disjoints. En vertu du théorème 9, tous les continus $M_{\{t_i\}}$ (de chacune de leurs $2^{\aleph_0}$ familles) sont homéomorphes. En particulier, si M est une dendrite, il en est donc de même de tout $M_{\{t_i\}}$.

En conséquence, si $\dim(X) = 0$, on peut, en vertu du corollaire 2, remplacer dans l'énoncé du théorème 9 le mot *sur-continu* par les mots *dendrite dont les bouts et les points de ramification appartiennent à X.*

La question est moins simple pour les X enfilables. En vertu du théorème 1, elle se réduit à des X compacts de dimension 0. Or un théorème permettant enfiler tout X compact de dimension 0 en un arc $\bar{L}$ tel que $\bar{L}-X$ se compose de segments ouverts disjoints (cf. le problème mentionné p. 72) permettrait, tout comme pour les dendrites, de remplacer dans l'énoncé du théorème 9 le mot *sur-continu* par le mot *arc* et avoir ainsi une généralisation considérable du théorème de Riesz-Denjoy. Mais à défaut de ce moyen de procéder, et celui consistant à enfermer dans des disques les arcs ouverts non-rectilignes (signalé p. 73 comme bien plus compliqué) n'ayant pas été utilisé, cette généralisation du théorème de Riesz-Denjoy reste un problème ouvert.

Ouvrages cités

[1] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914.

[2] B. Knaster, *Sur la fixation des décompositions*, Bull. Acad. Polon. des Sc., Cl. III, 4 (1956), p. 193-196.

[3] — et C. Kuratowski, *Sur les ensembles connexes*, Fund. Math. 2 (1921), p. 206-253.

[4] C. Kuratowski, *Topologie I*, Monogr. Matem., Warszawa, 1958.

[5] — *Topologie II*, Monogr. Matem., Warszawa-Wrocław, 1950.

[6] A. Lelek, *Fixations of sets in Euclidean spaces*, ce volume, p. 87-109.

[7] — et D. Zaremba, *Dimensions of irreducible continua and fixations of components in compact spaces*, ce volume, p. 81-86.

[8] D. Zaremba, *Sur la fixation et l'enfilage des ensembles compacts*, Bull. Acad. Polon. des Sc., Cl. III, 1 (1961), p. 13-15.

INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO
INSTITUT MATHÉMATIQUE DE L'UNIVERSITÉ DE WROCŁAW

Reçu par la Rédaction le 24. 6. 1961

Dimensions of irreducible continua and fixations of components in compact spaces

by

A. Lelek and **D. Zaremba** (Wrocław)

This note establishes a relation (see Theorem 1 below) between the dimension of a continuum X (i.e. connected compact metric space), irreducible between two points, and the dimension of fibres in X (1). As applications there are given two theorems on the existence, in a compact metric space X, of a compact subset which has dimension less than an integer n and intersects every component of X provided that all these components have large diameters or converge to points of a compact set with dimension less than n, and X satisfies a certain condition (see Theorems 3 and 4). That condition holds for subsets of polyhedra, and in this way generalizations of the results of D. Zaremba [4] concerning plane sets are obtained (see Corollaries 1 and 2).

LEMMA. *If $\varepsilon > 0$, X is a compact metric space and $n \leqslant \dim_p X$, then there exists a continuum $C \subset X$ such that $\delta(\{p\} \cup C) < \varepsilon$ and $n \leqslant \dim C$.*

Proof. Let Q be a closed solid sphere in X with centre p and radius $\varepsilon/4$. Hence $n \leqslant \dim Q$ and we infer (see [2], p. 106) that there is a component C of Q, satisfying $n \leqslant \dim C$. Thus C is the desidered continuum.

