disjoints. En vertu du théorème 9, tous les continus $M_{00}$ (de chacune de leurs 2° familles) sont homéomorphes. En particulier, si $M$ est une dendrite, il en est donc de même de tout $M_{00}$.

En conséquence, si $\dim(X) = 0$, on peut, en vertu du corollaire 2, remplacer dans l'énoncé du théorème 9 le mot sur-continu par les mots dendrite dont les bouts et les points de ramifications appartiennent à $X$.

La question est moins simple pour les $X$ enfilables. En vertu du théorème 1, elle se réduit à des $X$ compacts de dimension 0. Or un théorème permettant d'ender $X$ compact de dimension 0 en un arc $L$ tel que $L - X$ se compose de segments ouverts disjoints (cf. le problème mentionné p. 72) permettrait, tout comme pour les dendrites, de remplacer dans l'énoncé du théorème 9 le mot sur-continu par le mot arc et avoir ainsi une généralisation considérable du théorème de Riesz-Denjoy.

Mais à défaut de ce moyen de procéder, et celui consistant à enfermer dans des disques les arcs ouverts rectilignes (signalé p. 73 comme bien plus compliqué) n'ayant pas été utilisé, cette généralisation du théorème de Riesz-Denjoy reste un problème ouvert.

Ouvrages cités


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Dimensions of irreducible continua and fixations of components in compact spaces

by

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This note establishes a relation (see Theorem 1 below) between the dimension of a continuum $X$ (i.e. connected compact metric space), irreducible between two points, and the dimension of fibres in $X$ (1). As applications there are given two theorems on the existence, in a compact metric space $X$, of a compact subset which has dimension less than an integer $n$ and intersects every component of $X$ provided that all these components have large diameters or converge to points of a compact set with dimension less than $n$, and $X$ satisfies a certain condition (see Theorems 3 and 4). That condition holds for subsets of polyhedra, and in this way generalizations of the results of D. Zaremba (4) concerning plane sets are obtained (see Corollaries 1 and 2).

LEMMA. If $\varepsilon > 0$, $X$ is a compact metric space and $n \leq \dim_X x$, then there exists a continuum $C \subset X$ such that $d((p) - C) < \varepsilon$ and $n \leq \dim C$.

Proof. Let $Q$ be a closed solid sphere in $X$ with centre $p$ and radius $\varepsilon/4$. Hence $n \leq \dim Q$ and we infer (see [2], p. 106) that there is a component $C$ of $Q$, satisfying $n \leq \dim C$. Thus $C$ is the desired continuum.

THEOREM 1. If $X$ is a continuum irreducible between two points $a, b$ and $2 \leq n \leq \dim _X x$, then there exists a fibre $F_C X$ such that $n \leq \dim F$.

Proof. Let $g : X \to \mathcal{O}$ be such a continuous mapping of $X$ into the segment $\mathcal{O} = [0, 1]$ (2) that the sets $g^{-1}(t)$ coincide with the fibres of $X$ (see [2], p. 159). If $g(X)$ is a one-point set, then $F = X$ and Theorem 1 follows. Thus we may assume that $g(X) = \mathcal{O}$, $g(a) = 0$ and $g(b) = 1$.

Since $X$ is a compact metric space and $n \leq \dim _X X$, the Menger Theorem (see [2], p. 66) implies the inequality $n \leq \dim [X - X_{(n-1)}]$, where $X_{(n-1)} = \{x \in X : \dim _X x \leq n-1\}$. But $X_{(n-1)}$ is a $G_\delta$-set in $X$ (see [1], p. 164), whence $X - X_{(n-1)} = Y_1 \cup Y_2 \cup \ldots$, where $Y_i$ are closed

\begin{itemize}
  \item[(1)] Fibres of an irreducible continuum correspond to its "branches" in the sense of [3], p. 139. The notation from [1] and [2] is adopted here.
  \item[(2)] If $t_i < t_{i+1}$ are real numbers, then we denote by $[t_i, t_{i+1}]$ the closed interval $[t_i, t_{i+1}]$.
\end{itemize}

