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Theorem 6.4 can be improved. We note that in the case of a denumerable locally finite algebra $A$ we can take $n = \omega$, and that $n = 2^n$ is always a solution of the equation \( \sum_{k=0}^{n} k^n = n \).

References


A complete first-order logic with infinitary predicates

by

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It is well known that the first order predicate logic (with or without an identity symbol) has the following two properties:

$(*)$ each proof involves only finitely many formulas;

$(**)$ a set of formulas is consistent if and only if it is satisifiable (Gödel Completeness Theorem) $^1$.

In this paper we shall (in § 1) introduce a formal system $L$ which has predicates with infinitely many argument places and quantifiers over infinite sets of variables, but which has only finitary propositional connectives and no identity symbol, and which satisfies $(*)$. The system $L$ is patterned after the finitary first-order system $F_1$ of Church in [1], and our notion of satisfaction is the natural extension of Tarski's definition (e.g. in [26], p. 193). Our main result, the Completeness Theorem (Theorem 3.1), is that $L$ also satisfies $(**)$ $^2$. The methods of proof are based upon the proofs of Henkin, and of Rasiowa and Sikorski, of the Gödel Completeness Theorem.

Generalisations of the Löwenheim-Skolem Theorem and of the Compactness Theorem to $L$ (in § 3) will follow easily from 3.1. It is to be expected that many of the other familiar applications of the Gödel Completeness Theorem to first-order theory of models will eventually be generalized to the theory of models of the system $L$.

In § 4 we shall give some examples which indicate the difficulties encountered when one attempts to make various improvements of our main result.

In § 5 we shall introduce a more general formal system $L^0$ which has, in addition to the expressions of $L$, functions and terms with in-

\footnote{1} See [5], [9], [17], [26], and [31].

\footnote{2} The main results of this paper were announced in abstracts [14].

\footnote{3} For an expository discussion of several applications of the Gödel completeness theorem, and for a historical account and references, we refer to [18]. For related results concerning the infinitary logics of $L$—which do not have property $(*)$—see [8] and [28]. We shall not here be concerned with the systematic development of the theory of models for $L$ in the spirit of [31] or of [28].
We shall make free use of the Axiom of Choice, often in the form that any set can be well-ordered.

This paper is self-contained except for the Appendix, where we assume a familiarity with [3].

The author has benefited from discussions with C. C. Chang, Leon Henkin, and Donald Monk in connection with this paper.

**Terminology and notation.** The symbols $\subseteq, \cup, \cap, \vee, \wedge$ have their usual set-theoretic significance. The one-element set containing $x$ is denoted by $\{x\}$. $\mathbb{X} \rightarrow \mathbb{Y}$ is the set-theoretic difference of $\mathbb{X}$ and $\mathbb{Y}$. Ordered pairs and ordered triples will be written $\langle x, y \rangle$ and $\langle x, y, z \rangle$ respectively; in practice, we shall often omit the commas, thus $\langle xy \rangle$ and $\langle xyz \rangle$, and rely on the context to make the meaning clear. We shall denote by $\mathbb{Y}^X$ the set of all functions $f$ on $X$ into $\mathbb{Y}$; thus if $f \in \mathbb{Y}^X$, we have $f = \{\langle x, f(x) \rangle \mid x \in X\}$. If $f$ and $g$ are functions with domains $X$ and $Y$ respectively, we write

$$g \circ f = \{\langle x, g(f(x)) \rangle \mid x \in X \text{ and } f(x) \in Y\}.$$ 

If $f$ is a one-to-one function, we write

$$f^{-1} = \{\langle y, x \rangle \mid \langle x, y \rangle \in f\}.$$ 

For any function $f \in \mathbb{Y}^X$ and any set $Z$, we shall write $f[Z] = \{f(x) \mid x \in X \cap Z\}$, and

$$f \upharpoonright Z = \{\langle x, f(x) \rangle \mid x \in X \cap Z\} \cup \{\langle x, y \rangle \mid x \in Z
\setminus X\}.$$ 

Thus $f[Z]$ always has domain $Z$, and $\upharpoonright Z$ is the identity function on $Z$.

We assume that the ordinals are defined in such a way that each ordinal coincides with the set of all smaller ordinals. We shall use the small Greek letters $\alpha, \beta, \gamma, \ldots$ to denote arbitrary ordinals. The sum of the ordinals $\alpha, \beta$ shall be denoted by $\alpha + \beta$. The smallest infinite ordinal is denoted by $\omega$, and the finite ordinals are identified with the natural numbers. Thus 0 is the empty set. The letters $\kappa, \lambda, \mu, \nu, \ldots$ denote arbitrary natural numbers.

We shall denote arbitrary cardinal numbers by small German letters $m, n, p, \ldots$. The sum of the system $(m_i)_{i \in \mathcal{I}}$ of cardinals is denoted by $\sum m_i$, and the sum of the two cardinals $m, n$ is denoted by $m + n$. The cardinal $m$ to the $n$th power is denoted by $m^n$; we rely on context to distinguish the cardinal power from the function set $\mathbb{Y}^X$. The power, or cardinality, of a set $X$ is denoted by $|X|$, thus $\mathbb{Y}^X = |X|^{|X|}$. Recall that $\aleph_0 = \kappa$. The smallest cardinal which is greater than $m$ is denoted by $m^+$.

For any set $X$ and cardinal $m$, we define

$$S_m(X) = \{x \mid x \subseteq X \text{ and } \aleph_m \not\in m\}.$$
§ 1. A formal system with infinitary predicates. We shall begin by introducing some purely syntactical notions, i.e., notions which do not deal with our intended interpretation of the infinite logic, but only with its formal structure.

We shall construct our formal system $L$ within the underlying set theory. By a symbol we shall mean any set which is neither an ordered pair nor an ordered triple.

Let $V$ and $P$ be two disjoint sets of symbols, $\mu$ a function whose domain is $P$ and whose range is a set of ordinals, and $m$ a cardinal.

Throughout this paper we shall assume that $V$, $\mu$, and $m$ satisfy the following conditions (*):

1. $V$ is infinite;
2. $m \leq V$;
3. For each $p \in P$, $\mu(p) \leq V$.

We construct the logic $L(V, \mu, m)$, or more briefly $L$, in the following way.

The symbols of $L$ include:

- the implication symbol $\rightarrow$;
- the falsity symbol $\top$;
- the universal quantification symbol $\forall$;
- the (individual) variables $v \epsilon V$;
- the predicates $p \epsilon P$.

We shall assume that the symbols $\rightarrow$, $\top$, $\forall$ are not members of $V$ or of $P$.

By an atomic formula in $L$ we mean an ordered pair $\langle \rho, x \rangle$, where $\rho \in P$ and $x \epsilon \mu(p)$.

$\phi$ is said to be a formula in $L$ if there exists a finite sequence $\phi_0, \phi_1, \ldots, \phi_n$, called a formation of $\phi$ in $L$, such that $\phi_0 \rightarrow \phi$ and, for each $m \leq n$, at least one of the following holds:

1. $\phi_0$ is an atomic formula in $L$;
2. $\phi_0 = \top$;
3. For some $k < m$, $\phi_0$ is the ordered triple $\langle \phi_k \rightarrow \phi_l \rangle$;
4. For some $k < m$ and $W \epsilon \mu(V)$, $\phi_0$ is the ordered triple $\langle \forall W \phi \rangle$.

It follows at once, from the fact that $V$, $\rho$, and $\langle V, \rightarrow, \top \rangle$ are disjoint sets of symbols, that for each $m \leq n$ exactly one of (1.1)-(1.4) hold.

The set of all formulas in $L$ is denoted by $F$.

**Lemma 1.1. (Induction principle).** Suppose that $G$ contains every atomic formula in $L$, that $\phi \epsilon G$, that $\langle \forall W \phi \rangle \epsilon G$ whenever $\phi \epsilon G$ and $W \epsilon \mu(V)$, and that $\langle \forall W \phi \rangle \epsilon G$ whenever $\phi \epsilon G$ and $W \epsilon \mu(V)$. Then $F \subseteq G$.

(*) Although these conditions are not required for the construction of the formal system $L$, they are required for most of our results. An indication of the difficulties which are encountered when assumption I is removed is given (for polyadic algebras of finite degree) by Monk in [19].

**Proof.** Suppose that $\phi_0, \ldots, \phi_n$ is a formation in $L$, and every formula which has a formation of length less than $n$ belongs to $G$. Then $\phi_n \epsilon G$ for each $k < n$. If $n = 1$, then either $\phi_0$ is atomic or $\phi_0 = \top$, so $\phi_0 \epsilon G$. If $\phi_0 = \langle \forall W \phi \rangle$ for some $k < n$, then $\phi_0, \phi_1 \epsilon G$, so $\phi_n \epsilon G$. If $\phi_0 = \langle \forall W \phi \rangle$ for some $k < n$ and $W \epsilon \mu(V)$, then $\phi_0 \epsilon G$, so $\phi_n \epsilon G$. It follows by induction that $P \subseteq G$.

