Theorem 6.4 can be improved. We note that in the case of a denumerable locally finite algebra A we can take $\mathfrak{n} = \omega$, and that $\mathfrak{n} = 2^{\mathfrak{m}}$ is always a solution of the equation $\sum_{\mathfrak{s}<\mathfrak{m}} \mathfrak{n}^{\mathfrak{s}} = \mathfrak{n}$.

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A complete first-order logic with infinitary predicates

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It is well known that the first order predicate logic (with or without an identity symbol) has the following two properties:

(∗) *each proof involves only finitely many formulas;*

(∗∗) *a set of formulas is consistent if and only if it is satisfiable* (Gödel Completeness Theorem) (¹).

In this paper we shall (in § 1) introduce a formal system L which has predicates with infinitely many argument places and quantifiers over infinite sets of variables, but which has only finitary propositional connectives and no identity symbol, and which satisfies (∗). The system L is patterned after the finitary first-order system F_1 of Church in [1], and our notion of satisfaction is the natural extension of Tarski's definition (e.g. in [26], p. 193). Our main result, the *Completeness Theorem* (Theorem 3.1), is that L also satisfies (∗∗) (²). The methods of proof are based upon the proofs of Henkin, and of Rasiowa and Sikorski, of the Gödel Completeness Theorem.

Generalizations of the Löwenheim-Skolem Theorem and of the Compactness Theorem to L (in § 3) will follow easily from 3.1. It is to be expected that many of the other familiar applications of the Gödel Completeness Theorem to first-order theory of models will eventually be generalized to the theory of models of the system L (³).

In § 4 we shall give some examples which indicate the difficulties encountered when one attempts to make various improvements of our main result.

In § 5 we shall introduce a more general formal system $L^{\#}$ which has, in addition to the expressions of L, functions and terms with in-

(¹) See [5], [9], [17], [20], and [21].

(²) The main results of this paper were announced in abstracts [14].

(³) For an expository discussion of several applications of the Gödel completeness theorem, and for a historical account and references, we refer to [15]. For related results concerning the infinitary logics of [27] — which do not have property (∗) — see [8] and [28]. We shall not here be concerned with the systematic development of the theory of models for L in the spirit of [21] or of [25].

finitely many argument places. We shall generalize our results concerning L, and in particular shall prove that the system $L^{\#}$ has both properties (*) and (**). Słomiński in [23] has introduced a formal system which is essentially the system $L^{\#}$ without any quantifiers, and he shows that this system satisfies both (*) and (**). Our Completeness Theorem 5.9 for $L^{\#}$ is a generalization of Słomiński's Completeness Theorem as well as of our Theorem 3.1.

In the immediately preceding paper [3], Daigneault and Monk give an independent, purely algebraic proof of the *Representation Theorem for polyadic algebras of infinite degree*, which may be regarded as an algebraic version of the Completeness Theorem (4). In the Appendix, we shall give a rough outline of the close relationship which exists between the logic L and the polyadic algebras of Halmos. Although this relationship is not as simple (due to peculiarities of the substitution operator) as one might expect, it does allow one to easily establish the equivalence of the Completeness Theorem with the Polyadic Representation Theorem (5).

The system L is one of a variety of logics with infinitely long formulas which have been considered in the literature (6). In most cases, such as those systems of logic which have infinitary propositional connectives, and even the system L with identity, it is known to be impossible to define a notion of proof in such a way that both (*) and (**) are satisfied (7). We are thus confronted with the special situation that the system L without identity behaves like ordinary first order logic, while L with identity behaves like the stronger first order logics with infinitary connectives (8).

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(4) We refer to [3] for a historical account of the Representation Theorem and for further references.

(5) It is implicit in Halmos [6], [7] that polyadic algebra corresponds to an appropriate infinitary logic. The correspondence between cylindric algebra and infinitary logics is discussed informally in [11].

(6) For an informal discussion of various possibilities see [10]. Predicate logics with infinitary connectives are introduced in [21], p. 21f, and [27], and further early references are given in these papers. A number of more recent papers involve the system in [27]; in particular, [12], [13], [8], [28], [16] deal with questions analogous to those touched upon in this paper.

(7) See [22] (propositional logic with infinitary connectives) and [10], [23] (L with identity). In some instances, for example in [22], [12], [13], [16], systems with infinitary connectives have been given notions of proof which satisfy (**) but not (*). Henkin in [10] gives a notion of proof which satisfies (*) for a system like L but with an identity and with only finitary quantifiers, and he proves that (**) holds for sets of formulas satisfying certain additional conditions.

(8) Słomiński has shown in [23] that the system $L^{\#}$ (or L) without any quantifiers already has this behavior, i.e. the system satisfies both (*) and (**) but the corresponding logic with identity cannot be given a notion of proof which satisfies both (*) and (**).

We shall make free use of the Axiom of Choice, often in the form that any set can be well-ordered.

This paper is self-contained except for the Appendix, where we assume a familiarity with [3].

The author has benefited from discussions with C. C. Chang, Leon Henkin, and Donald Monk in connection with this paper.

Terminology and notation. The symbols $\subseteq$, ϵ, $\cup$, $\bigcup$, $\cap$, $\bigcap$ have their usual set-theoretic significance. The one-element set containing x is denoted by $\{x\}$. $X\sim Y$ is the set-theoretic difference of X and Y. Ordered pairs and ordered triples will be written $\langle x, y\rangle$ and $\langle x, y, z\rangle$ respectively; in practice, we shall often omit the commas, thus $\langle xy\rangle$ and $\langle xyz\rangle$, and rely on the context to make the meaning clear. We shall denote by Y^X the set of all functions f on X into Y; thus if $f \in Y^X$, we have $f = \{\langle x, f(x)\rangle |\ x \in X\}$. If f and g are functions with domains X and Y respectively, we write

$$g\circ f = \{\langle x, g(f(x))\rangle |\ x\in X \text{ and } f(x)\in Y\}.$$

If f is a one-to-one function, we write

$$f^{-1} = \{\langle y, x\rangle |\ \langle x, y\rangle \in f\}.$$

For any function $f\in Y^X$ and any set Z, we shall write $f[Z] = \{f(x)|\ x\in X\cap \cap Z\}$, and

$$f\restriction Z = \{\langle x, f(x)\rangle |\ x\in X\cap Z\}\cup\{\langle z, z\rangle |\ z\in Z\sim X\}.$$

Thus $f\restriction Z$ always has domain Z, and $0\restriction Z$ is the identity function on Z.

We assume that the ordinals are defined in such a way that each ordinal coincides with the set of all smaller ordinals. We shall use the small Greek letters $\alpha, \beta, \gamma, \ldots$ to denote arbitrary ordinals. The sum of the ordinals α, β shall be denoted by $\alpha+\beta$. The smallest infinite ordinal is denoted by ω, and the finite ordinals are identified with the natural numbers. Thus 0 is the empty set. The letters $k, l, m, n, p, \ldots$ denote arbitary natural numbers.

We shall denote arbitrary cardinal numbers by small German letters $\mathfrak{m}, \mathfrak{n}, \mathfrak{p}, \ldots$ The sum of the system $(\mathfrak{m}_i)_{i\in I}$ of cardinals is denoted by $\sum_{i\in I}\mathfrak{m}_i$, and the sum of the two cardinals $\mathfrak{m}, \mathfrak{n}$ is denoted by $\mathfrak{m}+\mathfrak{n}$. The cardinal $\mathfrak{m}$ to the $\mathfrak{n}$th power is denoted by $\mathfrak{m}^{\mathfrak{n}}$; we rely on context to distinguish the cardinal power from the function set Y^X. The power, or cardinality, of a set X is denoted by $\overline{\overline{X}}$; thus $\overline{\overline{Y}}^{\overline{\overline{X}}} = \overline{\overline{Y^X}}$. Recall that $\overline{\overline{\omega}} = \aleph_0$. The smallest cardinal which is greater than $\mathfrak{m}$ is denoted by $\mathfrak{m}^+$.

For any set X and cardinal $\mathfrak{m}$, we define

$$S_{\mathfrak{m}}(X) = \{x|\ x\subseteq X \text{ and } \overline{\overline{x}} < \mathfrak{m}\}.$$

§ 1. A formal system with infinitary predicates. We shall begin by introducing some purely syntactical notions, i.e. notions which do not deal with our intended interpretation of the infinite logic, but only with its formal structure.

We shall construct our formal system L within the underlying set theory. By a *symbol* we shall mean any set which is neither an ordered pair nor an ordered triple.

Let V and P be two disjoint sets of symbols, μ a function whose domain is P and whose range is a set of ordinals, and $\mathfrak{m}$ a cardinal.

Throughout this paper we shall assume that V, μ, and $\mathfrak{m}$ satisfy the following conditions [9]:

I. *V is infinite;*

II. $\mathfrak{m} \leqslant \overline{\overline{V}}^{+}$;

III. *for each* $p \in P$, $\overline{\overline{\mu(p)}} \leqslant \overline{\overline{V}}$.

We construct the logic $L(V, \mu, \mathfrak{m})$, or more briefly L, in the following way. The *symbols of* L include:

the implication symbol $\rightarrow$;
the falsity symbol $\mathfrak{f}$;
the universal quantification symbol $\forall$;
the (individual) variables $v \in V$;
the predicates $p \in P$.

We shall assume that the symbols $\rightarrow$, $\mathfrak{f}$, $\forall$ are not members of V or of P.

By an *atomic formula* in L we mean an ordered pair $\langle p x\rangle$, where $p \in P$ and $x \in V^{\mu(p)}$.

Φ is said to be a *formula in L* if there exists a finite sequence $\Phi_0, \Phi_1, \ldots, \Phi_n$, called a *formation of Φ in L*, such that $\Phi_n = \Phi$ and, for each $m \leqslant n$, at least one of the following hold:

(1.1) *Φ_m is an atomic formula in L;*

(1.2) $\Phi_m = \mathfrak{f}$;

(1.3) *for some* $k, l < m$, *Φ_m is the ordered triple* $\langle \Phi_k \rightarrow \Phi_l\rangle$;

(1.4) *for some* $k < m$ *and* $W \in S_{\mathfrak{m}}(V)$, *$\Phi_m$ is the ordered triple* $\langle \forall W\, \Phi_k\rangle$.

It follows at once, from the fact that V, P, and $\{\forall, \rightarrow, \mathfrak{f}\}$ are disjoint sets of symbols, that for each $m \leqslant n$ *exactly* one of (1.1)-(1.4) hold. The set of all formulas in L is denoted by F.

LEMMA 1.1. (Induction principle). *Suppose that G contains every atomic formula in L, that $\mathfrak{f} \in G$, that $\langle \Phi \rightarrow \Psi\rangle \in G$ whenever Φ, $\Psi \in G$, and that $\langle \forall W\, \Phi\rangle \in G$ whenever $\Phi \in G$ and $W \in S_{\mathfrak{m}}(V)$. Then $F \subseteq G$.*

[9] Although these conditions are not required for the construction of the formal system L, they are required for most of our results. An indication of the difficulties which are encountered when assumption I is removed is given (for polyadic algebras of finite degree) by Monk in [19].