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It follows that $k > 0$. Choosing a point $p$ and a number $s$ with $p \in g^{-1}(s)$ and $s_1 < s < s_2$, we see $p$ to be an interior point of $g^{-1}((s_1, s_2))$. Therefore a number $s > 0$ exists such that $s < 1/k$ and $g(p, s) < s$ implies $g(x) \in [s_1, s_2]$ for every $x \in X$. Moreover, we have $n < \dim X$, by (3) for all $t$, thus we infer from the lemma that there is a continuum $C \subset X$ satisfying $\delta(p, C) < \varepsilon$ and $n < \dim C$, whence

$$\delta(C) < 1/k \quad \text{and} \quad g(C) \subset [s_1, s_2].$$

If we had $g(C) = [v_1, v_2]$, where $v_1 < v_2$, then (4) and (5) would imply $[s_1, s_2] \subset Z_2$ and such a number $v$ would exist that $v_1 < v < v_2$ and $v \in Z_2$.

The union $D = g^{-1}((0, v_1]) \cup C \cup g^{-1}([v_2, 1])$

would be a continuum contained in $X$ and containing the points $a$ and $b$, whence $D = X$ and $D$ would contain $g^{-1}(s)$. Thus so would $C$ and, by (3), we should get $1/k < \delta(g^{-1}(s)) \leq \delta(0)$, contrary to (5).

Hence $g(C)$ must be a one-point set ($v_1$) and putting $F = g^{-1}(v_1)$, we obtain a fibre $F$ satisfying $C \subset F$. The inequalities $n < \dim C < \dim F$ follow.

**Theorem 2.** If $X$ is an irreducible continuum and each continuum $C \subset X$ nowhere dense in $X$ satisfies $\dim C < n$ (where $1 \leq n$, then $\dim X = n$.

**Proof.** Suppose that $Y \subset X$ is a non-degenerate indecomposable continuum. Thus $\delta(Y) > 0$ and for any point $p \in Y$ we had $n+1 < \dim Y$, then there would exist a continuum $C \subset Y$ such that $\delta(p, C) < \delta(Y) < n+1 < \dim C$, according to the lemma. Therefore $C$ would be a proper subcontinuum of $Y$, and so it would have to be nowhere dense in $Y$ (see [2], p. 145) and also—in $X$, contrary to the hypothesis. It follows that $\dim X \leq n$ for every $p \in Y$, i.e. $\dim Y \leq n$.

But every fibre $F$ of $X$ is a union of countably many continua, each of them nowhere dense in $X$ or indecomposable (see [2], p. 153). Hence $\dim F \leq n$ (see [1], p. 176) and Theorem 1 implies the inequality $\dim X \leq n$.

**Theorem 3.** If $\varepsilon > 0$, $X$ is a compact metric space, each continuum $C \subset X$ nowhere dense in $X$ satisfies $\dim C < n$ (where $1 \leq n$) and every component $K$ of $X$ has a diameter $\delta(K) \leq \varepsilon$, then a compact subset $Y \subset X$ exists such that $\dim Y < n-1$ and $Y \neq K \neq 0$ for every component $K$ of $X$.

**Proof.** Let $(K_a : a \in A)$ be the collection of all the components of $X$. Thus there are points $p_a, q_a \in K_a$ such that $p_a$ and $q_a$ are distinct. Let $L_a \subset K_a$ be irreducible continua between the points $p_a$ and $q_a$ (see [2], p. 132). Hence

$$\varepsilon < 2 \delta(L_a) \quad \text{and} \quad \dim L_a \leq n$$

for every $a \in A$, according to Theorem 2.
Put
\[ Z = \bigcup_{a \in D} T_a \]
and let \( K \) be an arbitrary component of \( Z \). Then there is an index \( a_0 \in A \) such that \( K \subseteq K_{a_0} \), whence \( K \cap L_{a_0} = \emptyset \) for \( a_0 \neq a \in A \).

If \( n + 1 \leq \dim K \) held, then the Menger Theorem would imply \( n + 1 \leq \dim (K - K_{a_0}) \), whence \( (K - K_{a_0}) \cap L_{a_0} = \emptyset \), according to (6). For a point \( p \in (K - K_{a_0}) \cap L_{a_0} \), we should have \( n + 1 \leq \dim K \) and \( g(p, I_{a_0}) > 0 \). Hence, applying the lemma, we should obtain a continuum \( C \subset K \) such that \( \delta(p) < \delta(g(p, I_{a_0})) \) and \( n + 1 \leq \dim C \). Therefore \( C \cap L_{a_0} = \emptyset \) and one could infer from \( C \subset K \subset Z \) and \( C \cap L_{a_0} = \emptyset \) for \( a_0 \neq a \in A \), which
\[ C \subset \bigcup_{a \in D} T_a \subseteq \bigcup_{a \in D} L_a, \]
according to (7). Thus \( C \) would be nowhere dense in \( X \), contrary to the hypothesis.