The set $V(\phi)$ of free variables of a formula $\phi \epsilon F$ is defined recursively as follows:

1. If $\phi$ is an atomic formula $\langle \rho, x \rangle$, then $V(\phi)$ is the range of $\rho$;
2. If $\phi = \top$, then $V(\phi) = \emptyset$;
3. If $\phi = \langle \forall W \phi \rangle$, then $V(\phi) = V(\phi) \cup V(\top)$;
4. If $\phi = \langle \forall W \phi \rangle$, then $V(\phi) = V(\phi) \cup W$.

Similarly, the set $V_0(\phi)$ of bound variables of $\phi$ is defined by:

1. If $\phi$ is atomic, then $V_0(\phi) = \emptyset$;
2. If $\phi = \top$, then $V_0(\phi) = \emptyset$;
3. If $\phi = \langle \forall W \phi \rangle$, then $V_0(\phi) = V_0(\phi) \cup V_0(\phi)$;
4. If $\phi = \langle \forall W \phi \rangle$, then $V_0(\phi) = V_0(\phi) \cup W$.

Finally, the set $V(\phi)$ of all variables of $\phi$ is defined by:

$\langle V(\phi) = V(\phi) \cup V_0(\phi) \rangle$.

For each $\tau \in V^\tau$ and $\phi \epsilon F$, we define $S(\tau)(\phi)$ (which is intuitively the result of substituting each variable $v \epsilon \phi$ by $\tau(v)$) recursively as follows:

1. If $\phi$ is an atomic formula $\langle \rho, x \rangle$, then $S(\tau)(\phi) = \langle \rho, \tau x \rangle$;
2. If $\phi = \top$, then $S(\tau)(\phi) = \top$;
3. If $\phi = \langle \forall W \phi \rangle$, then $S(\tau)(\phi) = S(\tau)(\phi)$;
4. If $\phi = \langle \forall W \phi \rangle$, then $S(\tau)(\phi) = S(\tau)(W \phi)$.

Similarly, for each $\tau \in V^\tau$ and $\phi \epsilon F$, we define $S(\tau)(\phi)$ (intuitively the result of substituting each free variable $v \epsilon \phi$ by $\tau(v)$) recursively as follows:

1. If $\phi$ is an atomic formula $\langle \rho, x \rangle$, then $S(\tau)(\phi) = \langle \rho, \tau x \rangle$;
2. If $\phi = \top$, then $S(\tau)(\phi) = \top$;
3. If $\phi = \langle \forall W \phi \rangle$, then $S(\tau)(\phi) = S(\tau)(\phi)$;
4. If $\phi = \langle \forall W \phi \rangle$, then $S(\tau)(\phi) = S(\tau)(W \phi)$, where $a = \tau(\langle V^\tau \rangle \phi)$.
LEMMA 1.2. If \( \Phi \in F \) and \( \tau \in V' \), then \( S(\tau)\Phi \in F \) and \( S(\tau)\Phi \in F \).

Proof. By 1.1.

LEMMA 1.3. Suppose \( W, X \subseteq Y \), \( W \cap X = 0 \), \( \sigma \in V'' \), \( \tau \in (V' \setminus W)' \), and \( \Phi \in F \). Then

\[
S(\sigma \cup \tau)\Phi = S(\sigma)S(\tau)\Phi, \quad \text{and} \quad S(\sigma \cup \tau)\Phi = S(\sigma)S(\tau)\Phi.
\]

Proof. By 1.1.

We shall now give the rules of inference and the axioms schemata for \( L \).

RULE 1: From \( \Phi, \langle \Phi \rightarrow \Psi \rangle \), infer \( \Psi \), whenever \( \Phi, \Psi \in F \). (Modus ponens.)

RULE 2: From \( \Phi, \Psi \), infer \( \langle \Psi W \Phi \rangle \), whenever \( \Phi, \Psi \in F \). (Rule of generalization.)

RULE 3: From \( S(\tau)\Phi \), infer \( \langle \Psi W \Phi \rangle \), whenever \( \Phi \in F \), \( \tau \in (V' \setminus \Psi)'^{(\sigma)} \), and \( \tau \) is one to one. (Rule of substitution.)

RULE 4: From \( \Phi, \Psi \), infer \( S(\tau)\Phi \), whenever \( \Phi \in F \), \( \tau \in (V' \setminus \Psi)'^{(\sigma)} \), and \( \tau \) is one to one. (Rule of substitution.)

AXIOM 1: \( \langle \Phi \rightarrow (\Psi \rightarrow \Phi) \rangle \), whenever \( \Phi, \Psi \in F \).

AXIOM 2: \( \langle (\Phi \rightarrow (\Psi \rightarrow \Phi)) \rightarrow (\Phi \rightarrow \Psi) \rangle \), whenever \( \Phi, \Psi, \theta \in F \).

AXIOM 3: \( \langle (\Phi \rightarrow (\Psi \rightarrow \Phi)) \rangle \), whenever \( \Phi \in F \).

AXIOM 4: \( \langle \Psi W (\Phi \rightarrow \Psi') \rangle \rightarrow \langle \Phi \rightarrow (\Psi W \Phi') \rangle \), whenever \( \Phi, \Psi \in F \) and \( W \in S(\tau)(V' \setminus \Psi) \).

AXIOM 5: \( \langle \Psi W \Phi \rangle \rightarrow S(\tau)\Phi \), whenever \( \langle \Psi W \Phi \rangle \), \( \Phi \in F \), and \( \tau \in (V' \setminus \Psi)'^{(\sigma)} \).

By an axiom of \( L \) we mean any formula which is an instance of one of the axioms schemata 1.5 above.

By a proof of \( \Phi \) in \( L \) we mean a finite sequence of formulas \( \Phi_1, ..., \Phi_n \) in \( L \) such that \( \Phi_n = \Phi \) and, for each \( m \leq n \), one of the following holds:

(6.1) for some \( k \), \( 1 \leq m \), \( \Phi_m \) is inferred from \( \Phi_k \), \( \Phi_i \) by Rule 1;

(6.2) for some \( k < m \), \( \Phi_m \) is inferred from \( \Phi_k \) by Rule 2;

(6.5) for some \( k < m \), \( \Phi_m \) is inferred from \( \Phi_k \) by Rule 5;

(6.4) for some \( k < m \), \( \Phi_m \) is inferred from \( \Phi_k \) by Rule 4;

(6.5) \( \Phi_0 \) is an axiom of \( L \).

\( \Phi \) is said to be a theorem in \( L \) in symbols \( \vdash L \Phi \), if there exists a proof of \( \Phi \) in \( L \).

LEMMA 4. Suppose that \( \Phi_k, ..., \Phi_n \) is a proof in \( L \), \( X = V(\Phi_0) \cup ... \cup V(\Phi_n) \), and \( \tau \in V' \) is one to one. Then \( S(\tau)\Phi_k, ..., S(\tau)\Phi_n \) is also a proof in \( L \).

Proof. This results from the following easily verified facts, where \( V(\Phi) \cup V(\Psi) \cup V(\theta) \subseteq X \):

if \( \Phi \) is inferred from \( \Psi, \theta \) by Rule 1, then \( S(\tau)\Phi \) is inferred from \( S(\tau)\Psi, S(\tau)\theta \) by Rule 1;

if \( n \in \{2, 3, 4\} \) and \( \Phi \) is inferred from \( \Psi \) by Rule 1, then \( S(\tau)\Phi \) is inferred from \( S(\tau)\Psi \) by rule 1;

if \( n \in \{1, 2, 3, 4, 5\} \) and \( \Phi \) is an instance of Axiom 5, then \( S(\tau)\Phi \) is an instance of Axiom 5.

THEOREM 3. Suppose \( m < V \) and, for each \( p \in F \), \( \mu(p) < V \). Then any theorem in \( L \) has a proof which does not use Rule 4, i.e., a proof \( \Psi_1, ..., \Psi_n \) in which either \( (6.1), (6.2), (6.3), \) or \( (6.5) \) holds for each \( k \leq n \).

Proof. It is sufficient to prove that if \( (\Phi) \) has a proof \( \Psi_1, ..., \Psi_n \) in \( L \) which does not use Rule 4, and if \( \tau \in V(\Phi') \) is one to one, then \( S(\tau)\Phi \) has a proof in \( L \) which does not use Rule 4. It follows from the hypotheses that \( V(\Phi') \subseteq V \). Therefore there is a one to one function \( \sigma \in V' \) such that \( \tau \subseteq \sigma \) and \( \sigma \) is a constant. Clearly \( S(\tau)\Phi = S(\tau)\Phi \). Then by Lemma 1.4 and its proof, \( S(\tau)\Psi_1, ..., S(\tau)\Psi_n \) is a proof of \( S(\tau)\Phi \) in \( L \) which does not use Rule 4.

Notice the analogy between the notions of formation and formula and those of proof and theorem. An important feature of the logic \( L \) is the fact that all formations and all proofs are finite, although it is possible for infinitely many variables to occur in a single formula in \( L \).