Proof. Suppose that $\Phi_0, \ldots, \Phi_n$ is a formation in L, and every formula which has a formation of length less than n belongs to G. Then $\Phi_k \in G$ for each $k < n$. If $n = 1$, then either Φ_n is atomic or $\Phi_n = \mathfrak{f}$, so $\Phi_n \in G$. If $\Phi_n = \langle \Phi_k \rightarrow \Phi_l\rangle$ for some $k, l < n$, then $\Phi_k, \Phi_l \in G$, so $\Phi_n \in G$. If $\Phi_n = \langle \forall W\, \Phi_k\rangle$ for some $k < n$ and $W \in S_{\mathfrak{m}}(V)$, then $\Phi_k \in G$, so $\Phi_n \in G$. It follows by induction that $F \subseteq G$.

The set $V_f(\Phi)$ of *free variables of* a formula $\Phi \in F$ is defined recursively as follows:

(2.1) *if Φ is an atomic formula $\langle px\rangle$, then $V_f(\Phi)$ is the range of x;*

(2.2) *if* $\Phi = \mathfrak{f}$, *then* $V_f(\Phi) = 0$;

(2.3) *if* $\Phi = \langle \Psi \rightarrow \theta\rangle$, *then* $V_f(\Phi) = V_f(\Psi) \cup V_f(\theta)$;

(2.4) *if* $\Phi = \langle \forall W\, \Psi\rangle$, *then* $V_f(\Phi) = V_f(\Psi) \sim W$.

Similarly, the set $V_b(\Phi)$ of *bound variables* of Φ is defined by:

(3.1) *if Φ is atomic, then* $V_b(\Phi) = 0$;

(3.2) *if* $\Phi = \mathfrak{f}$, *then* $V_b(\Phi) = 0$;

(3.3) *if* $\Phi = \langle \Psi \rightarrow \theta\rangle$, *then* $V_b(\Phi) = V_b(\Psi) \cup V_b(\theta)$;

(3.4) *if* $\Phi = \langle \forall W\, \Psi\rangle$, *then* $V_b(\Phi) = V_b(\Psi) \cup W$.

Finally, the set $V(\Phi)$ of (all) *variables of* Φ is defined by:

$$V(\Phi) = V_f(\Phi) \cup V_b(\Phi)\,.$$

For each $\tau \in V^V$ and $\Phi \in F$, we define $S(\tau)\Phi$ (which is intuitively the result of substituting each variable v in Φ by $\tau(v)$), recursively as follows:

(4.1) *if Φ is an atomic formula $\langle px\rangle$, then* $S(\tau)\Phi = \langle p\, \tau \circ x\rangle$;

(4.2) *if* $\Phi = \mathfrak{f}$, *then* $S(\tau)\Phi = \mathfrak{f}$;

(4.3) *if* $\Phi = \langle \Psi \rightarrow \theta\rangle$, *then* $S(\tau)\Phi = \langle S(\tau)\Psi \rightarrow S(\tau)\theta\rangle$;

(4.4) *if* $\Phi = \langle \forall W\, \Psi\rangle$, *then* $S(\tau)\Phi = \langle \forall \tau[W]\, S(\tau)\Psi\rangle$.

Similarly, for each $\tau \in V^V$ and $\Phi \in F$, we define $S_f(\tau)\Phi$ (intuitively the result of substituting each free variable v of Φ by $\tau(v)$) recursively by:

(5.1) *if Φ is an atomic formula $\langle px\rangle$, then* $S_f(\tau)\Phi = \langle p\, \tau \circ x\rangle$;

(5.2) *if* $\Phi = \mathfrak{f}$, *then* $S_f(\tau)\Phi = \mathfrak{f}$;

(5.3) *if* $\Phi = \langle \Psi \rightarrow \theta\rangle$, *then* $S_f(\tau)\Phi = \langle S_f(\tau)\Psi \rightarrow S_f(\tau)\theta\rangle$;

(5.4) *if* $\Phi = \langle \forall W\, \Psi\rangle$, *then* $S_f(\tau)\Phi = \langle \forall W\, S_f(\sigma)\Psi\rangle$, *where* $\sigma = \big(\tau \restriction (V \sim W)\big) \restriction V$.

If $\tau \in \bigcup \{V^W \mid W \subseteq V\}$ and $\Phi \in F$, let $S(\tau)\Phi = S(\tau \restriction V)\Phi$, and $S_f(\tau)\Phi = S_f(\tau \restriction V)\Phi$.

Lemma 1.2. *If* $\Phi \in F$ *and* $\tau \in V^V$, *then* $S(\tau)\Phi \in F$ *and* $S_f(\tau)\Phi \in F$.

Proof. By 1.1.

Lemma 1.3. *Suppose* $W, X \subseteq V$, $W \cap X = 0$, $\sigma \in V^W$, $\tau \in (V \sim W)^X$, *and* $\Phi \in F$. *Then*

$$S(\sigma \cup \tau)\Phi = S(\sigma)S(\tau)\Phi\,,$$

and

$$S_f(\sigma \cup \tau)\Phi = S_f(\sigma)S_f(\tau)\Phi\,.$$

Proof. By 1.1.

We shall now give the rules of inference and the axiom schemata for *L*.

Rule 1: *From* Φ, $\langle\Phi \to \Psi\rangle$, *infer* Ψ, *whenever* Φ, $\Psi \in F$. (Modus ponens.)

Rule 2: *From* Φ, *infer* $\langle\forall W\, \Phi\rangle$, *whenever* $\langle\forall W\, \Phi\rangle \in F$. (Rule of generalization.)

Rule 3: *From* $S_f(\tau)\Phi$, *infer* Φ, *whenever* $\Phi \in F$, $\tau \in \big(V \sim V_b(\Phi)\big)^{V_f(\Phi)}$, *and* τ *is one to one.* (Rule of free substitution.)

Rule 4: *From* Φ, *infer* $S(\tau)\Phi$, *whenever* $\Phi \in F$, $\tau \in V^{V(\Phi)}$, *and* τ *is one to one.* (Rule of substitution.)

Axiom 1: $\langle\Phi \to \langle\Psi \to \Phi\rangle\rangle$, *whenever* Φ, $\Psi \in F$.

Axiom 2: $\langle\langle\theta \to \langle\Phi \to \Psi\rangle\rangle \to \langle\langle\theta \to \Phi\rangle \to \langle\theta \to \Psi\rangle\rangle\rangle$, *whenever* $\Phi, \Psi, \theta \in F$.

Axiom 3: $\langle\langle\langle\Phi \to \mathfrak{f}\rangle \to \mathfrak{f}\rangle \to \Phi\rangle$, *whenever* $\Phi \in F$.

Axiom 4: $\langle\langle\forall W\, \langle\Phi \to \Psi\rangle\rangle \to \langle\Phi \to \langle\forall W\, \Psi\rangle\rangle\rangle$, *whenever* Φ, $\Psi \in F$ *and* $W \in S_{\mathfrak{m}}\big(V \sim V_f(\Phi)\big)$.

Axiom 5: $\langle\langle\forall W\, \Phi\rangle \to S_f(\tau)\Phi\rangle$, *whenever* $\langle\forall W\, \Phi\rangle \in F$ *and* $\tau \in \big(V \sim$ $\sim V_b(\Phi)\big)^W$.

By an *axiom of L* we mean any formula which is an instance of one of the axiom schemata 1-5 above.

By a *proof of* Φ *in L* we mean a finite sequence of formulas $\Phi_0, \ldots, \Phi_n$ in *L* such that $\Phi_n = \Phi$ and, for each $m \leqslant n$, one of the following holds:

(6.1) *for some* k, $l < m$, Φ_m *is inferred from* Φ_k, Φ_l *by Rule* 1;

(6.2) *for some* $k < m$, Φ_m *is inferred from* Φ_k *by Rule* 2;

(6.3) *for some* $k < m$, Φ_m *is inferred from* Φ_k *by Rule* 3;

(6.4) *for some* $k < m$, Φ_m *is inferred from* Φ_k *by Rule* 4;

(6.5) Φ_m *is an axiom of L.*

Φ is said to be a *theorem in L*, in symbols $\vdash_L \Phi$, if there exists a proof of Φ in *L*.

Lemma 1.4. *Suppose that* $\Phi_0, \ldots, \Phi_n$ *is a proof in L*, $X = V(\Phi_0) \cup \ldots \cup$ $\cup V(\Phi_n)$, *and* $\tau \in V^X$ *is one to one. Then* $S(\tau)\Phi_0, \ldots, S(\tau)\Phi_n$ *is also a proof in L.*

Proof. This results from the following easily verified facts, where $V(\Phi) \cup V(\Psi) \cup V(\theta) \subseteq X$:

if Φ is inferred from Ψ, θ by Rule 1, then $S(\tau)\Phi$ is inferred from $S(\tau)\Psi$, $S(\tau)\theta$ by Rule 1;

if $n \in \{2, 3, 4\}$ and Φ is inferred from Ψ by Rule n, then $S(\tau)\Phi$ is inferred from $S(\tau)\Psi$ by rule n;

if $n \in \{1, 2, 3, 4, 5\}$ and Φ is an instance of Axiom n, then $S(\tau)\Phi$ is an instance of Axiom n.

Theorem 1.5. *Suppose* $\mathfrak{m} \leqslant \overline{\overline{V}}$ *and, for each* $p \in P$, $\overline{\overline{\mu(p)}} < \overline{\overline{V}}$. *Then any theorem in L has a proof which does not use Rule 4, i.e. a proof* $\Psi_0, \ldots, \Psi_n$ *in which either* (6.1), (6.2), (6.3), *or* (6.5) *holds for each* $k \leqslant n$.

Proof. It is sufficient to prove that if Φ has a proof $\Psi_0, \ldots, \Psi_m$ in *L* which does not use Rule 4, and if $\tau \in V^{V(\Phi)}$ is one to one, then $S(\tau)\Phi$ has a proof in *L* which does not use Rule 4. It follows from the hypotheses that $\overline{\overline{V(\Phi)}} < \overline{\overline{V}}$. Therefore there is a one to one function $\sigma \in V^V$ such that $\tau \subseteq \sigma$. Clearly $S(\sigma)\Phi = S(\tau)\Phi$. Then by Lemma 1.4 and its proof, $S(\sigma)\Psi_0, \ldots, S(\sigma)\Psi_n$ is a proof of $S(\tau)\Phi$ in *L* which does not use Rule 4.

Notice the analogy between the notions of formation and formula and those of proof and theorem. An important feature of the logic *L* is the fact that all formations and all proofs are finite, although it is possible for infinitely many variables to occur in a single formula in *L*.

Rule 1 and Axioms 1, 2, 3 form the basis of the classical finitary propositional logic, e.g. as developed in [1]. In case $\mathfrak{m} = 2$ and all of the values of μ are finite, *L* is a finitary first order predicate logic. In case $\mathfrak{m} \leqslant 1$ and all values of μ are zero, *L* is a finitary propositional logic.

As we have pointed out in the introduction, the system *L* is patterned after Church's system F_1. There are, however, certain differences. The rule of substitution, Rule 4, has no counterpart in F_1, and is included here to cope with the case in which the hypotheses of 1.5 are not satisfied. Note that for the ordinary first order logic, Theorem 1.5 is applicable and Rule 4 may be eliminated. In order to avoid a rather complicated condition involving the substitution of variables, we have stated Axiom 5 in a weaker, but simpler, form than the corresponding axiom of F_1, and to fill the gap we have included an additional rule of inference—the rule of free substitution, Rule 3.