Hence the inequality \( \dim K \leq n \) follows. It yields \( \dim Z \leq n \) (see [2], p. 106). Then there are open subsets \( G_1, \ldots, G_k \) of the set \( Z \) which constitute a finite cover of \( Z \) and satisfy \( \delta(G_i) < \varepsilon \) and \( \dim (G_i - G_j) \leq n - 1 \) for \( i = 1, 2, \ldots, k \). Putting
\[ Y = \overline{G_1} \cup \ldots \cup \overline{G_k}, \]
we see \( Y \) to be a compact set, \( Y \subset Z \subset X \) and \( \dim Y \leq n - 1 \). Finally, by (7), every continuum \( L_a \) contained in \( Z \) and, by (6), it must intersect the boundary of at least one of the sets \( G_1, \ldots, G_k \) (see [2], p. 80), whence \( 0 \neq T_a \cap L \cap K_a \) for \( a \in A \).

Since every nowhere dense subset of the \( n \)-dimensional Euclidean space \( C^n \) has a dimension less than \( n \) (see [2], p. 353), Theorem 3 implies

**Corollary 1.** If \( \varepsilon > 0 \), \( X \) is a compact subset of \( n \)-dimensional polyhedron (where \( 2 \leq n \)) and all the components of \( X \) have diameters greater than \( \varepsilon \), then a compact subset \( Y \subset X \) exists such that \( \dim Y \leq n - 2 \) and \( Y \) intersects every component of \( X \).

A simple proof of Corollary 1 has been given by D. Zaremba (see [5], p. 66, Theorem 3). In the case when \( n = 2 \) and \( X \) lies on the plane \( C^2 \), Corollary 1 becomes the theorem which has been announced by her (see [4], p. 14, Theorem 3). In the same case, it has been generalized by A. Lelek (see [3], p. 88, Theorem 4) to a theorem with a more general hypothesis instead of the assumption that sets intersected by \( Y \) are components of \( X \). The question arises whether Corollary 1 can be generalized to the following: if \( \varepsilon > 0 \) and there is given, in a compact subset \( X \) of an \( n \)-dimensional polyhedron (where \( 2 \leq n \)), a collection \( C \) of mutually disjoint continua having diameters greater than \( \varepsilon \), then a compact subset \( Y \subset X \) exists such that \( \dim Y \leq n - 2 \) and \( Y \) intersects every component of \( X \).

such that \( \dim Y \leq n - 2 \) and \( Y \) intersects every element of \( C \) (compare [3], p. 90).

Now, let us denote by \( C_X \) the collection of components of the compact metric space \( X \), and by \( \delta(A_C) \) the set of points \( p \in X \) such that there exists an infinite sequence \( C_1, C_2, \ldots \) (not of necessarily distinct) elements of \( C_X \) satisfying \( \delta(p) = \lim g_i \) (see [3], p. 88). Then the following theorem generalizes Theorem 3:

**Theorem 4.** If \( X \) is a compact metric space, each continuum \( C \subset X \) nowhere dense in \( X \) satisfies \( \dim C \leq n \) (where \( 1 \leq n \)) and \( \dim \delta(A_C) \leq n - 1 \), then a compact subset \( Y \subset X \) exists such that \( \dim Y \leq n - 1 \) and \( Y \cap C = \emptyset \) for every \( C \in C_X \).

**Proof.** Setting
\[ C_0 = \{ C : C \subset C_X, \ 1 \leq \delta(C) \}, \]
\[ C_m = \{ C : C \subset C_X, \ 1/(m + 1) \leq \delta(C) \leq 1/m \} \]
for \( m = 1, 2, \ldots \) and
\[ X_m = \bigcup_{C \in C_m} C \]
for \( m = 0, 1, \ldots \), we see each point \( p \in C \subset C_X \) to be the limit of a sequence \( p_1, p_2, \ldots \) of points such that \( p_i \in C_i \subset C_m \), whence \( 1/(m + 1) \leq \delta(C_i) \leq 1/m \) for \( i = 1, 2, \ldots \), according to (8). It follows that \( p \in C \) is a continuum (see [3], p. 111) containing \( p_i \), contained in \( X_{m+1} \) and \( 1/(m+1) \leq \delta(C) \). Therefore \( C \subset C_X \), whence \( 1/(m+1) \leq \delta(C) \).