Rule 1 and Axioms 1, 2, 3 form the basis of the classical finitary propositional logic, e.g., as developed in [1]. In case \( m = 2 \) and all of the values of \( \mu \) are finite, \( L \) is a finitary first order predicate logic. In case \( m \leq 1 \) and all values of \( \mu \) are zero, \( L \) is a finitary propositional logic.

As we have pointed out in the introduction, the system \( L \) is patterned after Church's system \( F_k \). There are, however, certain differences. The rule of substitution, Rule 4, has no counterpart in \( F_k \), and is included here to cope with the case in which the hypotheses of 1.5 are not satisfied. Note that for the ordinary formal order logic, Theorem 1.5 is applicable and Rule 4 may be eliminated. In order to avoid a rather complicated condition involving the substitution of variables, we have stated Axiom 3 in a weaker, but simpler, form than the corresponding axiom of \( F_k \), and to fill the gap we have included an additional rule of inference—the rule of free substitution, Rule 3.

For each \( \Phi, \Psi \in F \), let \( \langle \Phi \rightarrow (\Psi \rightarrow \Phi) \rangle \) denote the formula \( \langle \Phi \rightarrow (\Psi \rightarrow \Phi) \rangle \), which is called the conjunction of \( \Phi \) and \( \Psi \). The familiar commutative and associative laws for conjunction are provable in \( L \). If \( \Phi_1, ..., \Phi_n \in F \),
let $(\Phi_1 \land \cdots \land \Phi_n)$ denote the formula $((\Phi_1 \land \cdots \land \Phi_{n-1}) \land \Phi_n)$ if $n > 1$, the formula $\Phi_2$ if $n = 1$, and the formula $\top$ if $n = 0$.

We shall similarly introduce the other usual propositional connectives and quantifiers in the following definitions. For each $\Phi, \Psi \in F$, let

\[
\begin{align*}
(\Phi \lor \Psi) & \neg \neg (\neg \Phi \lor \Psi); \\
(\Phi \rightarrow \Psi) & \neg \neg (\neg \Phi \lor \Psi); \\
(\neg \Phi) & \neg \neg (\neg \neg \Phi); \\
(\exists \Phi \Psi) & \neg \neg (\exists \neg \Psi \neg \neg \Phi).
\end{align*}
\]

We shall write $\Gamma \vdash \neg \neg \Phi$ if $\Gamma \subseteq F$, $\Psi \in F$, and there exist $\Phi_1, \cdots, \Phi_n \in \Gamma$ such that $\neg \neg (\neg \Phi_1 \lor \cdots \lor \neg \Phi_n) \rightarrow \neg \neg \Phi$. A subset $\Gamma$ of $F$ is said to be inconsistent in $L$ if $\Gamma \vdash \neg \neg \top$, and otherwise $\Gamma$ is said to be consistent in $L$.

We shall now introduce some semantical, or model-theoretic, notions, i.e. notions dealing with our intended interpretation of the logic $L$.

By a structure of type $\mu$ we mean a system $\mathcal{A} = (\mathbb{A}, R_0)_{\mu \in \mathbb{P}}$ in which $\mathbb{A}$ is a non-empty set and, for each $p \in \mathbb{P}$, $R_p \subseteq \mathbb{A}^{X_p}$. $\mathcal{A}$ is said to be of power $n$ if the set $\mathbb{A}$ is of power $n$.

Hereafter $\mathcal{A}$ will always denote a structure $\mathcal{A} = (\mathbb{A}, R_0)_{\mu \in \mathbb{P}}$ of type $\mu$. Let $a \in \mathbb{A}^\mu$. The notion of a formula $\Phi$ being satisfied by $a$ in $\mathcal{A}$, in symbols $\models_{\mathcal{A}} \Phi[a]$, is defined recursively as follows:

(1) If $\Phi$ is a atomic formula $\langle px \rangle$, then $\models_{\mathcal{A}} \Phi[a]$ if $a = x \in R_p$;

(2) If $\Phi = \neg \Phi$, then $\models_{\mathcal{A}} \Phi[a]$;

(3) If $\Phi = (\Phi_1 \rightarrow \Phi_2)$, then $\models_{\mathcal{A}} \Phi[a]$ if either $\models_{\mathcal{A}} \Phi_1[a]$ or not $\models_{\mathcal{A}} \Phi_2[a]$;

(4) If $\Phi = (\exists x \Psi)$, then $\models_{\mathcal{A}} \Phi[a]$ if, for every $b \in \mathbb{A}^\mu$ such that $b \models (\exists x \Psi)[b]$, we have $\models_{\mathcal{A}} \Psi[b]$.

A formula $\Phi$ is said to be satisfiable in $\mathcal{A}$ if there exists a function $a : \mathbb{A}^\mu$ such that $\models_{\mathcal{A}} \Phi[a]$. Similarly, at $\Gamma$ a set of formulas is said to be satisfiable in $\mathcal{A}$ if there exists a function $a : \mathbb{A}^\mu$ such that, for each $\Phi \in \Gamma$, we have $\models_{\mathcal{A}} \Phi[a]$. $\Gamma$ is said to be satisfiable if it is satisfiable in some structure of type $\mu$.

A formula $\Phi$ is said to be valid in $\mathcal{A}$ if, for every function $a : \mathbb{A}^\mu$, we have $\models_{\mathcal{A}} \Phi[a]$. $\Phi$ is said to be valid if it is valid in every structure of type $\mu$.

We shall now state without proof a number of simple results which are analogous to familiar results in the ordinary predicate logic.

**Theorem 1.6.** Suppose that $\Phi, \Psi \in F$, $\Gamma \subseteq F$, $\mathcal{A}$ is a structure of type $\mu$, and $a \in \mathbb{A}^\mu$. Then

(i) $\models_{\mathcal{A}} (\Phi \lor \Psi)[a]$ if and only if $\models_{\mathcal{A}} \Phi[a]$ and $\models_{\mathcal{A}} \Psi[a]$.

(ii) If $\Phi \rightarrow \Psi$ and $\tau$ is one to one, then $\models_{\mathcal{A}} \Phi[\tau(a)]$ if and only if $\models_{\mathcal{A}} \Phi[a]$.

(iii) If $\models_{\mathcal{A}} (\neg \neg \Phi)[a]$ then $\models_{\mathcal{A}} \Phi[\tau(a)]$ if and only if $\models_{\mathcal{A}} \Phi[a]$.

(iv) $\Phi$ is satisfiable if and only if $\neg \neg \Phi$ is not valid.

(v) If $\models \neg \neg \Phi$, then $\Phi$ is valid.

(vi) If $\Gamma$ is satisfiable, then $\Gamma$ is consistent in $L$.

(vii) The empty set of formulas is consistent, i.e., it is not the case that $\models \neg \neg \top$.

(viii) If $\Gamma \models \neg \neg \Phi$ and $\models \neg \neg \theta[b]$ for each $b \in \mathcal{A}$, then $\models_{\mathcal{A}} \Phi[a]$.

(ix) If $\Gamma$ is satisfiable in a structure of power $n$, and if $n < n'$, then $\Gamma$ is satisfiable in a structure of power $n'$.

**§ 2. Lemmas for the Completeness Theorem.** Throughout this section, we shall assume that $\mathcal{A}^*$ denotes a set of symbols which includes $V$, but is disjoint from $P$ and contains none of the symbols $\rightarrow, \neg, \forall$. We shall let $E = (\forall \exists \mu, \mu, n)$, and let $E^*$ be the set of formulas of $E^*$. For certain purposes, it is intuitively helpful to regard the elements of $E^* \rightarrow E$ as "individual constants".

$\Phi$ is said to be a $V$-formula in $E^*$ in $\Phi \in E^*$ and $V(\Phi) \subseteq V$. $\Phi$ is said to be a $V$-sentence in $E^*$ in $\Phi \in E^*$, $V(\Phi) \subseteq V$, and $V(\Phi) \subseteq V \rightarrow V$.

**Lemma 2.1.** Let $E'$ be the set of all $V$-formulas in $E^*$. Then

$\overline{E'} = \sum_{a \in \mathbb{A}^n} \sum_{\Phi \in \mathbb{P}} \Phi[\overline{\mathbb{A}}] + \nu_n$.

Proof. The number of atomic formulas in $E'$ is clearly $\sum_{a \in \mathbb{A}^n} \Phi[\overline{\mathbb{A}}]$, so $\sum_{a \in \mathbb{A}^n} \Phi[\overline{\mathbb{A}}] \leq \overline{E'}$. The set $S(V) = \mathbb{P}$, in view of the assumption II. Hence $\sum_{a \in \mathbb{A}^n} \Phi[\overline{\mathbb{A}}] \leq \overline{E'}$. By (1.2) and (1.3), we have $\nu_n \leq \overline{E'}$.

Since each member of $E'$ is constructed from finitely many atomic formulas, members of $S(V)$, and symbols $V, \rightarrow, \neg$, we have $\overline{E'} \leq \sum_{a \in \mathbb{A}^n} \sum_{\Phi \in \mathbb{P}} \Phi[\overline{\mathbb{A}}] + \nu_n$.

