Invariant extensions of the Lebesgue measure

by

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In this note we present a positive answer to a problem of W. Sierpiński concerning invariant extensions of the Lebesgue measure formulated in a paper of E. Marczewski in 1935 (cf. [5]).

Before discussing the problem itself we wish to fix our notation and recall some notions.

By a *measure space* we understand a triple $(X, \mathcal{B}, \mu)$ consisting of the set X a Borel σ-field $\mathcal{B}$ of subsets of X and a countably additive measure μ defined on $\mathcal{B}$. We generally assume that for a measure space there exists (at least one) subset $M \in \mathcal{B}$ such that $0 < \mu(M) < \infty$.

A measure space $(X, \mathcal{B}, \mu)$ is called *separable* if there exists a countable family $\mathcal{A} \subset \mathcal{B}$ such that for any M in $\mathcal{B}$ and any $\varepsilon > 0$ there exists a set N in $\mathcal{A}$ such that $\mu(M \triangle N) < \varepsilon$ (1).

A measure space $(\overline{X}, \overline{\mathcal{B}}, \overline{\mu})$ is called an *extension* of a measure space $(X, \mathcal{B}, \mu)$ if $X = \overline{X}$, $\mathcal{B} \subset \overline{\mathcal{B}}$, $\mu(M) = \overline{\mu}(M)$ for every $M \in \mathcal{B}$. It is a *proper* extension in the case of $\mathcal{B} \neq \overline{\mathcal{B}}$.

Let $(X, \mathcal{B}, \mu)$ be a measure space and T a one-to-one transformation of X onto itself. We say that T is an *automorphism* of $(X, \mathcal{B}, \mu)$, if it is $\mathcal{B}$-measurable and measure-preserving, i.e., for each $M \in \mathcal{B}$, $T(M) \in \mathcal{B}$ and $\mu\big(T^{-1}(M)\big) = \mu(M)$.

Two automorphisms T' and T'' of a measure space $(X, \mathcal{B}, \mu)$ are called *equivalent* (notation $T' \sim T''$) if $\mu\big(\{x\colon\ T'(x) \neq T''(x)\}\big) = 0$.

Given a separable measure space $(X, \mathcal{B}, \mu)$. A maximal set of non-equivalent automorphisms of $(X, \mathcal{B}, \mu)$ has cardinal at most $2^{\aleph}$ (cf. [3], Lemma 2).

Given a measure space $(X, \mathcal{B}, \mu)$ and a family $\mathcal{F}$ of its automorphisms. An extension $(X, \overline{\mathcal{B}}, \overline{\mu})$ of $(X, \mathcal{B}, \mu)$ is said to be $\mathcal{F}$-*invariant* if every automorphism of the family $\mathcal{F}$ is an automorphism of $(X, \overline{\mathcal{B}}, \overline{\mu})$.

Suppose that $\mathcal{F}'$ is a set of automorphisms of a measure space $(X, \mathcal{B}, \mu)$. It is clear that

If $\mathcal{F}'$ is another family of automorphisms of the measure space $(X, \mathcal{B}, \mu)$ such that for each $T' \in \mathcal{F}'$ there exists $T \in \mathcal{F}$ such that $T' \sim T$, then any $\mathcal{F}$-invariant extension $(X, \mathcal{B}, \mu)$ is $\mathcal{F}'$-invariant.

(1) By $M \triangle N$ we mean the symmetric difference of the sets M and N.

An extension $(X, \overline{\mathcal{B}}, \overline{\mu})$ is simply called *invariant* if it is $\mathcal{F}$-invariant, $\mathcal{F}$ being the set of all the automorphisms of $(X, \mathcal{B}, \mu)$.

Denote by $(\mathcal{X}, \mathcal{L}, \nu)$ the measure space in which $\mathcal{X}$ is an Euclidean space, $\mathcal{L}$ the σ-field of the Lebesgue measurable subsets of $\mathcal{X}$ and ν the Lebesgue measure.