For each Φ, $\Psi \in F$, let $(\Phi \wedge \Psi)$ denote the formula $\langle\langle\Phi \to \langle\Psi \to \mathfrak{f}\rangle\rangle \to \mathfrak{f}\rangle$, which is called the *conjunction* of Φ and Ψ. The familiar commutative and associative laws for conjunction are provable in *L*. If $\Phi_1, \ldots, \Phi_n \in F$, let $(\Phi_1 \wedge \ldots \wedge \Phi_n)$ denote the formula $\big((\Phi_1 \wedge \ldots \wedge \Phi_{n-1}) \wedge \Phi_n\big)$ if $n > 1$, the formula Φ_1 if $n = 1$, and the formula $\langle \mathfrak{f} \to \mathfrak{f} \rangle$ if $n = 0$.

We shall similarly introduce the other usual propositional connectives and quantifiers in the following definitions. For each $\Phi, \Psi \in F$, let

$(\Phi \vee \Psi)$ denote $\langle\langle \Phi \to \mathfrak{f}\rangle \to \Psi\rangle$;

$(\Phi \leftrightarrow \Psi)$ denote $(\langle \Phi \to \Psi\rangle \wedge \langle \Psi \to \Phi\rangle)$;

$(\neg \Phi)$ denote $\langle \Phi \to \mathfrak{f}\rangle$;

and

$(\exists W \Phi)$ denote $(\neg \langle \forall W\, (\neg \Phi)\rangle)$.

We shall write $\Gamma \vdash_L \Psi$ if $\Gamma \subseteq F$, $\Psi \in F$, and there exist $\Phi_1, \ldots, \Phi_n \in \Gamma$ such that $\vdash_L \langle (\Phi_1 \wedge \ldots \wedge \Phi_n) \to \Psi\rangle$. A subset Γ of F is said to be *inconsistent* in L if $\Gamma \vdash_L \mathfrak{f}$, and otherwise Γ is said to be *consistent* in L.

We shall now introduce some semantical, or model-theoretic notions, i.e. notions dealing with our intended interpretation of the logic L.

By a *structure of type* μ we mean a system $\mathfrak{A} = (A, R_p)_{p \in P}$ in which A is a non-empty set and, for each $p \in P$, $R_p \subseteq A^{\mu(p)}$. $\mathfrak{A}$ is said to be of power $\mathfrak{n}$ if the set A is of power $\mathfrak{n}$.

Hereafter $\mathfrak{A}$ will always denote a structure $(A, R_p)_{p \in P}$ of type μ. Let $a \in A^V$. The notion of a formula Φ being *satisfied by* a in $\mathfrak{A}$, in symbols $\models_{\mathfrak{A}} \Phi[a]$, is defined recursively as follows:

(7.1) *if Φ is an atomic formula $\langle px\rangle$, then $\models_{\mathfrak{A}} \Phi[a]$ if $a \circ x \in R_p$;*

(7.2) *if $\Phi = \mathfrak{f}$, then not $\models_{\mathfrak{A}} \Phi[a]$;*

(7.3) *if $\Phi = \langle \Psi \to \theta\rangle$, then $\models_{\mathfrak{A}} \Phi[a]$ if either $\models_{\mathfrak{A}} \theta[a]$ or not $\models_{\mathfrak{A}} \Psi[a]$;*

(7.4) *if $\Phi = \langle \forall W \Psi\rangle$, then $\models_{\mathfrak{A}} \Phi[a]$ if, for every $b \in A^V$ such that $b \upharpoonright (V \sim W) = a \upharpoonright (V \sim W)$, we have $\models_{\mathfrak{A}} \Psi[b]$.*

A formula Φ is said to be *satisfiable in* $\mathfrak{A}$ if there exists a function $a \in A^V$ such that $\models_{\mathfrak{A}} \Phi[a]$. Similarly, at set Γ of formulas is said to be *satisfiable in* $\mathfrak{A}$ if there exists a function $a \in A^V$ such that, for each $\Phi \in \Gamma$, we have $\models_{\mathfrak{A}} \Phi[a]$. Γ is said to be *satisfiable* if it is satisfiable in some structure of type μ.

A formula Φ is said to be *valid in* $\mathfrak{A}$ if, for every function $a \in A^V$, we have $\models_{\mathfrak{A}} \Phi[a]$. Φ is said to be *valid* if it is valid in every structure of type μ.

We shall now state without proof a number of simple results which are analogous to familar results in the ordinary predicate logic.

THEOREM 1.6. *Suppose that $\Phi, \Psi \in F$, $\Gamma \subseteq F$, $\mathfrak{A}$ is a structure of type μ, and $a \in A^V$. Then*

(i) *$\models_{\mathfrak{A}} (\Phi \wedge \Psi)[a]$ if and only if $\models_{\mathfrak{A}} \Phi[a]$ and $\models_{\mathfrak{A}} \Psi[a]$.*

(ii) *If $\tau \in V^{V(\Phi)}$ and τ is one to one, then $\models_{\mathfrak{A}} S(\tau)\Phi[a]$ if and only if $\models_{\mathfrak{A}} \Phi[a \circ (\tau \upharpoonright V)]$.*

(iii) *If $\tau \in \big(V \sim V_b(\Phi)\big)^{V_f(\Phi)}$ then $\models_{\mathfrak{A}} S_f(\tau)\Phi[a]$ if and only if $\models_{\mathfrak{A}} \Phi[a \circ (\tau \upharpoonright V)]$.*

(iv) *Φ is satisfiable if and only if $(\neg \Phi)$ is not valid.*

(v) *If $\vdash_L \Phi$, then Φ is valid.*

(vi) *If Γ is satisfiable, then Γ is consistent in L.*

(vii) *The empty set of formulas is consistent, i.e. it is not the case that $\vdash_L \mathfrak{f}$.*

(viii) *If $\Gamma \vdash_L \Phi$ and $\models_{\mathfrak{A}} \theta[a]$ for each $\theta \in \Gamma$, then $\models_{\mathfrak{A}} \Phi[a]$.*

(ix) *If Γ is satisfiable in a structure of power $\mathfrak{n}$, and if $\mathfrak{n} < \mathfrak{n}'$, then Γ is satisfiable in a structure of power $\mathfrak{n}'$.*

§ 2. Lemmas for the Completeness Theorem. Throughout this section, we shall assume that V^* denotes a set of symbols which includes V, but is disjoint from P and contains none of the symbols $\to$, $\mathfrak{f}$, $\forall$. We shall let $L^* = L(V^*, \mu, \mathfrak{m})$, and let F^* be the set of formulas of L^*. For certain purposes, it is intuitively helpful to regard the elements of $V^* \sim V$ as "individual constants".

Φ is said to be a *V-formula in L^** in $\Phi \in F^*$ and $V_b(\Phi) \subseteq V$. Φ is said to be a *V-sentence in L^** in $\Phi \in F^*$, $V_b(\Phi) \subseteq V$, and $V_f(\Phi) \subseteq V^* \sim V$.

LEMMA 2.1. *Let F' be the set of all V-formulas in L^*. Then*

$$\overline{\overline{F'}} = \sum_{\mathfrak{n} < \mathfrak{m}} \overline{\overline{V}}^{\mathfrak{n}} + \sum_{p \in P} \overline{\overline{V^*}}^{\overline{\overline{\mu(p)}}} + \aleph_0 .$$

Proof. The number of atomic formulas in F' is clearly $\sum_{p \in P} \overline{\overline{V^*}}^{\overline{\overline{\mu(p)}}}$, so $\sum_{p \in P} \overline{\overline{V^*}}^{\overline{\overline{\mu(p)}}} \leqslant \overline{\overline{F'}}$. The set $S_{\mathfrak{m}}(V)$ has power $\sum_{\mathfrak{n} < \mathfrak{m}} \overline{\overline{V}}^{\mathfrak{n}}$, in view of the assumption II. Hence $\sum_{\mathfrak{n} < \mathfrak{m}} \overline{\overline{V}}^{\mathfrak{n}} \leqslant \overline{\overline{F'}}$. By (1.2) and (1.3), we have $\aleph_0 \leqslant \overline{\overline{F'}}$. Since each member of F' is constructed from finitely many atomic formulas, members of $S_{\mathfrak{m}}(V)$, and symbols $\forall$, $\to$, $\mathfrak{f}$, we have

$$\overline{\overline{F'}} \leqslant \sum_{\mathfrak{n} < \mathfrak{m}} \overline{\overline{V}}^{\mathfrak{n}} + \sum \overline{\overline{V^*}}^{\overline{\overline{\mu(p)}}} + \aleph_0 .$$

The desired conclusion follows.

LEMMA 2.2. *Let $\Gamma \subseteq F$. Then Γ is consistent in L^* if and only if it is consistent in L.*

Proof. It is obvious that, if $\Gamma \vdash_L \mathfrak{f}$, then $\Gamma \vdash_{L^*} \mathfrak{f}$, for any proof in L is also a proof in L^*.

Suppose that $\Gamma \vdash_{L^*} \mathfrak{f}$. Then for some $\Phi_1, \ldots, \Phi_n \in \Gamma$, we have

$$\vdash_{L^*} \langle (\Phi_1 \wedge \ldots \wedge \Phi_n) \to \mathfrak{f}\rangle .$$

Let $\Phi = \langle(\Phi_1 \wedge \ldots \wedge \Phi_n) \to \mathfrak{f}\rangle$. Suppose that $\Psi_0, \ldots, \Psi_m$ is a proof of Φ in L^*. Let

$$X = V(\Psi_0) \cup \ldots \cup V(\Psi_m)\,.$$

Then, $\overline{\overline{X}} \leqslant \overline{\overline{V}}$, in view of I, II, III. Let τ be a one to one function on X into V. Then by 1.4,

(1) $$S(\tau)\Psi_0, \ldots, S(\tau)\Psi_m$$

is a proof in L^*. Since each formula in (1) belongs to F, (1) is also a proof in L. Thus $\vdash_L S(\tau)\Phi$. Let $\sigma = \tau \upharpoonright V(\Phi)$. Then $S(\sigma)\Phi = S(\tau)\Phi$. Moreover, $S(\sigma^{-1})S(\sigma)\Phi = \Phi$. Since $\Phi \in F$, we have $V(\Phi) \subseteq V$, and thus σ^{-1} is a one to one function on $V\big(S(\sigma)\Phi\big)$ into V. It now follows by the rule of substitution that $\vdash_L \Phi$. We therefore have $\Gamma \vdash_L \mathfrak{f}$, and our proof is complete.

Lemma 2.3. *Suppose that Γ is a consistent set of V-sentences in L^*. Then there exists a maximal consistent set of V-sentences in L^* which includes Γ.*

Proof. Let Ψ be any function whose domain is an ordinal α and whose range is the set of all V-sentences in L^*. Let $\Gamma_0 = \Gamma$. Define the set Γ_β of V-sentences recursively for each $\beta \leqslant \alpha$ as follows:

if $\beta = \gamma + 1$,

$$\Gamma_\beta = \begin{cases} \Gamma_\gamma & \text{if} \quad \Gamma_\gamma \cup \{\Psi_\gamma\} \vdash_{L^*} \mathfrak{f}\,, \\ \Gamma_\gamma \cup \{\Psi_\gamma\} & \text{otherwise;} \end{cases}$$

if $\beta > 0$ and β is a limit ordinal,

$$\Gamma_\beta = \bigcup_{\gamma<\beta} \Gamma_\gamma\,.$$

It follows by transfinite induction that Γ_α is consistent in L. Clearly, $\Gamma \subseteq \Gamma_\alpha$ and Γ_α is a set of V-sentences. Finally, Γ_α is maximal, because if Φ is any V-sentence not belonging to Γ_α, then $\Phi = \Psi_\alpha$ for some $\beta < \alpha$, and since $\Gamma_{\beta+1} \neq \Gamma_\beta \cup \{\Psi_\beta\}$, we have $\Gamma_{\beta+1} \cup \{\Phi\} \vdash_L \mathfrak{f}$, and so $\Gamma_\alpha \cup \{\Phi\} \vdash_L \mathfrak{f}$.