THEOREM 1. *If X is a continuum irreducible between two points a, b and $2 \leqslant n \leqslant \dim X$, then there exists a fibre $F \subset X$ such that $n \leqslant \dim F$.*

Proof. Let $g\colon X \to \mathcal{I}$ be such a continuous mapping of X into the segment $\mathcal{I} = [0,1]$ (2) that the sets $g^{-1}(t)$ coincide with the fibres of X (see [2], p. 139). If $g(X)$ is a one-point set, then $F = X$ and Theorem 1 follows. Thus we may assume that $g(X) = \mathcal{I}$, $g(a) = 0$ and $g(b) = 1$.

Since X is a compact metric space and $n \leqslant \dim X$, the Menger Theorem (see [2], p. 66) implies the inequality $n \leqslant \dim[X - X_{(n-1)}]$, where $X_{(n-1)} = \{x\colon x \in X, \dim_x X \leqslant n-1\}$. But $X_{(n-1)}$ is a G_δ-set in X (see [1], p. 164), whence $X - X_{(n-1)} = Y_1 \cup Y_2 \cup \ldots$, where Y_i are closed

(1) *Fibres* of an irreducible continuum correspond to its "tranches" in the sense of [2], p. 139. The notation from [1] and [2] is adopted here.

(2) If $t_1 \leqslant t_2$ are real numbers, then we denote by $[t_1, t_2]$ the closed interval $\{t\colon t_1 \leqslant t \leqslant t_2\}$.

subsets of X. Therefore there is a Y_{i_0} such that $n \leqslant \dim Y_{i_0}$ (see [1], p. 176) and so a component K of Y_{i_0} must satisfy $n \leqslant \dim K$ (see [2], p. 106). If $g(K) = \{t_0\}$, then $K \subset g^{-1}(t_0)$ and the proof is completed by putting $F = g^{-1}(t_0)$. Thus we can consider only the case when $g(K) = [t_1, t_2]$ and $0 \leqslant t_1 < t_2 \leqslant 1$.

The fibres $g^{-1}(t)$ being continua (see [2], p. 139), g is a monotone transformation, whence the union

$$L = g^{-1}([0, t_1]) \cup K \cup g^{-1}([t_2, 1]) \tag{1}$$

is a continuum contained in X and containing the points a and b. Since X is an irreducible continuum between these points, we get $L = X$. It follows that $g^{-1}(t) \subset L$ for every $t \in \mathcal{I}$, which implies

$$g^{-1}(t) \subset K \subset Y_{i_0} \subset X - X_{(n-1)} \tag{2}$$

for $t_1 < t < t_2$, according to (1).

Let us put

$$\begin{aligned} Z_0 &= \{t\colon\ t_1 \leqslant t \leqslant t_2,\ 0 = \delta[g^{-1}(t)]\}, \\ Z_m &= \{t\colon\ t_1 \leqslant t \leqslant t_2,\ 1/m \leqslant \delta[g^{-1}(t)]\} \end{aligned} \tag{3}$$

for $m = 1, 2, \ldots$ Then $[t_1, t_2] = Z_0 \cup Z_1 \cup \ldots$ and it follows from the Baire Theorem that there are an integer $k \geqslant 0$ and real numbers s_1, s_2 satisfying $t_1 \leqslant s_1 < s_2 \leqslant t_2$ and

$$[s_1, s_2] \subset \bar{Z}_k. \tag{4}$$

If we had $k = 0$, then, for a number s_0 such that $s_1 < s_0 < s_2$ and $s_0 \in Z_0$, the fibre $g^{-1}(s_0)$ would be a one-point set $\{q\}$, by (3). Hence, for an arbitrary neighbourhood G of the point q in X, there would be such a number $\eta > 0$ that

$$g^{-1}([s_0 - \eta,\ s_0 + \eta]) \subset G,$$

according to the continuity of g (see [2], p. 35-36). Next, by (4) for $k = 0$, numbers $u_1, u_2 \in Z_0$ would exist such that

$$s_0 - \eta \leqslant u_1 < s_0 < u_2 \leqslant s_0 + \eta$$

and for $U = \{t\colon\ u_1 < t < u_2\}$ we should have $s_0 \in U$, whence $q \in g^{-1}(U)$. Thus $g^{-1}(U)$ would be an open neighbourhood of q in X, contained in G and having the boundary

