Abstract covering theorems

by

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In [2] Karl Menger obtained relation-theoretic generalizations of the covering theorem for separable subsets of a topological space and the covering theorem for compact subsets of a separable space by applying the logic of relations to the proofs of these theorems given by Kuratowski and Sierpiński in [1]. Theorems II_1 and II_2 of [2] are intended to provide the abstract version of the covering theorem for separable sets; however, Theorem II_2 is not quite correct. It is our purpose to provide a correct version (Theorem 1 and its corollary) and to show how this may be used to obtain new, non-topological results: an interesting maximal principle for partially ordered sets (Theorem 2) and a condition guaranteeing that a group and each of its subsets is finitely generated (Theorem 3). These two applications indicate the broad scope of this relation-theoretic method.

We shall make use of some of the notions defined in [2] and which are given below.

Let A and B be two non-empty sets and R a binary relation defined between the elements of A and the elements of B so that for each $a \in A$ there exists some $b \in B$ such that aRb. We define a relation $\mathcal{R}$ between the subsets of A and of B: If $A' \subseteq A$ and $B' \subseteq B$ then $A'\mathcal{R}B'$ if and only if for each $a \in A'$ there is some $b \in B'$ such that aRb. If $A' \subseteq A$ and $b \in B$, then $A'(b)$ will be the set of all elements $a \in A'$ such that aRb.

Let $\mathcal{A}$ be a family of subsets of A and $\mathcal{B}$ a family of subsets of B. We shall say that a subset A' of A has the *covering property* (property M of [2]) if every subset B' of B such that $A'\mathcal{R}B'$ contains a subset B'' such that both $A'\mathcal{R}B''$ and $B'' \in \mathcal{B}$. A subset A' of A has the *condensation property* (property CII of [2]) if in every subset A'' of A' such that $A'' \notin \mathcal{A}$ there is an element p such that for every $b \in B$, pRb implies that $A''(b) \notin \mathcal{A}$.

We shall employ the following hypotheses:

H_1: *If A' and B' are of the same power relative to R (by which we mean that there is a one-to-one correspondence γ between A' and B' such that if $a \in A'$, $b \in B'$, and $\gamma(a) = b$, then aRb) and if $A' \in \mathcal{A}$ then $B' \in \mathcal{B}$.*

H_2: *The empty set φ belongs to $\mathcal{A}$.*

H_3: *If $A' \in \mathcal{A}$ and $a \in A$, then $A' \cup \{a\}$ belongs to $\mathcal{A}$.*

THEOREM 1. *Under hypotheses* H_1, H_2, *and* H_3, *if* A' *has the condensation property, then* A' *has the covering property.*

COROLLARY. *Under hypotheses* H_1, H_2, *and* H_3, *if* A' *has the condensation property, then every subset of* A' *has the covering property.*

This theorem and its corollary correspond, respectively, to Theorem II_2 of [2] and its corollary, with the exception that in [2] hypothesis H_3 is not present. The following simple example shows, however, that if we omit H_3, the conclusion of Theorem 1 need not hold. Let $A = B = \{a, b\}$. Define the relation R by aRa and bRb, so that $A\mathcal{R}B$. Set $\mathcal{A} = \{\varphi\}$ and $\mathcal{B} = \{\varphi, \{a\}, \{b\}\}$. It is easy to verify that H_1 and H_2 are satisfied but not H_3 and that A has the condensation property. Yet, there is no set $B' \in \mathcal{B}$ such that $A\mathcal{R}B'$; i.e., A does not have the covering property. Once we add the hypothesis H_3, as above, the proofs given in [2] carry through with only minor corrections and, therefore, we shall not reproduce them here.

We now apply Theorem 1 to non-topological situations and prove a maximal principle and a theorem on groups.

THEOREM 2. *Let* R *be an ordering (i.e., an anti-reflexive, transitive, binary relation) of a non-empty set* S *with the following property: every infinite subset* $S' \subseteq S$ *contains an element* p *such that if* pRs *for any element* $s \in S$, *then infinitely many elements of* S' *are in the* R*-relation with* s. *Then* S *contains a maximal element (i.e., an element* m *such that* mRs *for no element* $s \in S$).