Lemma 2.3 is analogous to, and is proved in the same way, as Lindenbaum's Theorem (see [26], p. 98 ff.; [21], p. 39). Like Lindenbaum's Theorem, Lemma 2.3 may also be obtained constructively from the prime ideal theorem in Boolean algebra (cf. [24]), by a well-known method using the so-called "Lindenbaum algebra" formed from equivalence classes of formulas (cf. [20], [26], p. 348).

Suppose that $V^* \neq V$, and Γ is a set of V-setences in L^*. We shall denote by $\mathfrak{A}(\Gamma, V)$ the structure $\mathfrak{A}$ of type μ such that

(8.1) $$A = V^* \sim V;$$

and

(8.2) $$\textit{for each } p \in P, \quad R_p = \{x \in A^{\mu(p)} \mid \langle px\rangle \in \Gamma\}.$$

Lemma 2.4. *Assume that*

(i) *Γ is a maximal consistent set of V-sentences in L^*;*

(ii) *for any V-sentence $\langle \forall W\, \Psi\rangle$, there exists $\tau \in (V^* \sim V)^W$ such that $\langle S_f(\tau)\Psi \to \langle \forall W\, \Psi\rangle\rangle \in \Gamma$.*

Then for each V-formula Φ in L^* and each function $b \in (V^* \sim V)^V$, we have

$$\models_{\mathfrak{A}(\Gamma,V)} \Phi[b \upharpoonright V^*] \textit{ if and only if } S_f(b)\Phi \in \Gamma\,.$$

Proof. Let $\mathfrak{A} = \mathfrak{A}(\Gamma, V)$. Let H be the set of all V-formulas $\Phi \in F$ such that for any function $b \in (V^* \sim V)^V$, we have

$$\models_{\mathfrak{A}} \Phi[b \upharpoonright V^*] \quad \text{if and only if} \quad S_f(b)\Phi \in \Gamma\,.$$

We shall show by induction that every V-formula belongs to H.

By (8.2), (5.1), and (7.1), every atomic V-formula belongs to H. By (7.2) and (5.2), $\mathfrak{f} \in H$. If $\Phi = \langle \Psi \to \theta\rangle$ and $\Psi,\ \theta \in H$, then it follows easily, from (5.3), (7.3), propositional logic, and (i), that $\Phi \in H$.

Suppose finally that $b \in (V^* \sim V)^V$, $\Phi = \langle \forall W\, \Psi\rangle$, $W \in S_m(V)$, and $\Psi \in H$. Let us write $c \approx b$ whenever $c \in (V^* \sim V)^{V^*}$ and $c \upharpoonright (V^* \sim W) = b \upharpoonright (V^* \sim W)$.

Suppose $S(b)\Phi \in \Gamma$. By (5.4), we have

$$S_f(b)\Phi = \langle \forall W\, S_f\big(b \upharpoonright (V^* \sim W)\big)\, \Psi\rangle\,.$$

Since $W \subseteq V$ and by 1.3, we have

$$S_f(c)\Psi = S_f(c \upharpoonright W)\, S_f\big(b \upharpoonright (V^* \sim W)\big)\, \Psi$$

whenever $c \approx b$. Then by Axiom 5 and (i), we have

$$S_f(c)\Psi \in \Gamma \quad \text{whenever} \quad c \approx b\,.$$

Since $\Psi \in H$, we have

$$\models_{\mathfrak{A}} \Psi[c] \quad \text{whenever} \quad c \approx b\,.$$

Then by (7.4), $\models_{\mathfrak{A}} \Phi[b \upharpoonright V^*]$.

Conversely suppose that $\models_{\mathfrak{A}} \Phi[b \upharpoonright V^*]$. By (5.4), we have $S_f(b)\Phi = \langle \forall W\, S_f\big((b \upharpoonright (V^* \sim W)\big)\, \Psi\rangle$. It follows from (ii) that there exists a function $\tau \in (V^* \sim V)^W$ such that

$$\langle S_f(\tau)\, S_f\big(b \upharpoonright (V^* \sim W)\big)\, \Psi \to S_f(b)\Phi\rangle \in \Gamma\,.$$

Let $d = \tau \cup \big(b \upharpoonright (V^* \sim W)\big)$. Then by 1.3,

$$S_f(d)\Psi = S_f(\tau)\, S_f\big(b \upharpoonright (V^* \sim W)\big)\, \Psi\,.$$

Also, $d \approx b$, and it follows from 7.4 that $\models_{\mathfrak{A}} \Psi[d]$. Since $\Psi \in H$ and $(d \restriction V) \restriction V^* = d$, we have $S_f(d)\Psi \in \Gamma$. But by (i) and since $\langle S_f(d)\Psi \rightarrow S_f(b)\Phi \rangle \in \Gamma$, we have $S_f(b)\Phi \in \Gamma$.

This verifies that $\Phi \in H$, and completes the proof that H is the set of all V-formulas.

§ 3. The Completeness Theorem.

THEOREM 3.1. (Completeness Theorem) *If Γ is a consistent set of formulas in L, then Γ is satisfiable in some structure $\mathfrak{A}$ of type μ.*

Moreover, if we have

(a) $$\overline{\overline{F}} \leqslant \mathfrak{n},$$

and

(b) $$\text{for each } p \in P, \quad \mathfrak{n} = \mathfrak{n}^{\overline{\overline{\mu(p)}}},$$

then $\mathfrak{A}$ may be taken to be of power $\mathfrak{n}$.

Proof. Let β be the initial ordinal of power $\mathfrak{n}$.

Let $V' \subseteq V$ and $\overline{\overline{V'}} = \overline{\overline{V \sim V'}}$, and let τ_0 be a one to one function on V into V'; this can be done in view of I. Let

$$\Gamma_0 = \{S(\tau_0)\Phi \mid \Phi \in \Gamma\}.$$

Thus Γ_0 is a set of V'-formulas in L. Then by the consistency of Γ and the rule of substitution, it follows that Γ_0 is consistent in L. Let τ_1 be a one to one function on V into $V \sim V'$, and let

$$\Gamma_1 = \{S_f(\tau_1)\Phi \mid \Phi \in \Gamma_0\}.$$

Thus Γ_1 is a set of V'-sentences in L. We shall show that Γ_1 is consistent in L. Suppose that there exist $\Phi_1, \ldots, \Phi_n \in \Gamma_0$ such that

$$\vdash_L \langle (S_f(\tau_1)\Phi_1 \wedge \ldots \wedge S_f(\tau_1)\Phi_n) \rightarrow \mathfrak{f} \rangle .$$

Then by (5.3), we have

$$\vdash_L S_f(\tau_1) \langle (\Phi_1 \wedge \ldots \wedge \Phi_n) \rightarrow \mathfrak{f} \rangle .$$

By the rule of free subsitution, it follows that

$$\vdash_L \langle (\Phi_1 \wedge \ldots \wedge \Phi_n) \rightarrow \mathfrak{f} \rangle .$$

But this contradicts the consistency of Γ_0, and hence Γ_1 is consistent in L.

For each $v \in V'$ and $\gamma < \beta$, let us choose new symbols $w_{v\gamma}$. Let

$$V^* = V \cup \{w_{v\gamma} \mid v \in V;\ \gamma < \beta\},$$

let $L^* = L(V^*, \mu, \mathfrak{m})$, and let G^* be the set of V'-sentences in L^*. By (a) and Lemma 2.1, we have

$$\sum_{\mathfrak{p} < \mathfrak{m}} \overline{\overline{V}}^{\mathfrak{p}} \leqslant \mathfrak{n}, \quad \overline{\overline{P}} < \mathfrak{n}, \quad \sum_{p \in P} \overline{\overline{V}}^{\overline{\overline{\mu(p)}}} \leqslant \mathfrak{n}, \quad \text{and} \quad \aleph_0 \leqslant \mathfrak{n}.$$

Since $\overline{\overline{V^*}} = \overline{\overline{V}} + \mathfrak{n}$, we have, by the above observations and (b), that $\sum_{p \in P} \overline{\overline{V^*}}^{\overline{\overline{\mu(p)}}} \leqslant \mathfrak{n}$. Then by 2.1, $\overline{\overline{G^*}} \leqslant \mathfrak{n}$. We may therefore choose a function Ψ which has domain β and range G^*. For each $\gamma \in \beta$, let $\sigma_\gamma \in V^{*V'}$ be the function such that

$$\sigma_\gamma(v) = w_{v\gamma} \quad \text{for each} \quad v \in V'.$$

It follows from (a) and (b) that, for each $\gamma \in \beta$, the set $\bigcup_{\alpha < \gamma} V(\Psi(\alpha))$ has power $< \mathfrak{n}$. Therefore there exists a one-to-one function $\lambda \in \beta^\beta$ such that, for each $\gamma \in \beta$, we have

$$\sigma_{\lambda(\gamma)}[V'] \cap \bigcup_{\alpha \leqslant \gamma} V(\Psi(\gamma)) = 0.$$

For each $\gamma \in \beta$, let $\theta(\gamma)$ be the member of G^* defined by the following condition:

$$\theta(\gamma) = \begin{cases} S_f(\sigma_{\lambda(\gamma)})\Phi, & \text{if} \quad \Psi(\gamma) = \langle \forall W \Phi \rangle, \\ \Psi(\gamma), & \text{if} \quad \Psi(\gamma) \text{ is not of the form } \langle \forall W \Phi \rangle. \end{cases}$$

Let

$$\Gamma_2 = \Gamma_1 \cup \{\langle \theta(\gamma) \rightarrow \Psi(\gamma) \rangle \mid \gamma \in \beta\};$$

Thus Γ_2 is a set of V'-sentences in L^*. We shall show that Γ_2 is consistent in L^*.

First of all, it follows from Lemma 2.2 that, since Γ_1 is consistent in L, Γ_1 is also consistent in L^*.