$$\overline{g^{-1}(U)} - g^{-1}(U) \subset g^{-1}(\bar{U}) - g^{-1}(U) = g^{-1}(\bar{U} - U) = g^{-1}(u_1) \cup g^{-1}(u_2),$$

where each of the sets $g^{-1}(u_1)$ and $g^{-1}(u_2)$ would consist of at most one point, according to (3). The boundary of $g^{-1}(U)$ in X would be composed of at most two points and so we should have $\dim_q X \leqslant 1 \leqslant n-1$, $q \in g^{-1}(s_0)$ and $t_1 < s_0 < t_2$, contrary to (2).

It follows that $k > 0$. Choosing a point p and a number s with $p \in g^{-1}(s)$ and $s_1 < s < s_2$, we see p to be an interior point of $g^{-1}([s_1, s_2])$. Therefore a number $\varepsilon > 0$ exists such that $\varepsilon < 1/k$ and $\varrho(p, x) < \varepsilon$ implies $g(x) \in [s_1, s_2]$ for every $x \in X$. Moreover, we have $n \leqslant \dim_p X$, by (2) for $t = s$; thus we infer from the lemma that there is a continuum $C \subset X$ satisfying $\delta(\{p\} \cup C) < \varepsilon$ and $n \leqslant \dim C$, whence

$$\delta(C) < 1/k \quad \text{and} \quad g(C) \subset [s_1, s_2]. \tag{5}$$

If we had $g(C) = [v_1, v_2]$, where $v_1 < v_2$, then (4) and (5) would imply $[v_1, v_2] \subset \bar{Z}_k$ and such a number v would exist that $v_1 < v < v_2$ and $v \in Z_k$. The union

$$D = g^{-1}([0, v_1]) \cup C \cup g^{-1}([v_2, 1])$$

would be a continuum contained in X and containing the points a and b, whence $D = X$ and D would contain $g^{-1}(v)$. Thus so would C and, by (3), we should get $1/k \leqslant \delta[g^{-1}(v)] \leqslant \delta(C)$, contrary to (5).

Hence $g(C)$ must be a one-point set $\{v_0\}$ and putting $F = g^{-1}(v_0)$, we obtain a fibre F statisfying $C \subset F$. The inequalities $n \leqslant \dim C \leqslant \dim F$ follow.

THEOREM 2. *If X is an irreducible continuum and each continuum $C \subset X$ nowhere dense in X satisfies $\dim C \leqslant n$ (where $1 \leqslant n$), then $\dim X \leqslant n$.*

Proof. Suppose that $Y \subset X$ is a non-degenerate indecomposable continuum. Thus $\delta(Y) > 0$ and if for any point $p \in Y$ we had $n+1 \leqslant \dim_p Y$, then there would exist a continuum $C \subset Y$ such that $\delta(\{p\} \cup C) < \delta(Y)$ and $n+1 \leqslant \dim C$, according to the lemma. Therefore C would be a proper subcontinuum of Y, and so it would have to be nowhere dense in Y (see [2], p. 145) and also—in X, contrary to the hypothesis. It follows that $\dim_p Y \leqslant n$ for every $p \in Y$, i.e. $\dim Y \leqslant n$.

But every fibre F of X is a union of countably many continua, each of them nowhere dense in X or indecomposable (see [2], p. 153). Hence $\dim F \leqslant n$ (see [1], p. 176) and Theorem 1 implies the inequality $\dim X \leqslant n$.