Applying Theorem 3 for \( s = 1/(m+1) \) and \( X = X_m \), we get a compact subset \( Y \subset C_X \) such that \( \dim Y \leq n - 1 \) and \( Y \cap C \neq \emptyset \) for every \( C \in C_X \), \( m = 0, 1, \ldots \). But since \( C \subset C_X \subset C_X \), all these \( C \) are also components of \( X_m \), whence \( C \subset C_X \). Therefore it follows that
\[ Y = \bigcup_{C \in C_m} C \]
intersects every component of \( X \), \( \dim Y \leq n - 1 \) (see [1], p. 116) and, by \( Y \subset C_X \), to complete the proof it is enough to show that \( Y \) is a closed set.

Indeed, the sets on the right in (10) being closed, let \( q = \lim q_i \), where \( q_i \in Y \) and \( \lim q_i = q \). Then, by (9), there exist sets \( C_1 \subset C_m \) and points \( r_i \in C_i \) such that \( q = \lim r_i \). Thus (8) implies \( \delta(C_i) \leq 1/m_i \) for \( i = 1, 2, \ldots \), whence \( \lim \delta(C_i) = 0 \) and so \( (q) = \lim C_i \). It follows, by \( C_m \subset C_X \), that \( q \in \delta(A_C) \). Hence the set \( Y \) is closed, according to (10).

As previously, Theorem 4 yields

**Corollary 2.** If \( X \) is a compact subset of \( n \)-dimensional polyhedron (where \( 2 \leq n \)) and \( \dim \delta(A_C) \leq n - 2 \), then a compact subset \( Y \subset X \) exists such that \( \dim Y \leq n - 2 \) and \( Y \) intersects every component of \( X \).

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(4) Thus \( X = C_X \) in the terminology of [3].
In the special case of \( n = 2 \) and of \( X \) being a plane set, Corollary 2 constitutes the theorem which has been announced by D. Zarembsa (see [4], p. 14, Theorem 4) and generalized in another direction by A. Lelek (see [3], p. 88, Theorem 7).

References


Fixations of sets in Euclidean spaces

by

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Results and problems. The fixation of a collection \( C \) of sets is here understood to mean a set intersecting each element of \( C \). Various fixations have been considered in connection with upper and lower semicontinuous decompositions, but they may also be studied separately.

It is the aim of this paper to examine three kinds of fixation for collections \( C \) consisting of sets contained in the Euclidean \( n \)-dimensional space \( \mathbb{C}^n \), and I am especially interested in the cases \( n = 2 \) and \( 3 \). These three kinds of fixation correspond to the following three properties of the collection \( C \), respectively:

(I) There exists a 0-dimensional compact set \( Z \subset \mathbb{C}^n \) such that \( Z \cap C \neq \emptyset \) for every \( C \in C \).

(II) There exists an arc \( A \subset \mathbb{C}^n \) such that \( A \cap C \neq \emptyset \) for every \( C \in C \).

(III) There exists, for each \( N > 0 \), a finite sequence \( Z_1, \ldots, Z_k \) of closed and mutually disjoint subsets of \( \mathbb{C}^n \) such that \( \delta(Z_i) < N \) \((\ast)\) for \( i = 1, \ldots, k \) and

\[
(Z_1 \cup \ldots \cup Z_k) \cap C \neq \emptyset
\]

for every \( C \in C \).

Property (III) restricted to upper semicontinuous decompositions is equivalent to the existence of fixation in the sense of Knaster [3].

Now, let \( C^* \) denote the union of all sets belonging to \( C \), i.e.

\[
C^* = \bigcup_{C \in C} C.
\]

We have the following theorems:

Theorem 1. (I) implies (II).

Theorem 2. (I) implies (III).

Theorem 1 is an immediate consequence of the Denjoy-Biesz Theorem (see [4], p. 385). Theorem 2 is obvious.

Theorem 3 (D. Zarembsa). If \( C^* \) is a compact set and every \( C \in C \) is a component of \( C^* \), then (II) implies (I) (see [6], p. 14).

\((\ast)\) \( \delta(Z) \) denotes the diameter of the set \( Z \).