Now suppose $\Gamma_2 \vdash_{L^*} \mathfrak{f}$. Then there exist $\Phi_1, \ldots, \Phi_n \in \Gamma_2$ such that

$$\vdash_{L^*} \langle (\Phi_1 \wedge \ldots \wedge \Phi_n) \rightarrow \mathfrak{f} \rangle,$$

but every proper subset of $\{\Phi_1, \ldots, \Phi_n\}$ is consistent in L^*. We cannot have $\Phi_1, \ldots, \Phi_n \in \Gamma_1$ because Γ_1 is consistent in L^*. It follows that $n > 0$ and that

$$\vdash_{L^*} \langle (\Phi_1 \wedge \ldots \wedge \Phi_{n-1}) \rightarrow (\neg \Phi_n) \rangle .$$

Let $\Phi = (\Phi_1 \wedge \ldots \wedge \Phi_{n-1})$; thus Φ is consistent in L^*. Let α be the greatest ordinal $\gamma \in \beta$ such that

$$\langle \theta(\gamma) \rightarrow \Psi(\gamma) \rangle \in \{\Phi_1, \ldots, \Phi_n\};$$

such an ordinal clearly exists. We may assume

$$\Phi_n = \langle \theta(\alpha) \rightarrow \Psi(\alpha) \rangle .$$

We must have $\theta(\alpha) \neq \Psi(\alpha)$, for otherwise $\vdash_{L^*} \Phi_n$, and hence we would have $\vdash_{L^*} \langle \Phi \rightarrow \mathfrak{f} \rangle$, contradicting our assumption. For some $W \in S_{\mathfrak{m}}(V')$ and some V'-formula Ψ_0 in L^*, we have

$$\Psi(\alpha) = \langle \forall W \Psi_0 \rangle \quad \text{and} \quad \theta(\alpha) = S_f(\sigma_{\lambda(\alpha)})\Psi_0 .$$

By propositional logic,

$$\vdash_{L^*}\langle(\neg\Phi_n)\to\theta(a)\rangle .$$

Then

$$\vdash_{L^*}\langle(\Phi)\to\theta(a)\rangle .$$

Since $V'\cap V_f(\Phi)=0$, we have, by (5.3),

$$\langle\Phi\to\theta(a)\rangle=S_f(\sigma_{\lambda(a)})\langle\Phi\to\Psi_0\rangle .$$

In view of the fact that $\big(V(\Phi)\cup V(\Phi_n)\big)\cap\sigma_{\lambda(a)}[V']=0$, and hence that $V_f(\langle\Phi\to\Psi_0\rangle)\cap\sigma_{\lambda(a)}[V']=0$, it follows that the function $\sigma_{\lambda(a)}\restriction V_f(\langle\Phi\to\Psi_0\rangle)$ is one to one. Then by the rule of free substitution,

$$\vdash_{L^*}\langle\Phi\to\Psi_0\rangle .$$

By generalization, we have

$$\vdash_{L^*}\langle\forall W\langle\Phi\to\Psi_0\rangle\rangle .$$

Since $V'\cap V_f(\Phi)=0$ and $W\subseteq V'$, we have, by Axiom 4 and modus ponens,

$$\vdash_{L^*}\langle\Phi\to\Psi(a)\rangle .$$

Then by propositional logic,

$$\vdash_{L^*}\langle\Phi\to\Phi_n\rangle .$$

But now, since $\{\Phi_1,\dots,\Phi_n\}$ is inconsistent, $\{\Phi_1,\dots,\Phi_{n-1}\}$ must be inconsistent, which contradicts our assumption.

We conclude that Γ_2 is consistent in L^*. By Lemma 2.3, there is a maximal consistent set Γ_3 of V'-sentences in L^* which included Γ_2.

Let us consider the structure $\mathfrak{A}=\mathfrak{A}(\Gamma_3,V')$ of the type μ. We have $A=V^*\sim V'$, and thus $\overline{\overline{A}}=\mathfrak{n}+\overline{\overline{V}}$. By Lemma 2.4, it follows that for each function $b\in A^{V'}$ and each V'-formula Φ in L^*, we have

$$\models_{\mathfrak{A}}\Phi[b\restriction V^*]\quad\text{if and only if}\quad S_f(b)\Phi\in\Gamma_3 .$$

Recall that $\tau_1\in(V\sim V')^{V'}$ (so $\tau_1\in A^{V'}$), that Γ_0 is a set of V'-formulas in L, that for any $\Phi_0\in\Gamma_0$ we have $S_f(\tau_1)\Phi_0\in\Gamma_1$, and that $\Gamma_1\subseteq\Gamma_3$. Let $\Phi_0\in\Gamma_0$. Then

$$\models_{\mathfrak{A}}\Phi_0[\tau_1\restriction V^*] .$$

Then, with respect to L, we have

$$\models_{\mathfrak{A}}\Phi_0[\tau_1\restriction V] .$$

Let $a=\tau_1\circ\tau_0$; thus $a\in A^V$. Let $\Phi\in\Gamma$. Then $S(\tau_0)\in\Gamma_0$, and thus

$$\models_{\mathfrak{A}}S(\tau_0)\Phi[\tau_1\restriction V] .$$

By Theorem 1.6 (ii), and because $a=(\tau_1\restriction V)\circ\tau_0$, we have

$$\models_{\mathfrak{A}}\Phi[a] .$$

Therefore Γ is satisfied in $\mathfrak{A}$ by a.

If $\mu\notin\{0\}^P$, then since $\sum_{p\in P}\overline{\overline{V}}^{\overline{\overline{\mu(p)}}}\leqslant\mathfrak{n}$, we have $\overline{\overline{V}}\leqslant\mathfrak{n}$. Therefore $\overline{\overline{A}}=\mathfrak{n}+\overline{\overline{V}}=\mathfrak{n}$, so $\mathfrak{A}$ is of power $\mathfrak{n}$.

Suppose $\mu\in\{0\}^P$; then for each $p\in P$, we have either $R_p=A^0=\{0\}$ or $R_p=0$. It follows that for any non-empty set B and any $b\in B^V$, Γ is satisfied in the structure $(B,R_p)_{p\in P}$ by b. Thus if we choose B to be a set of power $\mathfrak{n}$, then Γ is satisfiable in the structure $(B,R_p)_{p\in P}$ of power $\mathfrak{n}$.

Our proof is complete.

Corollary 3.2. *A set of formulas in L is satisfiable if and only if it is consistent. Moreover, a formula in L is valid if and only if it is provable.*

Proof. By 1.5 (iii), (v), (vi), and by 3.1.

Theorem 3.3. *Theorem* 3.1 *remains true when condition* (a) *is replaced by*

(a′) $$\aleph_0+\overline{\overline{\Gamma}}\leqslant\mathfrak{n} .$$

Proof. For each $\Phi\in F$, define Φ_0 recursively as follows:

$$\langle px\rangle_0=\langle px\rangle;$$
$$\mathfrak{f}_0=\mathfrak{f};$$
$$\langle\Psi\to\theta\rangle_0=\langle\Psi_0\to\theta_0\rangle;$$
$$\langle\forall W\Psi\rangle_0=\langle\forall W\cap V(\Psi_0)\Psi_0\rangle .$$

It is easily seen that $(\Psi\leftrightarrow\Phi_0)$ is valid. For each $\Phi\in F$, define $P(\Phi)$ recursively by:

$$P(\langle px\rangle)=\{p\};$$
$$P(\mathfrak{f})=0;$$
$$P(\langle\Psi\to\theta\rangle)=P(\Psi)\cup P(\theta);$$
$$P(\langle\forall W\Psi\rangle)=P(\Psi) .$$

Let $\Gamma_0=\{\Phi_0|\ \Phi\in\Gamma\}$, let $P_0=\{P(\Phi_0)|\ \Phi\in\Gamma\}$, and let $V_0=\bigcup\{V(\Phi_0)|\ \Phi\in\Gamma\}$. Let $\mathfrak{m}_0$ be the smallest infinite cardinal $\mathfrak{p}$ such that, for each $p\in P_0$, $\overline{\overline{\mu(p)}}<\mathfrak{p}$. Let $L_0=L(V,\mu\restriction P_0,\mathfrak{m}_0)$, and let F_0 be the set of formulas in L_0. Then $\Gamma_0\subseteq F_0$. Since $P(\Phi_0)$ is finite for each $\Phi_0\in\Gamma_0$, we have $\overline{\overline{V(\Phi_0)}}\leqslant\mathfrak{m}_0$ for each $\Phi_0\in\Gamma_0$. It follows from hypothesis (b) of Theorem 3.1 that $\sum_{\mathfrak{p}<\mathfrak{m}}\mathfrak{n}^{\mathfrak{p}}=\mathfrak{n}$; therefore $\mathfrak{m}_0\leqslant\mathfrak{n}$, $\overline{\overline{V}}_0\leqslant\mathfrak{n}$, and $\sum_{\mathfrak{p}<\mathfrak{m}_0}\overline{\overline{V}}_0^{\mathfrak{p}}\leqslant\mathfrak{n}$. Moreover, we have $\overline{\overline{P}}_0\leqslant\aleph_0+\overline{\overline{\Gamma}}\leqslant\mathfrak{n}$. Hence by Lemma 2.1 and (b), $\overline{\overline{F}}_0\leqslant\mathfrak{n}$.

By Theorem 3.1, there is a structure $\mathfrak{A}=(A,R_p)_{p\in P}$ (say of power $\overline{\overline{F}}^{\overline{\overline{F}}}$) in which Γ is satisfiable. Then Γ_0 is satisfiable in $(A,R_p)_{p\in P}$. By Theorem 1.5, Γ_0 is consistent in L_0, and hence, by Theorem 3.1, Γ_0 is satisfiable in a structure $\mathfrak{B}=(B,S_p)_{p\in P_0}$ of type $\mu\restriction P_0$ and power $\mathfrak{n}$. Finally, if we put $S_p=0$ for $p\in P\sim P_0$, then Γ is satisfiable in the structure $(B,S_p)_{p\in P}$ of type μ and power $\mathfrak{n}$. This completes the proof.

Corollary 3.4. (Compactness Theorem). *Suppose that $\Gamma \subseteq F$ and every finite subset of Γ is satisfiable. Then Γ is satisfiable.*

Proof. By Theorem 1.5, every finite subset of Γ is consistent. It follows that Γ is consistent. Therefore, by Theorem 3.1, Γ is satisfiable.

The statement of the Compactness Theorem is purely semantical in nature, i.e. it does not refer to the axioms and rules of inference of L. As in the case of the usual first order predicate logic, it is possible to give a direct semantical proof of the compactness theorem which is much shorter than the above syntactical proof. For example, one can prove the Compactness Theorem by a perfectly straightforward modification of the proof using ultraproducts outlined in [18].

Corollary 3.5. *Suppose that $\Gamma \subseteq F$, Γ is satisfiable, and $\mathfrak{n}$ satisfies conditions* (a′) *of* 3.3 *and* (b) *of* 3.1. *Then Γ is satisfiable in a structure of power $\mathfrak{n}$.*

Proof. By Theorem 1.6, Γ is consistent. The conclusion follows by Theorem 3.3.

Corollary 3.5 is the analogue of the Löwenheim-Skolem Theorem [10] for ordinary first order predicate logic. Like Corollary 3.4, it is purely semantical in nature. A direct semantical proof of Corollary 3.5 is also easy to construct by considering known proofs in ordinary logic. For example, one may give a proof using "Skolem functions" based on the argument in [29], Theorem 2.1.

§ 4. Complementary examples. In this section, we shall sometimes write (a, b) for the two-termed sequence $\{\langle 0, a\rangle, \langle 1, b\rangle\}$. It is well known from ordinary first order predicate logic that Theorem 3.1 is no longer true if condition (a) is removed. In fact, for each cardinal $\mathfrak{n}$ we can easily give a consistent set Γ of $\mathfrak{n}^+$ formulas, in a logic L with $\overline{\overline{V}} = \mathfrak{n}^+$ and a single binary predicate symbol, which is not satisfied in any structure of power $\mathfrak{n}$. Obviously, condition (a′) cannot be removed from Theorem 3.3 for the same reason. In the examples below, think of q as "equals".

Example 4.1. This shows that in Theorem 3.3, hypothesis (b) cannot be removed. Γ will imply there is a one-one function with domain 2^β.