THEOREM 3. *If $\varepsilon > 0$, X is a compact metric space, each continuum $C \subset X$ nowhere dense in X satisfies $\dim C \leqslant n$ (where $1 \leqslant n$) and every component K of X has a diameter $\delta(K) \geqslant \varepsilon$, then a compact subset $Y \subset X$ exists such that $\dim Y \leqslant n-1$ and $Y \cap K \neq 0$ for every component K of X.*

Proof. Let $\{K_a\colon\ a \in A\}$ be the collection of all the components of X. Thus there are points $p_a, q_a \in K_a$ such that $\varrho(p_a, q_a) = \varepsilon$ for every $a \in A$. Let $I_a \subset K_a$ be an irreducible continuum between the points p_a and q_a (see [2], p. 132). Hence

$$\varepsilon \leqslant \delta(I_a) \quad \text{and} \quad \dim I_a \leqslant n \tag{6}$$

for every $a \in A$, according to Theorem 2.

Put

$$Z = \overline{\bigcup_{\alpha \in A} I_\alpha} \tag{7}$$

and let K be an arbitrary component of Z. Then there is an index $\alpha_0 \in A$ such that $K \subset K_{\alpha_0}$, whence $K \cap I_\alpha = 0$ for $\alpha_0 \neq \alpha \in A$.

If $n+1 \leqslant \dim K$ held, then the Menger Theorem would imply $n+1 \leqslant \dim[K - K_{(n)}]$, whence $[K - K_{(n)}] - I_{\alpha_0} \neq 0$, according to (6). For a point $p \in [K - K_{(n)}] - I_{\alpha_0}$, we should have $n+1 \leqslant \dim_p K$ and $\varrho(p, I_{\alpha_0}) > 0$. Hence, applying the lemma, we should obtain a continuum $C \subset K$ such that $\delta(\{p\} \cup C) < \varrho(p, I_{\alpha_0})$ and $n+1 \leqslant \dim C$. Therefore $C \cap I_{\alpha_0} = 0$ and one could infer from $C \subset K \subset Z$ and $C \cap I_\alpha \subset K \cap I_\alpha = 0$ for $\alpha_0 \neq \alpha \in A$ that

$$C \subset \overline{\bigcup_{\alpha \in A} I_\alpha} - \bigcup_{\alpha \in A} I_\alpha ,$$

according to (7). Thus C would be nowhere dense in X, contrary to the hypothesis.

Hence the inequality $\dim K \leqslant n$ follows. It yields $\dim Z \leqslant n$ (see [2], p. 106). Then there are open subsets $G_1, \ldots, G_k$ of the set Z which constitute a finite cover of Z and satisfy $\delta(G_i) < \varepsilon$ and $\dim(\overline{G_i} - G_i) \leqslant n-1$ for $i = 1, 2, \ldots, k$. Putting

$$Y = (\overline{G_1} - G_1) \cup \ldots \cup (\overline{G_k} - G_k) ,$$

we see Y to be a compact set, $Y \subset Z \subset X$ and $\dim Y \leqslant n-1$. Finally, by (7), every continuum I_α is contained in Z and, by (6), it must intersect the boundary of a least one of the sets $G_1, \ldots, G_k$ (see [2], p. 80), whence $0 \neq Y \cap I_\alpha \subset Y \cap K_\alpha$ for $\alpha \in A$.

Since every nowhere dense subset of the n-dimensional Euclidean space $\mathcal{E}^n$ has a dimension less than n (see [2], p. 353), Theorem 3 implies

COROLLARY 1. *If $\varepsilon > 0$, X is a compact subset of n-dimensional polyhedron (where $2 \leqslant n$) and all the components of X have diameters greater than ε, then a compact subset $Y \subset X$ exists such that $\dim Y \leqslant n-2$ and Y intersects every component of X.*

A simple proof of Corollary 1 has been given by D. Zaremba (see [5], p. 66, Theorem 3). In the case when $n = 2$ and X lies on the plane $\mathcal{E}^2$, Corollary 1 becomes the theorem which has been announced by her (see [4], p. 14, Theorem 3). In the same case, it has been generalized by A. Lelek (see [3], p. 88, Theorem 6) to a theorem with a more general hypothesis instead of the assumption that sets intersected by Y are components of X. The question arises whether Corollary 1 can be generalized to the following: *if $\varepsilon > 0$ and there is given, in a compact subset X of n-dimensional polyhedron (where $2 \leqslant n$), a collection $\boldsymbol{C}$ of mutually disjoint continua having diameters greater than ε, then a compact subset $Y \subset X$ exists such that $\dim Y \leqslant n-2$ and Y intersects every element of $\boldsymbol{C}$* (compare [3], p. 90)?

