Representation of a finite graph by a set of intervals on the real line

by C. G. Lekkerkerker and J. Ch. Boland (Amsterdam)

1. Introduction. Let there be given a finite family of sets \( A_1, A_2, \ldots, A_n \). The sets may be thought of as subsets of a given set. For each pair of indices \( i, j \) (\( i \neq j \)) the sets \( A_i, A_j \) may overlap or may not overlap. We wish to establish necessary and sufficient conditions in order that the family \( \{A_i\} \) be representable by a family of intervals \( a_1, \ldots, a_n \) on the real line, in such a way that

\[ a_i \cap a_j \neq \emptyset \quad \text{if and only if} \quad A_i \cap A_j \neq \emptyset, \]

\( \emptyset \) denoting the empty set. It is immaterial whether we take the intervals \( a_i \) to be open or closed.

An equivalent but more transparent formulation of the problem is obtained, if we take what is known in algebraic topology as the one-dimensional skeleton of the nerve of the family \( \{A_i\} \). This is a graph \( G \) consisting of \( n \) points \( a_1, \ldots, a_n \), such that, for each two indices \( i, j \) (\( i \neq j \), \( i, j = 1, 2, \ldots, n \)), the two points \( a_i, a_j \) are joined if and only if the corresponding sets \( A_i, A_j \) meet. Two points \( a_i, a_j \) which are joined will also be called neighbouring points, or neighbours, and we shall write \( a_i \cong a_j \). Clearly, the relation \( \cong \) is symmetric. Our problem then takes the following form:

**Problem.** To decide for which graphs \( G = \{a_1, a_2, \ldots, a_n\} \) it is possible to assign to each point \( a_i \) an interval \( a_i \) on the real line, in such a way that

\[ a_i \cap a_j \neq \emptyset \quad \text{if and only if} \quad a_i \cong a_j \quad (i \neq j; \ i, j = 1, \ldots, n). \]

Any graph possessing the above property will be called **representable** (by intervals).

We note that it is convenient for our purposes to define the relation \( \cong \) only for (certain) pairs of distinct points. More generally, one could write \( a_i \cong a_j \) if the corresponding sets \( A_i, A_j \) meet, including the case that the indices are equal. Then, clearly, the relation \( \cong \) is reflexive. Now, we can conceive a graph \( G \) abstractly as a set on which an arbitrary binary
relation \( r \) is defined. From this standpoint, we are dealing here exclusively with the case that the relation \( r \) is symmetric and non-reflexive.

In this paper we shall prove two theorems each of which gives an answer to the problem stated above (theorems 3 and 4). They are of the following type. Let the concepts of subgraph, path, irreducible cycle, neighbour of a path be defined as in section 2; we emphasize that we use the concept of a subgraph in a rather restricted sense. Then we have

I. A finite graph \( G \) is representable by intervals if and only if it fulfills the two following conditions

(a) \( G \) does not contain an irreducible cycle with more than three points,

(b) if \( a, a, a \) are three points of \( G \), which are mutually distinct and no two of which are neighbouring points, then at least one \( a \) is a neighbour of every path connecting the two other points.

II. A finite graph \( G \) is representable by intervals if and only if it does not contain a subgraph which is one of the graphs \( I, II, III, IV, V \), \( V \), listed in fig. 5.

In our considerations an important rôle will be played by the notion of a simplicial point of a graph (see definition 1). Such a point can be seen as an end-point of the graph. It turned out that graphs which are subjected to the single condition (a) always contain simplicial points (see theorems 1, 2 and lemma 6). In other words, there always exist such points in \( G \) if \( G \) is admitted to contain triangles but not irreducible cycles of "length" greater than 3.

In the last section of this paper practical methods will be sketched by which we can decide whether a given graph is representable. These methods will be based on proposition I formulated above. We shall derive upper bounds for the number of operations needed for the verification of (a) and (b). A remarkable fact is that in the case of the condition (a) the larger number of operations is required. In general, this number is of the order \( O(n^2) \), whereas, if only (a) is known to hold true, the verification of (b) does not need more than \( O(n) \) operations.

