Since $(h_m|W)_*$ is an epimorphism, there exists a mapping $\psi\colon S_l\to W$ such that $\varphi\simeq h_m\psi$ in $h_m(W)$. It follows from (5) and (6) that $\psi\simeq 0$ in $r_j^{-1}(V')$, whence

$$\varphi\simeq h_m\psi\simeq 0 \quad\text{in}\quad h_m r_j^{-1}(V') = r_{j+m}^{-1}(V')\subset U\cap F_{j+m} = U\cap F_i$$

by (4) and (9). This completes the proof of the theorem.

Remarks. In the case where all $f(X_i)$ are single points ($i = 1, 2, \ldots$) the theorem is a simple consequence of the Smale theorem (see [4], p. 604). Easy examples show that each of conditions (i)-(iii) is necessary. Analyzing our proof one sees that condition

(a) the diameters of $f(X_i)$ converge to zero when $i\to\infty$,

given by (iii), has been used only when we assert that the decomposition of $f(X)$ into the sets $f(X_i)$, where $i\geqslant j$, and the remaining points, is upper semicontinuous for every $j = 1, 2, \ldots$ (cf. p. 227). The last statement is, however, equivalent to (a).

The following example shows that condition (a) cannot be replaced by a weaker one, namely by

(b) there is a homeomorphism h of the union $f(X_1)\cup f(X_2)\cup\ldots$ such that the diameters of $hf(X_i)$ converge to zero when $i\to\infty$.

Indeed, let I^2 be the unit square on the plane and X_i—an arc in I^2 composed of two straight segments one of which joins the points $(1/(2i-1), 0)$ and $(1/2i, 1)$ and another—the points $(1/2i, 0)$ and $(1/2i, 1)$ for $i = 1, 2, \ldots$ Consider the identification of points belonging to the same X_i and having the same ordinate. This determines a mapping f of I^2 such that all X_i and $f(X_i)$ are AR-sets ($i = 1, 2, \ldots$), f is 1-1 outside of $X_1\cup X_2\cup\ldots$ and the image $f(I^2)$ is not LC^1. Here (b) holds, but (a) does not.

The preceding theorem states a local regularity of images of compacta under those mappings of special kind that are investigated here. This allows us to deduce corollaries, some of which say that also a total regularity of spaces is preserved.

COROLLARY 1. *Let X, f and n satisfy the hypotheses of the theorem (see p. 225). If $\dim f(X)\leqslant n$, then $f(X)$ is an ANR-set.*

For if an n-dimensional compact metric space is LC^n, it is an ANR-set (see [3], p. 289).

COROLLARY 2. *Let X, f and n satisfy the hypotheses of the theorem. If $x\in X$, then f induces an epimorphism*

$$f_*\colon \pi_l(X, x)\to\pi_l\big(f(X), f(x)\big)$$

of homotopy groups for $l = 0, 1, \ldots, n$.

In fact, according to (iii), the decomposition of $f(X)$ into the sets $f(X_i)$, where $i = 1, 2, \ldots$, and the points belonging to $f(X_0)$, is upper semicontinuous. It thus induces a continuous mapping g of $f(X)$ such that the inverses of points under g are elements of this decomposition, and so they are n-connected ANR-sets by (ii). Consequently, the inverse of a point under the superposition gf is a single point or one of the sets $X_1, X_2, \ldots$ by (iii). It is therefore an $(n-1)$-connected ANR-set by (i). Since both compacta X and $f(X)$ are LC^n by our theorem, it follows from the Smale theorems that

$$g_*\colon \pi_l\big(f(X), f(x)\big)\to\pi_l\big(gf(X), gf(x)\big)$$

is a monomorphism (see [4], p. 607, Theorem 8) and

$$(gf)_*\colon \pi_l(X, x)\to\pi_l\big(gf(X), gf(x)\big)$$

is an epimorphism (see [4], p. 608-609, Theorems 9 and 11) for $l = 0, \ldots, n$. Hence f_* is an epimorphism, as $(gf)_* = g_* f_*$.

COROLLARY 3. *Let X, f and n satisfy the hypotheses of the theorem* (see p. 225). *If X is n-connected, then $f(X)$ is also n-connected.*

This instantly follows from Corollary 2.

COROLLARY 4. *Let X, f and n satisfy the hypotheses of the theorem. If X is n-connected and $\dim f(X) \leqslant n$, then $f(X)$ is an* AR*-set.*

For $f(X)$ is an n-connected ANR-set by Corollaries 1 and 3 (see [3], p. 289).

COROLLARY 5. *Let X, f and n satisfy the hypotheses of the theorem with conditions* (i) *and* (ii) *replaced by condition*

(iv) *X_i and $f(X_i)$ are n-connected* ANR*-sets for $i = 1, 2, \ldots$*

If $x \in X$, then f induces an isomorphism

$$f_*: \pi_l(X, x) \approx \pi_l(f(X), f(x))$$

of homotopy groups for $l = 0, 1, \ldots, n$.

Indeed, taking the mapping g, as previously after Corollary 2, we see that the inverses of points under g as well as under gf are n-connected ANR-sets by (iv). It follows from the same Smale theorems that both g_* and $(gf)_*$ are isomorphisms, whence $f_* = g_*^{-1}(gf)_*$ is an isomorphism.

COROLLARY 6. *Let X, f and n satisfy the hypotheses of the theorem with conditions* (i) *and* (ii) *replaced by condition* (iv). *If $\dim X \leqslant n$ and $\dim f(X) \leqslant n$, then f is a homotopy equivalence, and so X and $f(X)$ are of the same homotopy type.*

In fact, the compacta X and $f(X)$ are both ANR-sets by Corollary 1. Then X has a finite number of components $C_1, \ldots, C_k$ each of which is a 0-connected ANR-set with dimension less than or equal to n. By (iv), all sets X_i are connected $(i = 1, 2, \ldots)$, whence the sets $f(C_1), \ldots, f(C_k)$ are mutually disjoint according to (iii); thus they are components of $f(X)$, i.e. 0-connected ANR-sets with dimensions less than or equal to n. It follows that the hypotheses of the theorem (see p. 225) are satisfied for C_j and $f|C_j$ instead of X and f $(j = 1, \ldots, k)$, respectively, and with conditions (i) and (ii) replaced by condition (iv). In view of the well-known fact that every m-dimensional ANR-set is dominated by an m-dimensional finite simplicial complex, we conclude from Corollary 5 and from the Whitehead theorem (see [5], p. 1133) that each $f|C_j$ is a homotopy equivalence for $j = 1, \ldots, k$. Hence f is also a homotopy equivalence.

COROLLARY 7. *Let f be a continuous mapping of a space X and let $X_1, X_2, \ldots$ be subsets of X satisfying condition* (iii) *and*

(v) *X_i and $f(X_i)$ are* AR*-sets for $i = 1, 2, \ldots$*

If the image $f(X)$ has a finite dimension and X is an ANR*-set* (*or* AR*-set*), *then $f(X)$ is also an* ANR*-set* (*or* AR*-set*). *Further, if X is an* ANR*-set and both X and $f(X)$ have finite dimensions, then they are of the same homotopy type.*

Since (v) implies (iv) for every $n = 0, 1, \ldots$, Corollary 7 directly follows from Corollaries 1, 4 and 6.

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Reçu par la Rédaction le 18. 4. 1962