Following E. Marczewski (cf. [5]) we call an extension $(\mathcal{X}, \mathcal{B}, \mu)$ of $(\mathcal{X}, \mathcal{L}, \nu)$ perfect if it is $\mathfrak{M}$-invariant, $\mathfrak{M}$ being the family of the isometries of $\mathcal{X}$.

In his paper (cf. [5]) E. Marczewski quotes the following problem of W. Sierpiński:

Given: a perfect extension $(\mathcal{X}, \mathcal{B}, \mu)$ of the measure space $(\mathcal{X}, \mathcal{L}, \nu)$. Does there exists a proper extension $(\mathcal{X}, \overline{\mathcal{B}}, \overline{\mu})$ of $(\mathcal{X}, \mathcal{B}, \mu)$ which is again a perfect extension of $(\mathcal{X}, \mathcal{L}, \nu)$?

We present here a positive answer to this problem and indeed we propose a theorem formulated and proved in terms of general measure spaces which, by the use of a well-known set theoretical hypothesis (weaker than the continuum hypothesis), will imply a positive answer to the problem.

THEOREM. *Let $(X, \mathcal{B}, \mu)$ be a measure space such that*

(i) *the cardinal $\overline{\overline{X}}$ is less than the first (weakly) inaccessible cardinal* (cf. [6]) [2];

(ii) *for each $M \subset X$ such that $\overline{\overline{M}} < \overline{\overline{X}}$ we have $M \in \mathcal{B}$ and $\mu(M) = 0$. Further, let $\mathcal{F}$ be a group of automorphisms of $(X, \mathcal{B}, \mu)$ such that*

(iii) $\overline{\overline{\mathcal{F}}} \leqslant \overline{\overline{X}}$.

Then there exists an $\mathcal{F}$-invariant proper extension $(X, \overline{\mathcal{B}}, \overline{\mu})$ of the measure space $(X, \mathcal{B}, \mu)$.

In order to derive the answer to the problem of W. Sierpiński from the above theorem we note that if the continuum hypothesis is assumed, then every perfect extension $(\mathcal{X}, \mathcal{B}, \mu)$ of the measure space $(\mathcal{X}, \mathcal{L}, \nu)$ satisfies conditions (i) and (ii). Since the group of isometries of $\mathcal{X}$ has cardinal $\mathfrak{c} = \overline{\overline{\mathcal{X}}}$, then also condition (iii) is satisfied and the existence of a proper extension of $(\mathcal{X}, \overline{\mathcal{B}}, \overline{\mu})$ which is a perfect extension of $(\mathcal{X}, \mathcal{L}, \nu)$ follows.

It seems worth while to note that the theorem and Lemma 2 of [3], quoted above, imply a slightly stronger form of what was stated above. In fact,

if $(\mathcal{X}, \mathcal{B}, \mu)$ is an invariant extension of the measure space $(\mathcal{X}, \mathcal{L}, \nu)$, then there exists a proper extension of $(\mathcal{X}, \mathcal{B}, \mu)$ which is an invariant extension of $(\mathcal{X}, \mathcal{L}, \nu)$.

The general formulation of the theorem enables us to deduce the following corollary, which might be of some interest in the theory of topological groups.

If G is a locally compact separable topological group, $\mathcal{B}$ the σ-field of Haar measurable subsets of G, μ the Haar measure and $\mathfrak{S}$ the group of automorphisms of $(G, \mathcal{B}, \mu)$ defined by the left (right) translations, then each $\mathfrak{S}$-invariant extension of $(G, \mathcal{B}, \mu)$ has a proper $\mathfrak{S}$-invariant extension.

The proof of the theorem in its essential part is based on the idea of absolutely invariant sets introduced by S. Banach in 1932 (cf. [1]) and applied later by P. R. Halmos and J. von Neumann [2], and S. Kakutani and J. C. Oxtoby [3].

Needless to say, the axiom of choice is used freely in the proof of the theorem and its corollaries.

Proof of the theorem. We obtain the proof of the theorem in the following two simple steps.