The desired conclusion follows:

**Lemma 2.2.** Let $\Gamma \subseteq F$. Then $\Gamma$ is consistent in $E^*$ if and only if it is consistent in $L$.

Proof. It is obvious that, if $\Gamma \vdash \neg \neg \Phi$, then $\Gamma \vdash \neg \neg \Phi$, for any proof in $L$ is also a proof in $E^*$. Suppose that $\Gamma \vdash \neg \neg \Phi$. Then for some $\Phi_1, \cdots, \Phi_n \in \Gamma$, we have $\models (\top \land \Phi \land \cdots \land \Phi_n)$. 


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LEMMA 2.4. Assume that
(i) \( \Gamma \) is a maximal consistent set of \( \mathcal{V} \)-sentences in \( \mathcal{L}^* \);
(ii) for any \( \mathcal{V} \)-sentence \( \langle \mathcal{V} W \mathcal{F} \rangle \), there exists \( \tau \in \langle \mathcal{V}^* \mathcal{W} \mathcal{F} \rangle \) such that \( \langle S_1(\tau) \mathcal{W} \mathcal{F} \rangle \mathcal{W} \mathcal{F} \) is consistent with \( \langle S_1(\tau) \mathcal{W} \mathcal{F} \rangle \mathcal{W} \mathcal{F} \).

Then for each \( \mathcal{V} \)-formula \( \Phi \) in \( \mathcal{L}^* \) and each function \( b \in \langle \mathcal{V}^* \mathcal{W} \mathcal{F} \rangle \), we have

\[
\models_{\mathcal{W} \mathcal{F}} \Phi[b] \mathcal{W} \mathcal{F} \] if and only if \( S_1(b) \Phi \in \Gamma \).

Proof. Let \( \mathcal{A} = \mathcal{A}(\Gamma, \mathcal{F}) \). Let \( \mathcal{H} \) be the set of all \( \mathcal{V} \)-formulas \( \Phi \mathcal{F} \) such that for any function \( b \in \langle \mathcal{V}^* \mathcal{W} \mathcal{F} \rangle \), we have

\[
\models_{\mathcal{W} \mathcal{F}} \Phi[b] \mathcal{W} \mathcal{F} \] if and only if \( S_1(b) \Phi \in \Gamma \).

We shall show by induction that every \( \mathcal{V} \)-formula belongs to \( \mathcal{H} \). By (8.2) and (7.1), every atomic \( \mathcal{V} \)-formula belongs to \( \mathcal{H} \).

By (7.3) and (5.3), \( \models_{\mathcal{F}} \mathcal{F} \). If \( \Phi \mathcal{F} = \langle \mathcal{V} \rangle \mathcal{F} \) and \( \mathcal{A} \models_{\mathcal{F}} \mathcal{F} \), then it follows easily, from (5.3) and (7.3), propositional logic, and (i), that \( \Phi \models_{\mathcal{F}} \).

Suppose finally that \( b \in \langle \mathcal{V}^* \mathcal{W} \mathcal{F} \rangle \), \( \Phi = \langle \mathcal{V} \mathcal{W} \mathcal{F} \rangle \), \( \mathcal{W} \mathcal{F} \models_{\mathcal{F}} \mathcal{F} \), and \( \mathcal{A} \models_{\mathcal{F}} \mathcal{F} \). Let us write \( \models_{\mathcal{F}} \) whenever \( \mathcal{A} \models_{\mathcal{F}} \).

Suppose \( S_1(b) \Phi \in \Gamma \). By (5.4), we have

\[
S_1(b) \Phi = \langle \mathcal{V} \mathcal{W} \mathcal{F} \rangle \mathcal{W} \mathcal{F} \] and \( S_1(b) \Phi \in \mathcal{H} \).

Since \( \mathcal{W} \subseteq \mathcal{V} \) and by 1.3, we have

\[
S_1(c) \mathcal{F} = S_1(c \mathcal{F} \mathcal{W} \mathcal{F}) = S_1(b \mathcal{W} \mathcal{F} \mathcal{W} \mathcal{F}) \] whenever \( c \models_{\mathcal{F}} \).

Then by (7.4), \( \models_{\mathcal{F}} \Phi[b] \mathcal{W} \mathcal{F} \).

Conversely suppose that \( \models_{\mathcal{F}} \Phi[b] \mathcal{W} \mathcal{F} \). By (5.4), we have \( S_1(b) \Phi = \langle \mathcal{V} \mathcal{W} \mathcal{F} \rangle \mathcal{W} \mathcal{F} \mathcal{W} \mathcal{F} \). It follows from (ii) that there exists a function \( \tau \in \langle \mathcal{V}^* \mathcal{W} \mathcal{F} \rangle \) such that

\[
\langle S_1(\tau) \mathcal{W} \mathcal{F} \rangle \mathcal{W} \mathcal{F} \mathcal{W} \mathcal{F} \models_{\mathcal{F}} \mathcal{F} \mathcal{F} \]

Let \( d = \tau \mathcal{W} \mathcal{F} \mathcal{W} \mathcal{F} \). Then by 1.3,

\[
S_1(b) \mathcal{F} = S_1(c \mathcal{F} \mathcal{W} \mathcal{F} \mathcal{W} \mathcal{F}) \]

Therefore, by 1.3, the structure \( \mathcal{A} \) is of type \( \mu \) such that

\[
A = \mathcal{V} \models \mathcal{F} \quad \text{and} \quad \mathcal{F} \models \mathcal{F} \]

for each \( p \in P \), \( R_p = \{ \mathcal{A} \models \langle \mathcal{V} \mathcal{W} \mathcal{F} \rangle \} \).

\[
\quad \text{and} \quad \mathcal{F} \models \mathcal{F} \]

for each \( p \in P \), \( R_p = \{ \langle \mathcal{V} \mathcal{W} \mathcal{F} \rangle \} \).
Also, \( d = b \), and it follows from (7.a) that \( \models \exists y \forall \psi[d] \). Since \( \psi \in \Gamma \) and 
\( (d \upharpoonright \psi)[\forall \psi'] \models \psi' = d \), we have \( \exists y \psi[d] \models \Gamma_1 \). But by (i) and since \( \exists y \psi[\exists \psi'] \models \Gamma_1 \), we have \( \exists y \psi[d] \models \Gamma \).

This verifies that \( \Phi \in \Gamma_1 \), and completes the proof that \( \Gamma \) is the set of all \( \forall \)-formulas.

§ 3. The Completeness Theorem.

Theorem 3.1. (Completeness Theorem) If \( \Gamma \) is a consistent set of formulas in \( L \), then \( \Gamma \) is satisfiable in some structure \( \mathfrak{A} \) of type \( \mu \).

Moreover, if we have 
\( \Phi \equiv \psi \), 
and
\( \phi \in \mathcal{P} \), 
then \( \mathfrak{A} \) may be taken to be of power \( n \).

Proof. Let \( \beta \) be the initial ordinal of power \( n \).

Let \( \exists y \psi \subseteq \exists y \psi' \) and \( \exists y \psi' \models \exists y \psi \), and let \( \gamma \) be a one-to-one function on \( \exists y \psi \) into \( \exists y \psi' \); this can be done in view of I. Let 
\( \Gamma_0 = \{ \exists y \psi[d] \phi_0 \mid \phi_0 \in \Gamma \} \).

Thus \( \Gamma_0 \) is a set of \( \forall \)-formulas in \( L \). Then by the consistency of \( \Gamma \) and the rule of substitution, it follows that \( \Gamma_0 \) is consistent in \( L \). Let \( \bar{\gamma} \in \gamma \) be a one-to-one function on \( \exists y \psi \) into \( \exists y \psi' \); this can be done in view of I. Let 
\( \Gamma_0 = \{ \exists y \psi[d] \phi_0 \mid \phi_0 \in \Gamma_0 \} \).

Thus \( \Gamma_1 \) is a set of \( \forall \)-sentences in \( L \). We shall show that \( \Gamma_1 \) is consistent in \( L \). Suppose that there exist \( \phi_1, \ldots, \phi_n \in \Gamma_0 \) such that 
\[ \vdash \exists y \psi[d_1] \phi_1 \land \ldots \land \exists y \psi[d_n] \phi_n. \]

Then by (5.3), we have 
\[ \vdash \exists y \psi[d_1] \phi_1 \land \ldots \land \exists y \psi[d_n] \phi_n. \]

By the rule of free substitution, it follows that 
\[ \vdash \exists y \psi[d_1] \phi_1 \land \ldots \land \exists y \psi[d_n] \phi_n. \]

But this contradicts the consistency of \( \Gamma_0 \), and hence \( \Gamma_1 \) is consistent in \( L \).