Let β be any infinite ordinal. Let $V = \beta + \beta$, and $\mathfrak{m} = \overline{\overline{\beta}}^+$. Let $P = \{p, q\}$, $\mu(p) = \beta$, and $\mu(q) = 2$. Finally, let Γ be the following set of formulas (omitting outer brackets):

$$\exists\{0,1\}\,(\neg q(0,1));$$

$$\forall \beta \sim \{0\}(\exists\{0\}\langle p\, 0 \restriction \beta\rangle);$$

[10] For references concerning the Löwenheim-Skolem Theorem, see [29]. For generalizations to the infinitary system in [27], see [8].

for each $\gamma \in \beta \sim \{0\}$, the formula

$$\forall \beta + \beta \langle (\langle p\, 0 \restriction \beta\rangle \wedge \langle py\rangle) \rightarrow \langle q(\gamma, y_\gamma)\rangle\rangle,$$

where $y_\alpha = 0$ if $\alpha = 0$, and $y_\alpha = \beta + \alpha$ if $0 < \alpha < \beta$.

It is then clear that $\overline{\overline{\Gamma}} = \overline{\overline{\beta}}$. However, $\overline{\overline{\beta}} < \overline{\overline{\beta}}^{\overline{\overline{\mu(p)}}}$, so (b) fails with $\mathfrak{n} = \overline{\overline{\beta}}$. Moreover, Γ is satisfiable in a structure of power $2^{\overline{\overline{\beta}}}$ but in no smaller structure. In fact, if $\mathfrak{A} = (A, R_p, R_q)$ is a structure in which Γ is satisfied, $a, b \in A$, and $(a, b) \notin R_q$, then there is a one-to-one function f on $\{a, b\}^{\beta \sim \{0\}}$ into A such that $f(x) = c$ implies $x \cup \langle 0, c\rangle \in R_p$, so $\overline{\overline{A}} \geqslant 2^{\overline{\overline{\beta}}}$.

Example 4.2. This example shows that in Theorem 3.1, hypothesis (b) cannot be removed. Here the function will have domain n^ω.

Let $\mathfrak{n}_0$ be infinite and, for each $n < \omega$, let $\mathfrak{n}_{n+1} = 2^{\mathfrak{n}_n}$. Let $\mathfrak{n} = \sum_{n<\omega} \mathfrak{n}_n$. Let $V = \omega + \omega$, and $\mathfrak{m} = \aleph_1$. Let $\overline{\overline{\beta}} = \mathfrak{n}$, $P = \{p, q, r_\gamma \mid \gamma < \beta\}$, $\mu(p) = \omega$, $\mu(q) = 2$, and, for each $\gamma < \beta$, $\mu(r_\gamma) = 1$. It follows from Lemma 2.1 that $\overline{\overline{F}} = \mathfrak{n}$. Notice, however, that $\mathfrak{n} < \mathfrak{n}^{\overline{\overline{\mu(p)}}}$, so (b) fails. Let Γ' be the following set of formulas: for each $\gamma \in \beta$, the formula

$$\exists\{0\}\langle r_\gamma(0)\rangle;$$

for each distinct $\gamma, \gamma' \in \beta$, the formula

$$\forall\{0,1\}\langle (r_\gamma(0) \wedge r_{\gamma'}(1)) \rightarrow (\neg q(0,1))\rangle;$$

$$\forall \omega \sim \{0\}(\exists\{0\}\langle p\, 0 \restriction \omega\rangle);$$

for each $n \in \omega \sim \{0\}$, the formula

$$\forall \omega + \omega \langle (\langle p\, 0 \restriction \omega\rangle \wedge \langle py\rangle) \rightarrow \langle q(n, y_n)\rangle\rangle,$$

where $y_m = 0$ if $m = 0$, and $y_m = \omega + m$ if $0 < m < \omega$.

Then Γ' is satisfiable in a structure of power $\mathfrak{n}^{\aleph_0}$, but in no smaller structure. Since $\mathfrak{n} < \mathfrak{n}^{\aleph_0}$, Γ' is not satisfiable in any model of power $\mathfrak{n}$. Thus the conclusion of 3.1 does not hold.

§ 5. A formal system with infinitary functions. In this section we shall see that the Completeness Theorem can be generalized to apply to formal systems which have infinitary functions as well as infinitary predicates. We shall state the relevant definitions and, since all the theorems may be proved by a straightforward generalization of the proofs of corresponding results for the system L, we shall omit all proofs.

Let Q be a set of symbols which is disjoint from the sets V, P, and $\{\rightarrow, \mathfrak{f}, \forall\}$, and let ν be a function whose domain is Q and whose range is a set of ordinals.

In addition to the assumptions I, II, III of § 1, we shall also assume:

IV. *for each* $q \in Q$, $\overline{\overline{\nu(q)}} \leqslant \overline{\overline{V}}$.

We construct the logic $L^{\#}(V, \mu, \nu, \mathfrak{m})$, or more briefly $L^{\#}$, as follows. The *symbols* of $L^{\#}$ include the symbols of L and, in addition, the function symbols $q \in Q$.

By the set T of *terms in* $L^{\#}$ we mean the least set U such that each of the following hold:

(#1.1) $$V \subseteq U;$$

(#1.2) *if* $q \in Q$ *and* $x \in U^{\nu(q)}$, *then the ordered pair* $\langle qx \rangle \in U$.

By an *atomic formula* in $L^{\#}$ we mean an ordered pair $\langle px \rangle$ where $p \in P$ and $x \in T^{\mu(p)}$.

The notion of a *formula* Φ in $L^{\#}$, and of a *formation of* Φ *in* $L^{\#}$, is defined exactly as in the case of L, except that we begin with atomic formulas in $L^{\#}$. In the definition of a formula in L given in § 1, we need only replace L everywhere by $L^{\#}$ (cf. conditions (1.1)-(1.4)). The set of all formulas in $L^{\#}$ is denoted by $F^{\#}$.

Lemma 5.1. *Lemma* 1.1 (the induction principle) *remains true if we replace* L, F *everywhere by* $L^{\#}$, $F^{\#}$ *respectively*.

The set $V(t)$ of *variables of* a term $t \in T$ is defined recursively by

(#2.1) *if* $t \in V$, *then* $V(t) = \{t\}$;

(#2.2) *if* $t = \langle qx \rangle$, *then* $V(t) = \bigcup_{a \in \nu(t)} V(x_a)$.

Let $X \subseteq V$. We define the set $T(X)$ of *terms in* X by:

$$T(X) = \{t \mid t \in T,\ V(t) \subseteq X\}.$$

Thus we have $T = T(V)$.

The set $V_f(\Phi)$ of *free variables of* a formula $\Phi \in F^{\#}$ is defined in exactly the same way as for formulas in L except that condition (2.1) is replaced by

(2.1′) *if* Φ *is an atomic formula* $\langle px \rangle$, *then* $V_f(\Phi) = \bigcup_{a \in \mu(p)} V(x_a)$;

conditions (2.2)-(2.4) remain unchanged.

The set $V_b(\Phi)$ of *bound variables of* a formula $\Phi \in F^{\#}$ is defined exactly as for formulas in L, by the conditions (3.1)-(3.4).

For each $\tau \in T^V$, we define $\overline{\tau} \in T^T$ recursively by:

(#4.1) $$\tau \subseteq \overline{\tau};$$

(#4.2) *if* $q \in Q$ *and* $x \in T^{\nu(q)}$, *then* $\overline{\tau}(\langle qx \rangle) = \langle q\, \overline{\tau} \circ x \rangle$.

For each $\tau \in V^V$ and $\Phi \in F^{\#}$, the formula $S(\tau)\Phi$ is defined in exactly the same way as for formulas in L, except that the condition (4.1) is replaced by:

(4.1′) *if* Φ *is an atomic formula* $\langle px \rangle$, *then*
$$S(\tau)\Phi = \langle p\, \overline{\tau} \circ x \rangle;$$

conditions (4.2)-(4.4) remain unchanged.

For each $\tau \in T^V$ and $\Phi \in F^{\#}$, the formula $S_f(\tau)\Phi$ is defined recursively by the condition:

(5.1′) *if* Φ *is an atomic formula* $\langle px \rangle$, *then*
$$S_f(\tau)\Phi = \langle p\, \overline{\tau} \circ x \rangle;$$

and by conditions (5.2), (5.3), (5.4), which remain unchanged.

The rules of inference for $L^{\#}$ are exactly the same as the rules of inference for L, i.e. *Rules* 1-4 given in § 1 with $F^{\#}$ everywhere replacing F.

The axiom schemata for $L^{\#}$ are the following:

Axioms 1-4 *as stated in* § 1, *with* $F^{\#}$ *everywhere replacing* F;

Axiom #5. $\langle\langle \forall W\, \Phi \rangle \to S_f(\tau)\Phi \rangle$, *whenever*

$$\langle \forall W\, \Phi \rangle \in F^{\#} \quad \text{and} \quad \tau \in T\big(V \sim V_b(\Phi)\big)^W.$$

The notions of proof in $L^{\#}$ and theorem in $L^{\#}$ are defined exactly as we did for L in § 1, by means of (6.1)-(6.5), with $L^{\#}$ everywhere replacing L.

We write $\vdash_{L^{\#}} \Phi$ in case Φ is a theorem in $L^{\#}$.

Theorem 5.2. *Suppose* $\mathfrak{m} \leqslant \overline{\overline{V}}$ *and, for each atomic formula* Φ *in* $L^{\#}$, $\overline{\overline{V(\Phi)}} < \overline{\overline{V}}$. *Then any theorem in* $L^{\#}$ *has a proof which does not use Rule* 4.

In case $\mathfrak{m} = 0$, the system $L^{\#}$ is essentially the same as the system without quantifiers and without identity considered by Słomiński in [23].

The notation $\Gamma \vdash_{L^{\#}} \Psi$ has the same meaning as in § 1, and Γ is *inconsistent in* $L^{\#}$ if $\Gamma \vdash_{L^{\#}} \mathfrak{f}$.

By a *structure of type* (μ, ν) we mean a system $\mathfrak{A} = (A, R_p, R_q)_{p \in P, q \in Q}$ such that $(A, R_p)_{p \in P}$ is a structure of type μ and, for each $q \in Q$, $R_q \in A^{(A^{\nu(q)})}$.

Let $a \in A^V$. We define the function $\overline{a} \in A^T$ recursively as follows:

(#7.1) $$a \subseteq \overline{a};$$

(#7.2) *if* $q \in Q$ *and* $x \in T^{\nu(q)}$, *then* $\overline{a}(\langle qx \rangle) = R_q(\overline{a} \circ x)$.

The notion of a formula $\Phi \in F^{\#}$ being *satisfied by* a *in* $\mathfrak{A}$, in symbols $\models_{\mathfrak{A}} \Phi[a]$, is defined recursively in the same way as in § 1, except that condition (7.1) is replaced by:

(7.1′) *if* Φ *is an atomic formula* $\langle px \rangle$ *in* $L^{\#}$, *then* $\models_{\mathfrak{A}} \Phi[a]$ *if* $\overline{a} \circ x \in R_p$;

conditions (7.2)-(7.4) remain the same.

The notions of *satisfiability* and *validity* of formulas in $L^{\#}$ are defined exactly as in § 1.