Step 1. *Under the conditions of the theorem there exists a set $A \subset X$ such that*

(a) $A \notin \mathcal{B}$,

(b) *for each $T \in \mathcal{F}$ we have* $\mu(T(A) \triangle A) = 0$.

In order to construct the set A we well-order both the space X and the group $\mathcal{F}$ into transfinite sequences: the first into a sequence

$$x_1, \ldots, x_a, \ldots, \quad a < \omega_\xi,$$

where ω_ξ is the first ordinal of cardinal $\overline{\overline{X}}$, the second into a sequence

$$T_1, \ldots, T_a, \ldots, \quad a < \omega_\zeta,$$

where T_1 is the identity transformation and $\omega_\zeta \leqslant \omega_\xi$. Then for each a we define a set O_a as the set of the elements of X which are of the form $T_{\lambda_1}^{n_1}, \ldots, T_{\lambda_k}^{n_k}(x_\eta)$, where $\lambda_1, \ldots, \lambda_k$ runs over the finite sequences of ordinals $< \omega_\zeta$, $n_1, \ldots, n_k$ are integers and η is less than ω_ξ. Clearly

1. $\overline{\overline{O}}_a < \overline{\overline{X}}$, *so* $\mu(O_a) = 0$;
2. $O_a \subset O_{a+1}$;

[2] As we shall see in the proof, assumption (i) is needed only to ensure that the conditions of Ulam's theorem of the non-existence non-trivial measures, universal (defined on all the subsets of the set X) and vanishing on the one-point subsets on X are satisfied. Instead of this we could have simply assumed the assertion of Ulam's theorem, i.e. that there is no non-trivial universal measure vanishing on the one-point sets in X. As is known, if the continuum hypothesis is assumed, this is equivalent to the non-existence of a non-trivial universal measure taking only two values 0 and 1 and vanishing on the one-point sets in X. By the results obtained recently by A. Tarski and his pupils, this class is extremely large and a result of D. Scott states that the assumption that this class exhausts the class of all the cardinals is consitent.

3. *for each* T_β *with* $\beta \leqslant \alpha$, *the set* O_α *is invariant under* T_β (*that is* $T_\beta(O_\alpha) = O_\alpha$).

For each $\alpha < \omega_\xi$ we define a set Q_α putting $Q_1 = O_1$ and $Q_\alpha = O_{\alpha+1} \setminus O_\alpha$ for $\alpha > 1$. Properties 1, 2 imply the following properties of the sets Q_α, $\alpha < \omega_\xi$:

1′. $\mu(Q_\alpha) = 0$;

2′. $Q_\alpha \cap Q_\beta = \emptyset$ *for* $\alpha \neq \beta$.

Moreover,

3′. *for every subset* Ω *of the set of the ordinals* $< \omega_\xi$ *and for each* $T = T_\beta \in \mathcal{F}$ *we have* $\mu\big(T(\bigcup_{\alpha\in\Omega} Q_\alpha) \triangle (\bigcup_{\alpha\in\Omega} Q_\alpha)\big) = 0$,

4′. $\bigcup_{\alpha<\omega_\xi} Q_\alpha = X$.

To see 4′ we simply note that $O_{\alpha+1} = \bigcup_{\alpha\leqslant\beta} Q_\beta$ and that $\bigcup_{\alpha<\omega_\xi} Q_{\alpha+1} = X$.

In order to prove 3′ we note that the set

$$M = T_\beta(\bigcup_{\alpha\in\Omega} Q_\alpha) \triangle (\bigcup_{\alpha\in\Omega} Q_\alpha) = \big[T_\beta(\bigcup_{\substack{\alpha\in\Omega\\ \alpha\leqslant\beta}} Q_\alpha) \cup \big(\bigcup_{\substack{\alpha\in\Omega\\ \alpha\geqslant\beta}} T_\beta(Q_\alpha)\big)\big] \triangle (\bigcup_{\alpha\in\Omega} Q_\alpha).$$