For each \( \varphi \in \exists y \psi' \) and \( \gamma < \beta \), let us choose new symbols \( \omega_\varphi \). Let 
\[ \exists y \psi' = \exists y \psi \cup \{ \exists y \psi[d] \models \exists y \psi' \}, \]
let \( \exists y \psi' = \exists y \psi' \cup \{ \exists y \psi[d] \models \exists y \psi' \} \), and let \( \exists y \psi' \) be the set of \( \forall \)-sentences in \( \exists y \psi' \). By (a) and Lemma 2.1, we have 
\[ \sum \exists y \psi[d] \models \exists y \psi' \leq n, \]
\[ \exists y \psi[d] \models \exists y \psi' \leq n \text{, and } \psi[d] \models \exists y \psi' \leq n \text{.} \]
By propositional logic,
\[ \vdash \neg \Phi \rightarrow \neg \Theta(a) . \]
Then
\[ \vdash \Phi \rightarrow \Theta(a) . \]
Since \( V' \cap V(\Phi) = 0 \), we have, by (3.3),
\[ \langle \Phi \rightarrow \Theta(a) \rangle = S_{\{\Theta(a)\}}(\Phi \rightarrow \psi'_b) . \]
In view of the fact that \( \{\Phi(a) \cup \{\Theta(a)\} \cap \text{sem}(V') = 0 \), and hence that
\[ V(\Phi(a)) \cap \text{sem}(V') = 0 \), it follows that the function \( \text{sem}(a) \rightarrow V(\Phi(a)) \) is onto. Then by the rule of free substitution,
\[ \vdash \neg \Phi \rightarrow \psi'_b . \]
By generalization, we have
\[ \vdash \neg \lambda W \langle \Phi \rightarrow \psi'_b \rangle . \]
Since \( V' \cap V(\Phi) = 0 \) and \( W \subseteq V' \), we have, by Axiom 4 and modus ponens,
\[ \vdash \langle \Phi \rightarrow \Theta(a) \rangle . \]
Then by propositional logic,
\[ \vdash \Phi \rightarrow \Theta(a) . \]
But now, since \( (\Phi_1, ..., \Phi_n) \) is inconsistent, \( (\Phi_1, ..., \Phi_n) \) must be inconsistent, which contradicts our assumption.
We conclude that \( I_1 \) is inconsistent in \( L^* \). By Lemma 2.3, there is a maximal consistent set \( I_1 \) of \( V' \)-sentences in \( L^* \) which included \( I_\gamma \).
Let us consider the structure \( \mathcal{M} = (\mathcal{I}_1, V') \) of the type \( \mu \). We have \( A = V' \cup \mathcal{I}_1 \), and thus \( A = \mathcal{N} \cap \mathcal{M} \).
By Lemma 2.4, it follows that for each function \( \delta \in A' \) and each \( V' \)-formula \( \Phi \) in \( L^* \), we have
\[ |= \mathcal{M} \Phi(b, V') \] if and only if \( S_\delta(b) \Phi \in I_1 \).
Recall that \( \tau_1 \in (V' \cup \mathcal{I}_1)' \), that \( I_1 \) is a set of \( V' \)-formulas in \( L^* \) for which \( \Phi_1 \notin I_1 \), and that \( I_1 \subseteq I_\gamma \). Let \( \Phi_1 \in I_1 \). Then
\[ |= \mathcal{M} \Phi_1(b, V') \].
Now, with respect to \( L_0 \), we have
\[ |= \mathcal{M} \Phi_1(b, V') \].
Let \( a = \tau_1 = a \in A'. \) Let \( \Phi \in I_1 \). Then \( S_\delta(a) \Phi \in I_1 \), and thus
\[ |= \mathcal{M} S_\delta(a) \Phi(b, V') \].
By Theorem 1.6 (ii), and because \( a = \tau_1 \), \( V' = \tau_1 \), we have
\[ |= \mathcal{M} \Phi(a) \].
Therefore \( I_1 \) is satisfied in \( \mathcal{M} \) by \( a \).

A complete first-order logic with infinitary predicates

If \( \mu \notin \{0\} \), then since \( \sum_{\mu \notin \{0\}} \mu \leq n \), we have \( \overline{V} \leq n \). Therefore
\[ \overline{A} = \overline{V} + V = \overline{A} \leq n \] so \( \mathcal{M} \) is of power \( n \).

Suppose \( \mu \notin \{0\} \); then for each \( p \in P_0 \), we have either \( p_0 = \mu_0 = \{0\} \) or \( p_0 = 0 \). It follows that for any non-empty set \( B \) and any \( b \in B \), \( B \) is satisfied in the structure \( (B, \mathcal{R}_b)_b \) by \( b \). Thus if we choose \( B \) to be a set of power \( n \), then \( B \) is satisfied in the structure \( (B, \mathcal{R}_b)_b \) of power \( n \).

Our proof is complete.

**Corollary 3.2.** A set of formulas in \( L \) is satisfiable if and only if it is consistent. Moreover, a formula in \( L \) is satisfiable if and only if it is provable.

**Proof.** By 1.5 (iii), (v), (vi), and 3.1.

**Theorem 3.3.** Theorem 3.1 remains true when condition (a) is replaced by

\[ \mathcal{M} \mathcal{V} \rightarrow \mathcal{V} \leq n \] by \( a \).

**Proof.** For each \( \Phi \in F_1, \) define \( \Phi_1 \) recursively as follows:

\[ \langle px \rangle_1 = \langle px \rangle ; \]
\[ \mathcal{F}_1 = \mathcal{F} ; \]
\[ \langle \psi \rightarrow \psi \rangle_1 = \langle \psi \rightarrow \psi \rangle_1 ; \]
\[ \langle \forall W \psi \rangle_1 = \langle \forall W \psi \rangle_1 . \]

It is easily seen that \( \langle \forall W \psi \rangle_1 \) is valid. For each \( \Phi \in F_1 \), define \( P(\Phi) \) recursively by

\[ P(\langle px \rangle_1) = \langle px \rangle_1 ; \]
\[ \mathcal{F} \rightarrow \mathcal{F} = 0 \];
\[ P(\langle \psi \rightarrow \psi \rangle_1) = P(\langle \psi \rightarrow \psi \rangle_1) \cup P(\langle \psi \rightarrow \psi \rangle_1) ; \]
\[ P(\langle \forall W \psi \rangle_1) = P(\langle \forall W \psi \rangle_1) \].

Let \( I_2 = \langle \Phi_1 \Phi \rangle \), let \( I_3 = \langle \langle \psi \rangle_1 \rangle \), and let \( I_4 = \langle \psi \rangle_1 \). Let \( m_0 \) be the smallest infinite cardinal \( n \) such that, for each \( p \in P_0 \), \( 
(\mathcal{M}, \mathcal{N}) \) is of power \( n \). Therefore
\[ m_0 \leq m_0 \]
and \( \overline{V} \leq m_0 \) if and only if \( \mathcal{M} \mathcal{N} \rightarrow \mathcal{N} \leq m_0 \).

By Theorem 1.6, there is a structure \( \mathcal{N} = (\mathcal{A}, (\mathcal{R}_b)_b) \) by a (any of the power \( \mathcal{N} \)) in which \( I_1 \) is satisfiable. Then \( I_2 \) is satisfiable in \( (\mathcal{A}, (\mathcal{R}_b)_b) \). By Theorem

1.5, \( I_1 \) is consistent in \( \mathcal{N} \), and hence, by Theorem 3.1, \( I_2 \) is satisfiable in a structure \( \mathcal{N} = (\mathcal{A}, (\mathcal{R}_b)_b) \), of type \( \mu \) power \( n \). Finally, if we put \( \mathcal{M} = \mathcal{M} \mathcal{N} \rightarrow \mathcal{N} \); then \( I_1 \) is satisfiable in the structure \( (\mathcal{M}, (\mathcal{R}_b)_b) \) of type \( \mu \) and power \( n \). This completes the proof.
COROLLARY 3.4. (Compactness Theorem). Suppose that \( \Gamma \subseteq \mathcal{F} \) and every finite subset of \( \Gamma \) is satisfiable. Then \( \Gamma \) is satisfiable.

Proof. By Theorem 3.5, every finite subset of \( \Gamma \) is consistent. It follows that \( \Gamma \) is consistent. Therefore, by Theorem 3.1, \( \Gamma \) is satisfiable.

The statement of the Compactness Theorem is purely semantical in nature, i.e., it does not refer to the axioms and rules of inference of \( \mathcal{L} \). As in the case of the usual first-order predicate logic, it is possible to give a direct semantical proof of the compactness theorem which is much shorter than the above syntactical proof. For example, one can prove the Compactness Theorem by a perfectly straightforward modification of the proof using ultraproducts outlined in [18].

COROLLARY 3.5. Suppose that \( \Gamma \subseteq \mathcal{F}, \Gamma \) is satisfiable, and \( \mathfrak{u} \) satisfies conditions (a') of 3.2 and (b) of 3.1. Then \( \Gamma \) is satisfiable in a structure of power \( \mathfrak{u} \).

Proof. By Theorem 1.6, \( \Gamma \) is consistent. The conclusion follows by Theorem 3.3.

Corollary 3.5 is the analogue of the Löwenheim-Skolem Theorem \((\ast)\) for ordinary first-order predicate logic. Like Corollary 3.4, it is purely semantical in nature. A direct semantical proof of Corollary 3.5 is also easy to construct by considering known proofs in ordinary logic. For example, one may give a proof using “Skolem functions” based on the argument in [29], Theorem 2.1.