THEOREM 5.3. *Suppose that* $\Phi, \Psi \in F^{\#}$, $\Gamma \subseteq F^{\#}$, $\mathfrak{A}$ *is a structure of type* (μ, ν), *and* $a \in A^{V}$. *Then conditions* (i)-(ix) *of Theorem* 1.6 *remain valid with* $L^{\#}$ *everywhere replacing* L. *Moreover, we have*:

(iii′) *if* $\tau \in T\big(V \sim V_b(\Phi)\big)^{V_f(\Phi)}$, *then*

$$\models_{\mathfrak{A}} S_f(\tau)\Phi[a] \quad \textit{if and only if} \quad \models_{\mathfrak{A}} \Phi[\bar{a} \circ (\tau \upharpoonright V)]\,.$$

THEOREM 5.4 (Completeness Theorem). *If* Γ *is a consistent set of formulas in* $L^{\#}$, *then* Γ *is satisfiable in some structure* $\mathfrak{A}$ *of type* (μ, ν).

Moreover, if we have

(#a) $\quad \overline{\overline{F^{\#}}} \leqslant \mathfrak{n}$,

(#b) *for each* $p \in P$, $\quad \mathfrak{n} = \mathfrak{n}^{\overline{\overline{\mu(p)}}}$,

and

(#c) *for each* $q \in Q$, $\quad \mathfrak{n} = \mathfrak{n}^{\overline{\overline{\nu(q)}}}$,

then $\mathfrak{A}$ *may be taken to be of power* $\mathfrak{n}$.

The proof of Theorem 5.4 is an obvious modification of our proof of Theorem 3.1.

In the special case that $\mathfrak{m} = 0$, and thus $L^{\#}$ has no quantifiers, Theorem 5.4 reduces to the Completeness Theorem obtained by Słomiński in [23], (5.1). In fact, it is possible to give an alternative proof of Theorem 5.4 in general by applying the theorem of Słomiński and a suitable generalization of the Skolem Normal Form Theorem (cf. [1]). Such a proof of the Completeness Theorem would be somewhat more along the lines of Gödel's original proof in [5] than the proof we have given is.

Appendix. Relation to polyadic algebras. For the terminology and notation concerning polyadic algebras, we refer to the immediately preceding paper [3]. The results which follow show the relationship between the Completeness Theorem in this paper and the Representation Theorem in [3].

There is a natural correspondence between polyadic algebras and sets of sentences in the formal systems $L(\mu, V, \mathfrak{m})$. This correspondence depends on Lemmas A.1-A.4 below which we shall state without proofs; the proofs are straightforward but tedious.

Let V_1 be a set of symbols which is disjoint from P and V, which does not contain $\rightarrow$, $\mathfrak{f}$, or $\forall$, and such that $\overline{\overline{V_1}} = \overline{\overline{V}}$. Let $V^* = V \cup V_1$. Let τ_0 be a one-to-one function of V^* into V_1. Let G be the set of all sentences of L^*, and let H be the set of all formulas Φ in L^* such that $V(\Phi) \subseteq V$.

Let $\Gamma \subseteq G$. For each formula $\Phi \in H$, we define

$$\Phi/\Gamma = \{\Psi \mid \Psi \in H,\ \Gamma \vdash_{L^*} (\Psi \leftrightarrow \Phi)\}\,.$$

Clearly, for any $\Phi, \Psi \in H$, the sets Φ/Γ, Ψ/Γ are either equal or disjoint.

We shall write

$$H_\Gamma = \{\Phi/\Gamma \mid \Phi \in H\}\,.$$

For any $\Phi, \Psi \in H$, we define

$$(\Phi/\Gamma) +_\Gamma (\Psi/\Gamma) = (\Phi \vee \Psi)/\Gamma\,,$$
$$(\Phi/\Gamma) \cdot_\Gamma (\Psi/\Gamma) = (\Phi \wedge \Psi)/\Gamma\,,$$

and

$$-_\Gamma(\Phi/\Gamma) = (\neg\Phi)/\Gamma\,.$$

LEMMA A.1. *If* Γ *is consistent in* L^*, *then* $\mathbf{H}_\Gamma = (H_\Gamma, +_\Gamma, \cdot_\Gamma, -_\Gamma)$ *is a Boolean algebra* ([11]).

Following [6] and [3], we assume that "Boolean algebra" is defined in such a way that every Boolean algebra has at least two elements. Thus H_Γ is not a Boolean algebra if Γ is inconsistent in L^*.

We shall denote by 1_Γ the unit element of the Boolean algebra H_Γ. Note that

$$1_\Gamma = \{\Phi \mid \Phi \in H,\ \Gamma \vdash_{L^*} \Phi\}\,.$$

Thus Γ is consistent in L^* if and only if $1_\Gamma \neq H_\Gamma$.

For each $W \in S_{\mathfrak{m}}(V)$ and $\Phi \in F_1$ we shall define

$$\exists_\Gamma(W)(\Phi/\Gamma) = (\exists W\, \Phi)/\Gamma\,.$$

Finally, for each $\tau \in V^V$ and $\Phi \in H$, we define

$$S_\Gamma(\tau)(\Phi/\Gamma) = \big(S_f(\tau) S_f(\tau_0^{-1}) S(\tau_0)\Phi\big)/\Gamma\,.$$

LEMMA A.2. (i) *If* Φ *is atomic and* $\tau \in V^V$, *then*

$$S_\Gamma(\tau)(\Phi/\Gamma) = \big(S_f(\tau)\Phi\big)/\Gamma = \big(S(\tau)\Phi\big)/\Gamma\,.$$

(ii) *If* $\tau \in \big(V \sim V_b(\Phi)\big)^V$, *then*

$$S_\Gamma(\tau)(\Phi/\Gamma) = \big(S_f(\tau)\Phi\big)/\Gamma\,.$$

(iii) *If* $\tau \in V^V$ *is one to one, then*

$$S_\Gamma(\tau)(\Phi/\Gamma) = \big(S(\tau)\Phi\big)/\Gamma\,.$$

([11]) H_Γ is essentially the Lindenbaum algebra (cf. [26], p. 348).

LEMMA A.3. *If* $\mathfrak{m} = \overline{\overline{V}}^{+}$ *and* Γ *is consistent in* L^*, *then the quadruple*

$$(\boldsymbol{H}_\Gamma) = (\boldsymbol{H}_\Gamma, V, S_\Gamma, \exists_\Gamma)$$

is a polyadic algebra of degree V (12).

We shall denote by (X^I) the $\boldsymbol{O}$-valued functional polyadic algebra $(F(X^I, \boldsymbol{O}), I, S^I, \exists^I)$.

If Γ is satisfiable in the structure $\mathfrak{A}$ of type μ, we shall denote by $h_{\Gamma,\mathfrak{A}}$ the function h on H_Γ into $F(A^V, \boldsymbol{O})$ defined by the condition:

$$h(\Phi/\Gamma)(b) = 1 \quad \textit{if and only if} \quad \models_{\mathfrak{A}} \Phi[b \upharpoonright V^*], \qquad \textit{whenever } \Phi \in H \textit{ and } b \in A^V.$$

LEMMA A.4. *Suppose that* $\mathfrak{m} = \overline{\overline{V}}^{+}$ *and* Γ *is satisfiable in the structure* $\mathfrak{A}$ *of type* μ. *Then* $h_{\Gamma,\mathfrak{A}}$ *is a polyadic homomorphism of* $(\boldsymbol{H}_\Gamma)$ *into* (A^V).

LEMMA A.5. *Let* $(\boldsymbol{B}) = (\boldsymbol{B}, I, S, \exists)$ *be a polyadic algebra of infinite degree* $\mathfrak{d}$. *Let* $\overline{\overline{V}} = \mathfrak{d}$, $\overline{\overline{P}} = \overline{\overline{B}}$, $\mathfrak{m} = \mathfrak{d}^+$, *and* $\overline{\overline{\mu(p)}} = \mathfrak{d}$ *for each* $p \in P$. *Then there exists a set of sentences* $\Gamma \subseteq G$ *such that* $(\boldsymbol{H}_\Gamma)$ *is isomorphic to* $(\boldsymbol{B})$.

In Lemma A.5, let us suppose for convenience that $I = V$, $B = P$, and $\mu(p)$ is the same for each $p \in P$. Let x be a one to one function on $\mu(P)$ onto V. We may take for Γ the following set of sentences, where p, q range over P, τ ranges over V^V; and W ranges over $S_{\mathfrak{m}}(V)$:

$$\forall V(\langle \bar{p}x\rangle \leftrightarrow (\neg \langle px\rangle));$$

$$\forall V(\langle p+qx\rangle \leftrightarrow (\langle px\rangle \vee \langle qx\rangle));$$

$$\forall V(\langle p\cdot qx\rangle \leftrightarrow (\langle px\rangle \wedge \langle qx\rangle));$$

$$\forall V(\langle S(\tau)px\rangle \leftrightarrow S(\tau)\langle px\rangle);$$

$$\forall V(\langle \exists(W)px\rangle \leftrightarrow (\exists W \langle px\rangle)).$$

The required isomorphism of $(\boldsymbol{B})$ onto $(\boldsymbol{H}_\Gamma)$ is then given by the function g defined by the condition

$$g(p) = \langle px\rangle/\Gamma$$

for each $p \in P$. The proof that g is an isomorphism consists of a series of computations which amount to showing that the axioms and rules of inferences of L^*, and the axioms of polyadic algebras, "say the same things" under the proper interpretation.

(12) A more natural procedure up to this point would have been to work with the original system L instead of passing to L^*, and to consider the set F of formulas and a set Γ of sentences in L. However, in case there are formulas Φ in F such that $V_f(\Phi) = V$, this simpler procedure would leave us at a loss to define the substitution operator S_Γ so as to satisfy the axioms of polyadic algebra. Evidently, we need the extra variables in $V^* \sim V$ in order to avoid collisions of free and bound variables.

If X is a set of elements of a complete Boolean algebra $\boldsymbol{A}$, the greatest lower bound of X is denoted by $\prod X$. Thus the zero element of $\boldsymbol{A}$ is $\prod \boldsymbol{A}$.

THEOREM A.6 (Representation Theorem). *Let* $(\boldsymbol{B})$ *be a polyadic algebra of infinite degree* $\mathfrak{d}$ *and cardinality* $\mathfrak{c}$. *Then there exists a polyadic homomorphism* h *of* $(\boldsymbol{B})$ *into an* $\boldsymbol{O}$-*valued functional algebra* (A^V) *of degree* $\mathfrak{d}$ *whose domain* A *has power* $\mathfrak{d}^{\mathfrak{d}} + \mathfrak{c}$.

Moreover, if D *is a proper (not necessarily polyadic) dual ideal in the Boolean algebra* $\boldsymbol{B}$, *then* h *may be chosen so that* $\prod h[D] \neq \prod (A^V)$ (13).

Proof. By A.5 we have an isomorphism g of $(\boldsymbol{B})$ onto an algebra of the form $(\boldsymbol{H}_\Gamma)$, and where $\overline{\overline{V}} = \mathfrak{d}$, $\overline{\overline{P}} = \mathfrak{c}$, and $\mathfrak{m} = \mathfrak{d}^+$. We must have $\mathfrak{c} > 1$, and therefore $1_\Gamma \neq H$. Let $\Gamma_1 = g[D]$. Then $\Gamma \subseteq \Gamma_1$, and Γ_1 is consistent in L^*. By the Completeness Theorem and 2.1, Γ_1 is satisfiable in a structure $\mathfrak{A}$ of power $\mathfrak{d}^{\mathfrak{d}} + \mathfrak{c}$. By A.4, $h_{\Gamma,\mathfrak{A}}$ is a polyadic homomorphism of $(\boldsymbol{H}_\Gamma)$ into (A^V). There exists $b \in A^V$ such that $\models_{\mathfrak{A}} \Phi[b \upharpoonright V^*]$ holds for each $\Phi \in \Gamma_1$, and thus we have $h_{\Gamma,\mathfrak{A}}(\Phi/\Gamma)(b) = 1$ for each $\Phi \in \Gamma_1$. Let $h = (h_{\Gamma,\mathfrak{A}}) \circ g$. Then $(\prod h[D])(b) = 1$, and our proof is complete.