Now, let us denote by $\boldsymbol{C}_X$ the collection of components of the compact metric space X [3], and by $A(\boldsymbol{C}_X)$—the set of points $p \in X$ such that there exists an infinite sequence $C_1, C_2, \ldots$ of (not necessarily distinct) elements of $\boldsymbol{C}_X$ satisfying $(p) = \operatorname{Lim} C_i$ (see [3], p. 88). Then the following theorem generalizes Theorem 3:

THEOREM 4. *If X is a compact metric space, each continuum $C \subset X$ nowhere dense in X satisfies $\dim C \leqslant n$ (where $1 \leqslant n$) and $\dim A(\boldsymbol{C}_X) \leqslant n-1$, then a compact subset $Y \subset X$ exists such that $\dim Y \leqslant n-1$ and $Y \cap C \neq 0$ for every $C \in \boldsymbol{C}_X$.*

Proof. Setting

$$\begin{aligned} \boldsymbol{C}_0 &= \{C\colon\ C \in \boldsymbol{C}_X,\ 1 \leqslant \delta(C)\} , \\ \boldsymbol{C}_m &= \{C\colon\ C \in \boldsymbol{C}_X,\ 1/(m+1) \leqslant \delta(C) \leqslant 1/m\} \end{aligned} \tag{8}$$

for $m = 1, 2, \ldots$ and

$$X_m = \overline{\bigcup_{C \in \boldsymbol{C}_m} C} \tag{9}$$

for $m = 0, 1, \ldots$, we see each point $p \in C \in \boldsymbol{C}_{X_m}$ to be the limit of a sequence $p_1, p_2, \ldots$ of points such that $p_i \in C_i \in \boldsymbol{C}_m$, whence $1/(m+1) \leqslant \delta(C_i)$ for $i = 1, 2, \ldots$, according to (8). It follows that $p \in \operatorname{Li} C_i$ and so $K = \operatorname{Ls} C_i$ is a continuum (see [2], p. 111) containing p, contained in X_m and $1/(m+1) \leqslant \delta(K)$. Thus we have $K \subset C$, whence $1/(m+1) \leqslant \delta(C)$.

Applying Theorem 3 for $\varepsilon = 1/(m+1)$ and $X = X_m$, we get a compact subset $Y_m \subset X_m$ such that $\dim Y_m \leqslant n-1$ and $Y_m \cap C \neq 0$ for every $C \in \boldsymbol{C}_{X_m}$ $(m = 0, 1, \ldots)$. But since $C \subset X_m \subset X$ for every $C \in \boldsymbol{C}_m \subset \boldsymbol{C}_X$, all these C are also components of X_m, whence $\boldsymbol{C}_m \subset \boldsymbol{C}_{X_m}$. It follows that the set

$$Y = A(\boldsymbol{C}_X) \cup Y_0 \cup Y_1 \cup \ldots \tag{10}$$

intersects every component of X, $\dim Y \leqslant n-1$ (see [1], p. 176) and, by $Y \subset X$, to complete the proof it is enough to show that Y is a closed set.

Indeed, the sets on the right in (10) being closed, let $q = \lim q_i$, where $q_i \in Y_{m_i}$ and $\lim m_i = \infty$. Then, by (9), there exist sets $C_i \in \boldsymbol{C}_{m_i}$ and points $r_i \in C_i$ such that $q = \lim r_i$. Thus (8) implies $\delta(C_i) \leqslant 1/m_i$ for $i = 1, 2, \ldots$, whence $\lim \delta(C_i) = 0$ and so $(q) = \operatorname{Lim} C_i$. It follows, by $\boldsymbol{C}_{m_i} \subset \boldsymbol{C}_X$, that $q \in A(\boldsymbol{C}_X)$. Hence the set Y is closed, according to (10).