§ 4. Complementary examples. In this section, we shall sometimes write \((a, b)\) for the two-ordered sequence \((0, a, 1, b)\). It is well known from ordinary first-order predicate logic that Theorem 3.1 is not longer true if condition (a) is removed. In fact, for each cardinal \( \mathfrak{u} \), we may easily give a consistent set \( \Gamma \) of \( \mathfrak{u} \) formulas, in a logic \( \mathcal{L} \) with \( \mathfrak{u} \) and a single binary predicate symbol, which is not satisfied in any structure of power \( \mathfrak{u} \). Obviously, condition (a) cannot be removed from Theorem 3.3 for the same reason. In the examples below, think of \( q \) as “equals”.

EXAMPLE 4.1. This shows that in Theorem 3.3, hypothesis (b) cannot be removed. \( \Gamma \) will imply there is a one-one function with domain \( \mathfrak{u}^2 \).

Let \( \beta \) be any infinite ordinal. Let \( \mathfrak{V} = \beta + \beta \), and \( \mathfrak{V} = \beta \). Let \( P = (p, q) \), \( \mu(p) = \beta \), and \( \mu(q) = 2 \). Finally, let \( \Gamma \) be the following set of formulas (omitting outer brackets):

\[
\exists \langle 0, 1 \rangle \forall \beta \bowtie 0 \langle \exists \langle 0, 1 \rangle \rangle ;
\]

\[
\forall \beta \bowtie 0 \langle \exists \langle 0, 1 \rangle \rangle ;
\]

for each \( \gamma \bowtie \beta \bowtie 0 \), the formula

\[
\exists \beta \bowtie 0 \langle \exists \langle 0, 1 \rangle \rangle ;
\]

where \( y_0 = 0 \) if \( a = 0 \), and \( y_a = \beta \bowtie 0 \) if \( 0 < a < \beta \).

It is then clear that \( \Gamma \) is consistent. However, \( \beta \bowtie 2^{\mathfrak{u}} \), so (b) fails with \( \mathfrak{u} = \beta \). Moreover, \( \Gamma \) is satisfiable in a structure of power \( \mathfrak{u}^2 \) but in no smaller structure. In fact, if \( \mathfrak{U} = (A, R_\gamma) \) is a structure in which \( \Gamma \) is satisfied, a, b \in A, and \( (a, b) \notin R_\gamma \), then there is a one-to-one function \( f \) on \( (a, b)^{\mathfrak{u}} \) into \( A \) such that \( f(x) = y \) implies \( x \bowtie 0, \gamma \bowtie 0, \) so \( \mathfrak{U} \bowtie \mathfrak{u}^2 \).

EXAMPLE 4.2. This example shows that in Theorem 3.3, hypothesis (b) cannot be removed. Here the function will have domain \( \mathfrak{u}^2 \).

Let \( n_0 \) be infinite and, for each \( n < \omega \), let \( n_{n+1} = 2^n \). Let \( \mathfrak{u} = \sum n_{n+1} \). Let \( V = \omega \cup n_0 \), and \( \mathfrak{m} = \omega \). Let \( \beta \bowtie n \), \( P = (p, q, r, \gamma < \beta, \mu(p) = \omega, \mu(q) = 2) \), and, for each \( \gamma < \beta, \mu(r) = 1 \). It follows from Lemma 3.1 that \( \mathfrak{V} = \mathfrak{u} \). Notice, however, that \( \mathfrak{u} < \mathfrak{u}^2 \), so (b) fails. Let \( \Gamma' \) be the following set of formulas: for each \( \gamma \bowtie \beta \), the formula

\[
\exists \langle 0, 1 \rangle \langle r, 0, 1 \rangle ;
\]

for each distinct \( \gamma, \gamma' \bowtie \beta \), the formula

\[
\forall \omega \bowtie 0 \langle \exists \langle 0, 1 \rangle \rangle ;
\]

for each \( \mu \bowtie n \bowtie 0 \), the formula

\[
\forall \omega \bowtie 0 \langle (p, 0 \bowtie 0) \rangle ;
\]

where \( y_0 = 0 \) if \( n = 0 \), and \( y_\mu = \omega + \mu \) if \( 0 < \mu < \omega \).

When \( \Gamma' \) is satisfiable in a structure of power \( \mathfrak{u}^2 \), but in no smaller structure. Since \( \mathfrak{u} < \mathfrak{u}^2 \), \( \Gamma' \) is not satisfiable in any model of power \( \mathfrak{u} \). Thus the conclusion of 3.1 does not hold.

§ 5. A formal system with infinitary functions. In this section, we shall see that the Completeness Theorem can be generalized to apply to formal systems which have infinitary functions as well as infinitary predicate. We shall state the relevant definitions and, since all the theorems may be proved by a straightforward generalization of the proofs of corresponding results for the system \( \mathcal{L} \), we shall omit all proofs.

Let \( \mathcal{Q} \) be a set of symbols which is disjoint from the sets \( \mathcal{V}, \mathcal{P}, \) and \( \langle \rightarrow, \exists, \forall \rangle \), and let \( \nu \) be a function whose domain is \( \mathcal{Q} \) and whose range is a set of ordinals.
In addition to the assumptions I, II, III of § I, we shall also assume:

1. For each $q \in Q$, $r(q) \subseteq \bar{V}$.

We construct the logic $L^V(\mathcal{V}, \mu, \tau, m)$, or more briefly $L^V$, as follows. The symbols of $L^V$ include the symbols of $L$ and, in addition, the function symbols $q \in Q$.

By the set $T$ of terms in $L^V$ we mean the least set $U$ such that each of the following holds:

(1.1) If $q \in Q$ and $x \in U^m$, then the ordered pair $\langle qx \rangle \in U$.

By an atomic formula in $L^V$ we mean an ordered pair $\langle px \rangle$ where $p \in P$ and $x \in U^m$.

The notion of a formula $\phi$ in $L^V$, and of a formation of $\phi$ in $L^V$, is defined exactly as in the case of $L$ except that we begin with atomic formulas in $L^V$. In the definition of a formula in $L^V$ given in § I, we need only replace $L$ everywhere by $L^V$ (cf. conditions (1.1)-(1.4)). The set of all formulas in $L^V$ is denoted by $F^V$.

**Lemma 5.1.** Lemma 1.1 (the induction principle) remains true if we replace $L$, $F$ everywhere by $L^V$, $F^V$ respectively.

The set $V(t)$ of variables of a term $t \in T$ is defined recursively by:

(1.2) If $t \in V$, then $V(t) = \{t\}$;

(1.3) If $t = \langle qx \rangle$, then $V(t) = \bigcup_{q \in Q} V(a_q)$.

Let $X \subseteq V$. We define the set $T(X)$ of terms in $X$ by:

$$T(X) = \{t | t \in T, V(t) \subseteq X\}.$$  

Thus we have $T = T(V)$.

The set $V(\phi)$ of free variables of a formula $\phi \in F^V$ is defined exactly the same way as for formulas in $L$ except that condition (2.1) is replaced by:

(2.1) If $\phi$ is an atomic formula $\langle px \rangle$, then $V(\phi) = \bigcup_{q \in Q} V(a_q)$;

conditions (2.2)-(2.4) remain unchanged.

The set $F(\phi)$ of bound variables of a formula $\phi \in F^V$ is defined exactly as for formulas in $L$ by the conditions (3.1)-(3.4).

For each $t \in T^V$, we define $t \in T^V$ recursively by:

(4.1) If $q \in Q$ and $x \in T^Q$, then $t''(\langle qx \rangle) = \langle q \bar{t} \circ x \rangle$.
The notions of satisfiability and validity of formulas in $L^M$ are defined exactly as in §1.

**Theorem 5.3.** Suppose that $\Phi, \Psi \in L^M$, $\Gamma \subseteq L^M$, $\mathcal{U}$ is a structure of type $(\mu, \tau)$, and $a \in A^\mathcal{U}$. Then conditions (i)-(ix) of Theorem 1.6 remain valid with $L^M$ everywhere replacing $L$. Moreover, we have:

(iii') $\models \forall \Theta [\mathcal{U}] (\Phi)$ if and only if $\models \forall \Phi [\mathcal{U}] (\forall \Theta (\Phi \tau \Theta)).$

**Theorem 5.4 (Completeness Theorem).** If $\Gamma$ is a consistent set of formulas in $L^M$, then $\Gamma$ is satisfiable in some structure $\mathcal{U}$ of type $(\mu, \tau)$. Moreover, if we have

(a) $\mathcal{U} \models \Phi$ for each $\phi \in \Gamma$,

(b) $\mathcal{U} \models \neg \Phi$ for each $\phi \in \Gamma$,

and

(c) $\mathcal{U} \models \Phi$ for each $\phi \in \Gamma$,

then $\mathcal{U}$ may be taken to be of power $n$.