Using Theorem 3.3, we may improve the cardinality condition in Theorem A.6, using the notion of local degree.

LEMMA A.5′. *Let* $(\boldsymbol{B}) = (\boldsymbol{B}, I, S, \exists)$ *be a polyadic algebra with infinite degree* $\mathfrak{d}$, *local degree* $\mathfrak{d}'$, *and cardinality* $\mathfrak{c}$. *Then for some formal system* $L(\mu, V, \mathfrak{m})$ *such that* $\overline{\overline{V}} = \mathfrak{d}$, $\mathfrak{m} = \mathfrak{d}^+$, $\overline{\overline{P}} = \overline{\overline{B}}$, *and* $\overline{\overline{\mu(p)}} < \mathfrak{d}'$ *for each* $p \in P$, *there exists a set of sentences* $\Gamma \subseteq G$ *such that* $\overline{\overline{\Gamma}} = \mathfrak{c} + \sum_{\mathfrak{p} < \mathfrak{d}'} \mathfrak{d}^{\mathfrak{p}}$ *and* $(\boldsymbol{H}_\Gamma)$ *is isomorphic to* $(\boldsymbol{B})$.

The proof of A.5′ is a modification of the proof of A.5. For each $p \in P$, we may choose a one to one function x_p on $\mu(p)$ onto a subset of V which supports p. We now take for Γ the following set of sentences, where p_0, p_1, range over P, τ ranges over the set

$$\{\sigma \mid \sigma \in V^V, \overline{\overline{\{v \mid v \in V, \sigma(v) \neq v\}}} < \mathfrak{d}'\},$$

and W ranges over the set $S_{\mathfrak{d}'}(V)$:

$$\forall V(\langle \bar{p}x_{\bar{p}}\rangle \leftrightarrow (\neg \langle px_p\rangle));$$

$$\forall V(\langle p+qx_{p+q}\rangle \leftrightarrow (\langle px_p\rangle \vee \langle qx_q\rangle));$$

$$\forall V(\langle p\cdot qx_{p\cdot q}\rangle \leftrightarrow (\langle px_p\rangle \wedge \langle qx_q\rangle));$$

$$\forall V(\langle S(\tau)px_{S(\tau)p}\rangle \leftrightarrow S_f(\tau)\langle px_p\rangle);$$

$$\forall V(\langle \exists(W)px_{\exists(W)p}\rangle \leftrightarrow (\exists W \langle px_p\rangle)).$$

(13) A form of the Representation Theorem corresponding to the first paragraph of A.6 was announced independently both in [2] and in [14]; in the former, the cardinality condition was not acctually stated, but was implicit in the proof which was outlined. The first paragraph of A.6 may be proved using the Completeness Theorem only in the special case that Γ is a set of sentences, while the second paragraph of A.6 uses the Completeness Theorem in general.

It is then easily seen that $\overline{\overline{\Gamma}} = \mathfrak{c} + \sum_{\mathfrak{p}<\mathfrak{d}'} \mathfrak{d}^{\mathfrak{p}}$. The required isomorphism h is now defined by

$$h(p) = \langle p x_p \rangle / \Gamma$$

for each $p \in P$.

THEOREM A.6′. *Let $(\boldsymbol{B})$ be a polyadic algebra of infinite degree $\mathfrak{d}$, local degree $\mathfrak{d}'$ and cardinality $\mathfrak{c}$, and let $\mathfrak{n}$ be a cardinal such that $\mathfrak{n} \geqslant \mathfrak{c}$ and $\sum_{\mathfrak{p}<\mathfrak{d}'} \mathfrak{n}^{\mathfrak{p}} = \mathfrak{n}$. Then there exists a polyadic homomorphism h of $(\boldsymbol{B})$ into an $\boldsymbol{O}$-valued functional algebra (A^V) of degree $\mathfrak{d}$ whose domain A has power $\mathfrak{n}$.*

Moreover, if D is a proper dual ideal in $\boldsymbol{B}$, then h may be chosen so that $\prod h[D] \neq \prod (A^V)$ (14).

Proof. If $\mathfrak{c} < \mathfrak{d}$, then (by Theorem 3.12 in [3]) $(\boldsymbol{B})$ is degenerate, and the result is a trivial consequence of the Representation Theorem for Boolean Algebras (cf. [24]).

Suppose $\mathfrak{c} \geqslant \mathfrak{d}$. By A.5′, there is a formal system $L(\mu, V, \mathfrak{m})$ and a set $\Gamma \subseteq G$ which satisfy the conclusions of A.5′. Let g be an isomorphism of $(\boldsymbol{B})$ onto $(\boldsymbol{H}_\Gamma)$, and let $\Gamma_1 = g[\underline{D}]$. Clearly $\Gamma \subseteq \Gamma_1$. Since $\mathfrak{d} \leqslant \mathfrak{c}$, it follows that $\overline{\overline{\Gamma}} \leqslant \mathfrak{n}$. Also, we have $\mathfrak{n} = \mathfrak{n}^{\overline{\overline{\mu(p)}}}$ for each $p \in P$. Since $\mathfrak{c} > 1$, Γ_1 is consistent in L^*.

It now follows from Theorem 3.3 that Γ_1 is satisfiable in a structure $\mathfrak{A}$ of power $\mathfrak{n}$. Let $h = (b_{\Gamma,\mathfrak{A}}) \circ g$. As in A.6, we see that h is the desired homomorphism of $(\boldsymbol{B})$ into (A^V).

Using Theorem 4.4 in [3], we obtain the Representation Theorem in the form stated in [3], Theorem 6.4.

THEOREM A.6″. *Let $(\boldsymbol{B})$ be a polyadic algebra of infinite degree $\mathfrak{d}$, effective cardinality $\mathfrak{c}'$, and local degree $\mathfrak{d}'$, and let $\mathfrak{n}$ be a cardinal such that $\mathfrak{n} > \mathfrak{c}'$ and $\sum_{\mathfrak{p}<\mathfrak{d}'} \mathfrak{n}^{\mathfrak{p}} = \mathfrak{n}$. Then there exists a homomorphism of $(\boldsymbol{B})$ into an $\boldsymbol{O}$-valued functional algebra whose domain A has power $\mathfrak{n}$* (15).

Proof. Let $J \subseteq I$ and let J be the effective degree of $(\boldsymbol{B})$. Then the faithful compression $(\boldsymbol{B})_J$ of $(\boldsymbol{B})$ has power $\mathfrak{c}$, degree $\overline{\overline{J}}$, and local degree $\mathfrak{d}'$.

If $(\boldsymbol{B})$ is degenerate, then, as in A.6, the result is trivial.

Suppose $(\boldsymbol{B})$ is not degenerate. Then it is easily seen that J is infinite.

By A.6′, there is a homomorphism of $(\boldsymbol{B})_J$ into an $\boldsymbol{O}$-valued functional algebra of degree $\overline{\overline{J}}$ whose domain A is of power $\mathfrak{n}$. It follows from Theorem 4.4 of [3] that there is a homomorphism of $(\boldsymbol{B})$ into an $\boldsymbol{O}$-valued functional algebra of degree $\mathfrak{d}$ and with domain A. This completes the proof.

In the other direction, it is also possible to prove the Completeness Theorem by applying the Representation Theorem. It is most convenient to apply the version of the latter stated in A.6′. We shall give an outline of such a proof.

(14) The improved cardinality condition in A.6′ was suggested to the author by the results in [3].

(15) Theorem A.6″ is, because of the cardinality condition, stronger than the first paragraph of A.6 and of A.6′. The result A.6″ is originally due to Daigneault and Monk and is the same as Theorem 6.4 in [3].

LEMMA A.7. *Suppose that h is a polyadic homomorphism from the polyadic algebra $(\boldsymbol{H}_P)$, where $\Gamma \subseteq G$ and Γ is consistent in L^*, into an O-valued functional algebra (A^V) with non-empty domain A. Then there exists a structure $\mathfrak{A} = (A, R_p)_{p \in P}$ of type μ such that Γ is satisfiable in $\mathfrak{A}$ and $h = h_{\Gamma,\mathfrak{A}}$.*

In fact, the structure $\mathfrak{A}$ of Lemma A.7 may be defined in the following way: *for each $p \in P$, let $x \in V^{\mu(p)}$ be one to one, and let*

$$R_p = \{a \circ x \mid a \in A^V,\ h(\langle p x \rangle / \Gamma)(a) = 1\}.$$

Note that the relation R_p does not depend on our choice of the one to one function x.

We now outline a second proof of the Completeness Theorem.

Suppose that Γ is a consistent set of formulas in L and that

(a″) $$\overline{\overline{F}} \leqslant \mathfrak{n},$$

(b″) $$\sum_{\mathfrak{p}<\mathfrak{m}} \mathfrak{n}^{\mathfrak{p}} = \mathfrak{n},$$

(c″) $$\overline{\overline{\mu(p)}} < \mathfrak{m} \quad \textit{for each} \quad p \in P.$$

We wish to show that Γ is satisfiable in a structure of power $\mathfrak{n}$.

Let V^* be a set of symbols disjoint from P and not containing $\rightarrow$, $\mathfrak{f}$, $\forall$, such that $V \subseteq V^*$ and $\overline{\overline{V}} = \overline{\overline{(V^* \sim V)}}$. Then $\Gamma \subseteq H$. By Lemma 2.2 of this paper, Γ is also consistent in L^*. Since each formula $\Phi \in H$ has fewer than $\mathfrak{m}$ free variables, we may define $\exists_0$ on $\boldsymbol{H}_0$ in the obvious way, where 0 is the empty set of formulas, so that the quadruple $(\boldsymbol{H}_0)$ is a polyadic algebra. The set $D = \{\Phi/0 \mid \Phi \in \Gamma\}$ is a proper dual ideal in $\boldsymbol{H}_0$. $(\boldsymbol{H}_0)$ has local degree $\mathfrak{m}$ and cardinality $\leqslant \overline{\overline{F}}$. By the Representation Theorem A.6′, and by (a″), (b″), (c″), there exists a homomorphism h from $(\boldsymbol{H}_0)$ into an 0-valued functional algebra (A^V) whose domain A has power $\mathfrak{n}$, and such that $\prod h[D] \neq \prod (A^V)$. Choose an element $b \in A^V$ such that $(h(x))(b) = 1$ for all $x \in D$. By Lemma A.7, there exists a structure $\mathfrak{A}$ of type μ such that $h = h_{0,\mathfrak{A}}$. Then it follows that $\models_{\mathfrak{A}} \Phi[b \restriction V^*]$ for each $\Phi \in \Gamma$, and Γ is satisfiable in $\mathfrak{A}$.

The hypothesis (c″) used above was not assumed in Theorem 3.1; however, this hypothesis may easily be removed by syntactical methods.

A proof of the Completeness Theorem in the special case that Γ is a set of sentences may be given, along the same lines as the above argument, using only the first paragraph of A.6′, or else using A.6″. In such a proof we consider $(\boldsymbol{H}_\Gamma)$ instead of $(\boldsymbol{H}_0)$.

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