Since, by 3, $T_\beta(O_\alpha) = O_\alpha$ for all $\alpha \geqslant \beta$, we have also $T_\beta(Q_\alpha) = Q_\alpha$. Hence

$$M = [T_\beta(\bigcup_{\substack{\alpha\in\Omega\\ \alpha<\beta}} Q_\alpha) \cup (\bigcup_{\substack{\alpha\in\Omega\\ \alpha\geqslant\beta}} Q_\alpha)] \triangle (\bigcup_{\alpha\in\Omega} Q_\alpha) \subset T_\beta(\bigcup_{\alpha<\beta} Q_\alpha) \cup \bigcup_{\alpha<\beta} Q_\alpha \subset T_\beta(O_\beta) \cup O_\beta.$$

Thus, by 1, $\overline{\overline{M}} < \overline{\overline{X}}$ and hence $\mu(M) = 0$, as required.

Now we suppose that for each set Ω of ordinals α, $\alpha < \omega_\xi$, the set $\bigcup_{\alpha\in\Omega} Q_\alpha \in \mathcal{B}$. Let M be a set of $\mathcal{B}$ such that $0 < \mu(M) < \infty$. Then putting $m(\Omega) = \mu(\bigcup_{\alpha\in\Omega} Q_\alpha \cap M)$ we would obtain, in virtue of 1′, 2′, 4′, a σ-additive, atom-free, finite measure defined on all the subsets of the set of ordinals less than ω_ξ. But since $\overline{\omega}_\xi = \overline{\overline{X}}$ is less than the first inaccessible cardinal, this contradicts the well-known result of S. Ulam [6]. Thus there exists a set Ω such that $\bigcup_{\alpha\in\Omega} Q_\alpha \notin \mathcal{B}$. We put $A = \bigcup_{\alpha\in\Omega} Q_\alpha$.

Step 2. *We join the set* A *to the* σ-*field* $\mathcal{B}$, *that is we form the family* $\overline{\mathcal{B}} = \{(M \cap A) \cup (N \cap A')\colon M, N \in \mathcal{B}\}$ *and for each* $E \in \overline{\mathcal{B}}$ *we set*

$$\overline{\mu}(E) = \mu^*(E \cap A) + \mu_*(E \cap A'),$$

where by μ^* *and* μ_* *we mean the outer and the inner measure induced by* μ *respectivelly.*

As was proved by J. Łoś and E. Marczewski (cf. [4]), the family $\overline{\mathcal{B}}$ is indeed a σ-field and $\overline{\mu}$ is the measure defined on $\overline{\mathcal{B}}$ equal to μ on the sets of $\mathcal{B}$. Since also $A \in \overline{\mathcal{B}}$, the measure space $(X, \overline{\mathcal{B}}, \overline{\mu})$ is a proper extension of $(X, \mathcal{B}, \mu)$. To see that it is $\mathcal{F}$-invariant we note that, by 3′, for each $T \in \mathcal{F}$ we have $T(A) = A \setminus M \cup N$, $T(A') = A' \setminus P \cup R$ with $M, N, P, R \in \mathcal{B}$ and $\mu(M) = \mu(N) = \mu(P) = \mu(R) = 0$. Hence for each $E \in \overline{\mathcal{B}}$ also $T(E) \in \overline{\mathcal{B}}$. Moreover,

$$\begin{aligned}\overline{\mu}\big(T^{-1}(E)\big) &= \mu^*\big(T^{-1}(E) \cap A\big) + \mu_*\big(T^{-1}(E) \cap A'\big) \\ &= \mu^*\big(T^{-1}(E) \cap [T^{-1}(A) \setminus M \cup N]\big) + \mu_*\big(T^{-1}(E) \cap [T^{-1}(A') \setminus P \cup R]\big),\end{aligned}$$

where $\mu(M) = \mu(N) = \mu(P) = \mu(R) = 0$. Thus

$$\begin{aligned}\overline{\mu}\big(T^{-1}(E)\big) &= \mu^*\big(T^{-1}(E \cap A)\big) + \mu_*\big(T^{-1}(E \cap A')\big) \\ &= \mu^*(E \cap A) + \mu_*(E \cap A') = \overline{\mu}(E),\end{aligned}$$

as required.

References

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