The proof of Theorem 5.4 is an obvious modification of our proof of Theorem 3.1.

In the special case that $m = 0$, and thus $L^M$ has no quantifiers, Theorem 5.4 reduces to the Completeness Theorem obtained by Smirnov in [23], (5.1). In fact, it is possible to give an alternative proof of Theorem 5.4 in general by applying the theorem of Smirnov and a suitable generalization of the Skolem Normal Form Theorem (cf. [1]). Such a proof of the Completeness Theorem would be somewhat more along the lines of Gödel's original proof in [5] than the proof we have given is.

**Appendix. Relation to polyadic algebras.** For the terminology and notation concerning polyadic algebras, we refer to the immediately preceding paper [3]. The results which follow show the relationship between the Completeness Theorem in this paper and the Representation Theorem in [3].

There is a natural correspondence between polyadic algebras and sets of formulas in the formal systems $L(\mu, \nu, \omega)$. This correspondence depends on Lemmas A.1-A.4 below which we shall state without proofs; the proofs are straightforward but tedious.

Let $V_\mu$ be a set of symbols which is disjoint from $P$ and $V_\tau$, which does not contain $\rightarrow, \neg, \lor$, or $\land$, and such that $V_\mu = V$. Let $V^* = V \cup V_\mu$. Let $S^*$ be a one-to-one function of $V^*$ into $V$. Let $B$ be the set of all sentences of $L^*$, and let $H$ be the set of all formulas $\Phi$ in $L^*$ such that $V(\Phi) \subseteq V$.

Let $\Gamma \subseteq G$. For each formula $\Phi \in H$, we define $\Phi[\Gamma] = [\Phi \tau H, \Gamma, \Gamma \tau \neg (\Psi \tau \Phi)]$.

Clearly, for any $\Phi, \Psi \in H$, the sets $\Phi[\Gamma], \Psi[\Gamma]$ are either equal or disjoint.

We shall write $H_\Gamma = \{\Phi[\Gamma] : \Phi \in H\}$.

For any $\Phi, \Psi \in H$, we define

$$(\Phi[\Gamma] \tau (\Psi[\Gamma]) = (\Phi \tau \Psi)[\Gamma],$$

and

$$\neg (\Phi[\Gamma]) = (\neg \Phi)[\Gamma].$$

**Lemma A.1.** If $\Gamma$ is consistent in $L^*$, then $H_\Gamma = (H_\Gamma, \tau, \neg, \neg, \neg)$ is a Boolean algebra.$^{(1)}$

Following [6] and [3], we assume that "Boolean algebra" is defined in such a way that every Boolean algebra has at least two elements. Thus $H_\Gamma$ is not a Boolean algebra if $\Gamma$ is inconsistent in $L^*$.

We shall denote by $1_\Gamma$ the unit element of the Boolean algebra $H_\Gamma$.

Note that

$$1_\Gamma = \{\Phi \in H, \Gamma, \Gamma \tau \neg \Phi\}.$$

Thus $\Gamma$ is consistent in $L^*$ if and only if $1_\Gamma = H_\Gamma$.

For each $W \in S_m(\Gamma)$ and $\Phi \in F_\mu$ we shall define

$$H_\Gamma(W)(\Phi[\Gamma]) = (\neg W \Phi)[\Gamma].$$

Finally, for each $\tau \in V^*$ and $\Phi \in H$, we define

$$S_\Gamma(\tau)(\Phi[\Gamma]) = (S_\Gamma(\tau \neg) S_\Gamma(\neg \tau))(\Phi[\Gamma]).$$

**Lemma A.2.** (i) If $\Phi$ is atomic and $\tau \in V^*$, then $S_\Gamma(\tau)(\Phi[\Gamma]) = (S_\Gamma(\tau \neg))(\Phi[\Gamma])$.

(ii) If $\tau \in [V \tau V_\tau(\Phi[\Gamma])$, then $S_\Gamma(\tau)(\Phi[\Gamma]) = (S_\Gamma(\tau \neg))(\Phi[\Gamma])$.

(iii) If $\tau \in V^*$ is one to one, then $S_\Gamma(\tau)(\Phi[\Gamma]) = (S_\Gamma(\tau \neg))(\Phi[\Gamma])$.

---

$^{(1)}$ $H_\Gamma$ is essentially the Lindenbaum algebra (cf. [29], p. 348).
Lemma A.3. If \( m = \overline{V}^\perp \) and \( \Gamma \) is consistent in \( \mathcal{L}^* \), then the quadruple 
\[
(\mathcal{H}_C) = (\mathcal{H}_D, V, \mathcal{S}_D, \mathcal{M}_D)
\]
is a polyadic algebra of degree \( V \) (\( \mathcal{V} \)).

We shall denote by \( (\mathcal{X})_D \) the \( O \)-valued functional polyadic algebra 
\[
(\mathcal{V}(\mathcal{X})_D, \mathcal{O}), I, S, X
\]
If \( \Gamma \) is satisfiable in the structure \( \mathcal{W} \) of type \( \mu \), we shall denote by 
\[
h_{\mathcal{X}D}(\theta) \in \mathcal{H}_D \text{ into } \mathcal{V}(\mathcal{X})_D \text{ defined by the condition:}
\]
\[
h_{\mathcal{X}D}(\Gamma)(b) = 1 \text{ if and only if } |\mu \mathcal{O}[b]^E|,
\]
whenever \( \Phi \in H \) and \( b \in A^\perp \).

Lemma A.4. Suppose that \( m = \overline{V}^\perp \) and \( \Gamma \) is satisfiable in the structure \( \mathcal{W} \) of type \( \mu \). Then \( h_{\mathcal{X}D} \) is a polyadic homomorphism of \( (\mathcal{H}_D) \) into \( (\mathcal{A}^\perp) \).

Lemma A.5. Let \( (\mathcal{B}) = (\mathcal{B}, I, S, X) \) be a polyadic algebra of infinite degree \( b \). Let \( \overline{V} = \overline{b}, \overline{E} = \overline{E}, m = b^4 \), and \( \mu(p) = b \) for each \( p \in P \). Then there exists a set of sentences \( \Gamma \subseteq G \) such that \( (\mathcal{H}_D) \) is isomorphic to \( (\mathcal{B}) \).

In Lemma A.5, let us suppose for convenience that \( I = V, B = P \), and \( \mu(p) \) is the same for each \( p \in P \). Let \( x \) be a one to one function on \( \mu(V) \) onto \( V \). We may take for \( \Gamma \) the following set of sentences, where \( p, q \) range over \( P \), \( r \) ranges over \( \mathbb{W}(V) \); and \( W \) ranges over \( \mathbb{S}(V) \):

\[
\forall V(xz) + (\forall xz) = (\forall xz) \}
\]
\[
\forall V(x + qz) \equiv (\forall xz \lor \forall qz) \}
\]
\[
\forall V(x + qz) \equiv (\forall xz \lor \forall qz) \}
\]
\[
\forall V(xzr) \equiv (\forall xzr) \}
\]
\[
\forall V(xzWq) \equiv (\forall xzWq) \}
\]

The required isomorphism of \( (\mathcal{B}) \) onto \( (\mathcal{H}_D) \) is then given by the function \( f \) defined by the condition 
\[
g(p) = (\forall xz) \}
\]
for each \( p \in P \). The proof that \( g \) is an isomorphism consists of a series of computations which amount to showing that the axioms and rules of inferences of \( \mathcal{L}^* \), and the axioms of polyadic algebras, "say the same things" under the proper interpretation.

A more natural procedure up to this point would have been to work with the original system \( \mathcal{L} \) instead of passing to \( \mathcal{L}^* \), and to consider the set \( F \) of formulas and a set \( T \) of sentences in \( \mathcal{L} \). However, in case there are formulas \( \Phi \) in \( \mathcal{L} \) such that \( \Phi_p = V \), this simpler procedure would leave us at a loss to define the substitution operator \( S_p \) so as to simplify the axioms of polyadic algebras. Evidently, we need the extra variables in \( \mathcal{L}^* \) in order to avoid collisions of free and bound variables.

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*\( \mathcal{A} \) is a complete first-order logic with infinitary predicates.*
It is then easily seen that $\mathfrak{I}^+ = \mathfrak{C} + \sum p$ for each $p \in P$. The required isomorphism $\mathfrak{h}$ is now defined by

$$\mathfrak{h}(p) = \{p \times \mathfrak{P}\}$$. 

Moreover, if $D$ is a proper dual ideal in $\mathfrak{B}$, then $\mathfrak{h}$ may be chosen so that $\mathfrak{I} = \mathfrak{I}^+$. 

Proof. If $\mathfrak{c} \geq \mathfrak{b}$, then (by Theorem 3.12 in [3]) $\mathfrak{B}$ is degenerate, and the result is a trivial consequence of the Representation Theorem for Boolean Algebras (cf. [24]).