Proof. Apply Theorem 2 by letting $A = A'$ be the set of all elements $a \in S$ such that for some $s \in S$, aRs. If $s \in S - A'$, then s is a maximal element. If A' is empty then every element of S is maximal. Suppose that A' is not empty. Let $B = S$ and let the relation R of Theorem 2 be the ordering R of S. Clearly $A'\mathcal{R}B$. Let $\mathcal{A}$ be the family of all finite (or empty) subsets of A' and let $\mathcal{B}$ be the family of all finite (or empty) subsets of B. Each hypothesis H_1, H_2, H_3 is satisfied. The property assumed in Theorem 4 is equivalent to the condensation property. Hence, by Theorem 1, S must have the covering property; i.e., whenever $A'\mathcal{R}S'$, for any subset $S' \subseteq S$, there is a finite subset $S'' \subseteq S'$ such that $A'\mathcal{R}S''$. In particular, since $A'\mathcal{R}S$, there is a finite subset $F \subseteq S$ such that $A'\mathcal{R}F$. Clearly F is not empty. From $F \subseteq A'$ it would follow that fRf for some element $f \in F$ which, however, contradicts the assumption that R is anti-reflexive. Therefore there is some $m \in F$ such that $m \in S - A'$. This element m is then a maximal element in S.

In the group-theoretic application we denote, for any subset S of a group G, by $G:S$ the subgroup of G generated by S. Every subset $G' \subseteq G:S$ (even if G' is not a group) will be said to be *generated by* S. If a subset $G' \subseteq G$ is generated by a finite subset of G' then G' will be called *finitely generated.*

THEOREM 3. *In order that each subset of a group* G *(in particular,* G *itself) be finitely generated it is sufficient that each infinite subset* $G' \subseteq G$ *contain an element* p *with the following property: If* $p \in G:F$, *where* F *is any finite subset of* G, *then* $G:F$ *includes infinitely many elements of* G'.

Proof. We shall apply the Corollary of Theorem 1 by letting $A = G$ and letting B be the family of all finite subsets of G, including the empty set. Let $\mathcal{A} = B$ and let $\mathcal{B}$ be the family of all finite subsets of B. As in the previous theorem, each of the hypotheses H_1, H_2, and H_3 is satisfied. We now consider the case where $A' = A$ and $B' = B$. By aRb we mean that $a \in G:b$. Thus $A''\mathcal{R}B''$ means that $A'' \subseteq \bigcup_{b \in B''} G:b$. (Here, of course, we mean the set theoretic and not the group theoretic union.) Hence, $A''\mathcal{R}B''$ implies that $\bigcup_{b \in B''} b$ generates A'. The property assumed in Theorem 3 is equivalent to the condensation property for the group G. Therefore, by the Corollary of Theorem 1, G and each of its subsets has the covering property; i.e., if $G' \subseteq G$ and $\mathcal{S}$ is any family of finite subsets of G such that $G' \subseteq \bigcup_{S \in \mathcal{S}} G:S$, then there exists a finite subfamily $\mathcal{S}' \subseteq \mathcal{S}$ such that $G' \subseteq \bigcup_{S \in \mathcal{S}'} G:S$. In particular, if $\mathcal{S}$ is the family of all finite subsets of G', then $G' \subseteq \bigcup_{S \in \mathcal{S}} G:S$. Hence there is a finite subfamily $\mathcal{S}' \subseteq \mathcal{S}$ such that $G' \subseteq \bigcup_{S \in \mathcal{S}'} G:S$. It follows that G' is generated by $\bigcup_{S \in \mathcal{S}'} S$. Since each set S is finite and $\mathcal{S}'$ is a finite family, $\bigcup_{S \in \mathcal{S}'} S$ is finite and, moreover, a subset of G'. Hence G' is finitely generated. Thus a direct application of the Corollary of Theorem 1 yields that each subset $G' \subseteq G$ (in particular, G itself) is finitely generated.

References

[1] C. Kuratowski and W. Sierpiński, *Le théorème de Borel-Lebesque dans la théorie des ensembles abstraits*, Fund. Math. 2 (1921), pp. 172-178.

[2] K. Menger, *An abstract form of the covering theorems of topology*, Ann. of Math. 39 (1938), pp. 794-803.

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