As previously, Theorem 4 yields

COROLLARY 2. *If X is a compact subset of n-dimensional polyhedron (where $2 \leqslant n$) and $\dim A(\boldsymbol{C}_X) \leqslant n-2$, then a compact subset $Y \subset X$ exists such that $\dim Y \leqslant n-2$ and Y intersects every component of X.*

---

[3] Thus $X = \boldsymbol{C}_X^*$ in the terminology of [3].

In the special case of $n = 2$ and of X being a plane set, Corollary 2 constitutes the theorem which has been announced by D. Zaremba (see [4], p. 14, Theorem 4) and generalized in another direction by A. Lelek (see [3], p. 88, Theorem 7).

References

[1] C. Kuratowski, *Topologie I*, Warszawa 1958.
[2] — *Topologie II*, Warszawa 1961.
[3] A. Lelek, *Fixations of sets in Euclidean spaces*, Fund. Math., this volume, pp. 87-109.
[4] D. Zaremba, *Sur la fixation et l'enfilage des ensembles compacts*, Bull. Acad. Polon. Sciences Math. Astr. Phys. 9 (1961), pp. 13-15.
[5] — *Sur l'enfilage et la fixation des ensembles compacts*, this volume, pp. 65-80.

INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
MATHEMATICAL INSTITUTE OF THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 24. 6. 1961

Fixations of sets in Euclidean spaces

by

A. Lelek (Wrocław)

Results and problems. The *fixation* of a collection $\boldsymbol{C}$ of sets is here understood to mean a set intersecting each element of $\boldsymbol{C}$. Various fixations have been considered in connection with upper and lower semi-continuous decompositions, but they may also be studied separately.

It is the aim of this paper to examine three kinds of fixation for collections $\boldsymbol{C}$ consisting of sets contained in the Euclidean n-dimensional space $\mathcal{E}^n$, and I am especially interested in the cases $n = 2$ and 3. These three kinds of fixation correspond to the following three properties of the collection $\boldsymbol{C}$, respectively:

(I) *There exists a 0-dimensional compact set $Z \subset \mathcal{E}^n$ such that $Z \cap C \neq 0$ for every $C \in \boldsymbol{C}$.*

(II) *There exists an arc $A \subset \mathcal{E}^n$ such that $A \cap C \neq 0$ for every $C \in \boldsymbol{C}$.*

(III) *There exists, for each $\zeta > 0$, a finite sequence $Z_1, \ldots, Z_k$ of closed and mutually disjoint subsets of $\mathcal{E}^n$ such that $\delta(Z_i) < \zeta$ (1) for $i = 1, \ldots, k$ and*

$$(Z_1 \cup \ldots \cup Z_k) \cap C \neq 0$$

for every $C \in \boldsymbol{C}$.

Property (III) restricted to upper semicontinuous decompositions is equivalent to the existence of fixation in the sense of Knaster [2].

Now, let $\boldsymbol{C}^*$ denote the union of all sets belonging to C, i.e.

$$\boldsymbol{C}^* = \bigcup_{C \in \boldsymbol{C}} C .$$

We have the following theorems:

THEOREM 1. (I) *implies* (II).

THEOREM 2. (I) *implies* (III).

Theorem 1 is an immediate consequence of the Denjoy-Riesz Theorem (see [4], p. 385). Theorem 2 is obvious.

THEOREM 3 (D. Zaremba). *If $\boldsymbol{C}^*$ is a compact set and every $C \in \boldsymbol{C}$ is a component of $\boldsymbol{C}^*$, then* (II) *implies* (I) (see [6], p. 14).

(1) $\delta(Z)$ denotes the diameter of the set Z.