Suppose $\mathfrak{c} \geq \mathfrak{b}$. By $\mathfrak{A}^\mathfrak{c}$, there is a formal system $L(\mathfrak{C}, V, m)$ and a set $\mathfrak{C} \subseteq \mathfrak{C}$ which satisfy the conclusions of $\mathfrak{A}^\mathfrak{c}$. Let $\mathfrak{c}$ be an isomorphism of $\mathfrak{B}$ onto $\mathfrak{B}_p$, and let $\mathfrak{I} = \mathfrak{g}(D)$. Clearly $\mathfrak{I} \subseteq \mathfrak{I}_1$. Since $\mathfrak{b} \leq \mathfrak{c}$, it follows that $\mathfrak{I} \subseteq \mathfrak{I}_1$. Also, we have $\mathfrak{u} = \mathfrak{U}_{\mathfrak{C}}$ for each $p \in P$. Since $\mathfrak{c} > \mathfrak{b}$, $\mathfrak{I}_1$ is consistent in $\mathfrak{L}^*$. 

It now follows from Theorem 3.3 that $\mathfrak{I}_1$ is satisfiable in a structure $\mathfrak{K}$ of power $\mathfrak{p}$. Let $\mathfrak{h} = (\mathfrak{B}_p) \circ \mathfrak{g}$, as in $\mathfrak{A}^\mathfrak{c}$, we see that $\mathfrak{h}$ is the desired isomorphism of $\mathfrak{B}$ into $\mathfrak{I}^+$. 

By using Theorem 4.4 in [3], we obtain the Representation Theorem in the form stated in [3]. Theorem 6.1. 

THEOREM $\mathfrak{A}^\mathfrak{c}$. Let $\mathfrak{B}$ be a polyadic algebra of infinite degree $\mathfrak{b}$, effective cardinality $\mathfrak{c}$, and local degree $\mathfrak{b}'$, and let $n$ be a cardinal such that $n > \mathfrak{c}$ and $\sum p = n$. Then there exists a homomorphism of $\mathfrak{B}$ into an $\mathfrak{O}$-valued functional algebra whose domain $\mathfrak{A}$ has power $\mathfrak{p}$. 

Proof. Let $\mathfrak{J} \subseteq \mathfrak{I}$ and let $\mathfrak{J}$ be the effective degree of $\mathfrak{B}$. Then the faithful compression $(\mathfrak{B}_p)^{\mathfrak{J}}$ of $\mathfrak{B}$ has power $\mathfrak{c}$, degree $\mathfrak{J}$, and local degree $\mathfrak{b}'$. 

If $(\mathfrak{B})$ is degenerate, then, as in $\mathfrak{A}^\mathfrak{c}$, the result is trivial. 

Suppose $(\mathfrak{B})$ is not degenerate. Then it is easily seen that $\mathfrak{J}$ is infinite. 

By $\mathfrak{A}^\mathfrak{c}$, there is a homomorphism of $\mathfrak{B}_p$ into an $\mathfrak{O}$-valued functional algebra of degree $\mathfrak{J}$ whose domain $\mathfrak{A}$ is of power $\mathfrak{p}$. It follows from Theorem 4.4 of [3] that there is a homomorphism of $\mathfrak{B}$ into an $\mathfrak{O}$-valued functional algebra of degree $\mathfrak{b}$ and with domain $\mathfrak{A}$. This completes the proof. 

In the other direction, it is also possible to prove the Completeness Theorem by applying the Representation Theorem. It is most convenient to apply the version of the latter stated in $\mathfrak{A}^\mathfrak{c}$. We shall give an outline of such a proof.

LEMMA $\mathfrak{A}^\mathfrak{c}$. Suppose that $\mathfrak{h}$ is a polyadic homomorphism from the polyadic algebra $\mathfrak{H}_\mathfrak{P}$, where $\mathfrak{P}^\mathfrak{C} \subseteq \mathfrak{P}$ and $\mathfrak{I}$ is consistent in $\mathfrak{I}^+$, into an $\mathfrak{O}$-valued functional algebra $(\mathfrak{A}^\mathfrak{D})$ with non-empty domain $\mathfrak{A}$. Then there exists a structure $\mathfrak{K} = (\mathfrak{A}, \mathfrak{B}_p)^{\mathfrak{D}}$ of type $\mathfrak{P}$ such that $\mathfrak{I}$ is satisfiable in $\mathfrak{K}$ and $\mathfrak{h} = \mathfrak{h}_\mathfrak{K}$. 

In fact, the structure $\mathfrak{K}$ of Lemma $\mathfrak{A}^\mathfrak{c}$ may be defined in the following way: for each $p \in P$, let $x \in \mathfrak{V}^{\mathfrak{P}_p}$ be one to one, and let 

$$\mathfrak{B}_p = \{x \in \mathfrak{A} : \mathfrak{h}(\mathfrak{P}_p) \subseteq \mathfrak{K}(x) \}.$$ 

Note that the relation $\mathfrak{R}_p$ does not depend on our choice of the one to one function $x$. 

We now outline a second proof of the Completeness Theorem.

Suppose that $\mathfrak{I}$ is a consistent set of formulas in $\mathfrak{L}$ and that

$$\begin{align*}
\mathfrak{P} &\leq \mathfrak{N}, \\
\mathfrak{B} &\leq \mathfrak{B}_p, \\
\mathfrak{C} &\leq \mathfrak{C}_p, \\
\mu(\mathfrak{p}) &< \mathfrak{m} \text{ for each } p \in P.
\end{align*}$$

We wish to show that $\mathfrak{I}$ is satisfiable in a structure of power $\mathfrak{p}$. 

Let $\mathfrak{P}^*$ be a set of symbols disjoint from $\mathfrak{P}$ and not containing $\Rightarrow$, $\land$, $\exists$, such that $\mathfrak{V} \subseteq \mathfrak{P}^*$ and $\mathfrak{P} = (\mathfrak{P} \setminus \mathfrak{P}^*)$. Then $\mathfrak{I}^* \subseteq \mathfrak{I}$. By Lemma 2.2 of this paper, $\mathfrak{I}^*$ is also consistent in $\mathfrak{L}^*$. Since all formulas $\mathfrak{P} \in \mathfrak{H}$ has fewer than $\mathfrak{m}$ free variables, we may define $\mathfrak{K}_\mathfrak{P}$ on $\mathfrak{H}_\mathfrak{P}$ in the obvious way, where $\mathfrak{0}$ is the empty set of formulas, so that the quadruple $(\mathfrak{H}_\mathfrak{P})$ is a polyadic algebra. The set $\mathfrak{D} = \{\mathfrak{P}(\mathfrak{P}) : \mathfrak{P} \in \mathfrak{I}^*\}$ is proper dual in $\mathfrak{H}_\mathfrak{P}$. By the Representation Theorem $\mathfrak{A}^\mathfrak{c}$, and by $\mathfrak{A}^\mathfrak{c}$, $\mathfrak{A}^\mathfrak{c}$, there exists a homomorphism $\mathfrak{h}$ from $(\mathfrak{H}_\mathfrak{P})$ into an $\mathfrak{O}$-valued functional algebra $(\mathfrak{A}^\mathfrak{D})$ whose domain $\mathfrak{A}$ has power $\mathfrak{p}$ and such that $\mathfrak{h}(\mathfrak{D}) \subseteq \mathfrak{A}^\mathfrak{D}$. Choose an element $b \in \mathfrak{A}^\mathfrak{D}$ such that $\mathfrak{h}(\mathfrak{D}) = b$ for all $x \in \mathfrak{D}$. By Lemma $\mathfrak{A}^\mathfrak{c}$, there exists a structure $\mathfrak{K}$ of type $\mathfrak{D}$ such that $\mathfrak{h}(\mathfrak{b}) = b$. Then it follows that $\mathfrak{h}(\mathfrak{Q}(\mathfrak{P})^* \mathfrak{P}^*) = \mathfrak{P}(\mathfrak{P})$ for each $\mathfrak{Q} \in \mathfrak{I}$, and $\mathfrak{I}$ is satisfiable in $\mathfrak{K}$. 

The hypothesis $(\mathfrak{c}^*)$ used above was not assumed in Theorem 3.1; however, this hypothesis may easily be removed by syntactical methods.

A proof of the Completeness Theorem in the special case that $\mathfrak{I}$ is a set of sentences may be given, along the same lines as the above argument, using only the first paragraph of $\mathfrak{A}^\mathfrak{c}$, or else using $\mathfrak{A}^\mathfrak{c}^\mathfrak{c}$. 

In such a proof we consider $(\mathfrak{H}_\mathfrak{P})$ instead of $(\mathfrak{H}_\mathfrak{P})$. 

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(12) The improved cardinality condition in $\mathfrak{A}^\mathfrak{c}$ was suggested to the author by the results in [3]. 

(13) Theorem $\mathfrak{A}^\mathfrak{c}^\mathfrak{c}$ is, because of the cardinality condition, stronger than the first paragraph of $\mathfrak{A}^\mathfrak{c}$ and of $\mathfrak{A}^\mathfrak{c}$. The result $\mathfrak{A}^\mathfrak{c}^\mathfrak{c}$ is originally due to Daigneault and Monk and is the same as Theorem 6.1 in [3].
References