The problem formulated at the beginning of this introduction was put by the American biologist S. Benzer. He was concerned with the fine-structure of genes. The problem is whether the sub-elements of genes are linked together in a linear order. He could deal with this problem successfully for a certain microorganism. Of these microorganisms, there are a standard form and mutants, the latter arising if a certain connected portion of the genetic structure is blemished. By recombination tests, it is possible to decide whether the blemished parts of two given mutants overlap or not. Thus, for a large number of portions of the genetic structure, the experiments lead to data as to whether any two of these portions overlap or not. The problem is to decide whether these data are compatible with a linear structure of the gene.

Professor de Groot drew our attention to Benzer’s problem. He found the forbidden graphs with the exception of \( V_n \) (\( n > 1 \)) and his work was continued by the authors of this paper. The second author found and proved theorems 3 and 4. His proofs were simplified by the first author, who introduced in this context the notion of a simplicial point. Sections 4 and 7 are entirely due to the first author.

2. Notations and definitions. In the following \( G \) will always be a finite graph.

If \( a, b \) are two (distinct) neighbouring points of the graph \( G \), then we write \( a \neq b \).

The relation \( \neq \) is symmetric.

A subgraph of \( G \) is a graph \( H \) such that each point of \( H \) belongs to \( G \) and that, for two distinct points \( a, b \in H \), the relation \( a \neq b \) is true whenever it is true in \( G \). In other words, if \( G \) is conceived abstractly as a finite set of elements, together with a certain set of non-ordered pairs \( (a, b) \), then \( H \) is obtained from \( G \) by removing certain elements and those pairs for which at least one constituent does not belong to \( H \).

By the union of two subgraphs \( H_1 \) and \( H_2 \) of \( G \), such that subgraph \( H \) of \( G \) is meant which consists of the points belonging to at least one of \( H_1 \) and \( H_2 \). This union depends on \( H_1 \) if \( a, b \) are two points in \( H \) do not belong to the same graph \( H_1 \), then the relation \( a \neq b \) may or may not hold in \( H \), and this cannot be decided from the structure of \( H_1 \) and \( H_2 \) alone. We therefore write \( H = [H_1 \cup H_2] \). Only if no confusion can arise, we shall simply write \( H = H_1 \cup H_2 \).

The complement of a subgraph \( H \) of \( G \), is denoted by \( G \setminus H \); it is the subgraph of \( G \) consisting of the points in \( G \) which do not belong to \( H \) (c).

We have \( (H \setminus G \setminus H) = G \).

The graph consisting of a single point \( a \) is denoted by \( \{a\} \).

A point \( a \) will be called a neighbour of a subgraph \( H \) of \( G \) and we shall write \( a \in H \), if \( a \in H \) and \( a \neq b \) for some point \( b \in H \).

We further use the following terms:

- path: \( W = a_1 a_2 \ldots a_k \); any subgraph of \( G \), such that \( a_1 \neq a_{i+1} \) \((i = 1, \ldots, k-1)\); it is not required that \( a_1 \neq a_i \) for all \( j \neq i \);
- irreducible path: a path \( a_1 a_2 \ldots a_k \) such that \( a_i \neq a_j \) if \( i \neq j \) and \( a_i \neq a_k \); only if \( j = \pm 1 \);
- cycle: a path of the form \( a_1 a_2 \ldots a_k a_1 \);

(c) Confer the previous definition of a subgraph.
irreducible cycle: a cycle \( a_1a_2 \ldots a_k \) such that \( a_i \neq a_j \) if \( i \neq j \) and \( a_i \neq a_j \), only if \( j = i \pm 1 \) or \( i = (k-1) \).

- star \( S(a) = \{S \mid a \in S \} \) of a point \( a \in G \): the subgraph of \( G \) consisting of \( a \) and all neighbours of \( a \).
- star \( S(H) = \{S \mid H \subseteq S \} \) of a subgraph \( H \subseteq G \): the subgraph of \( G \) whose points are given by the points of \( H \) and the neighbours of \( H \).

**Definition 1.** Let \( a \in G \). Then \( a \) is called a **simplex** point of \( G \), if \( S(a) \) is a simplex.

**Definition 2.** A graph \( G \) is called acyclic, if it does not contain an irreducible cycle with more than three points.

**Definition 3.** A graph \( G \) is called **asteroidal** if it contains three distinct points \( a, b, c \) and three paths \( W_1, W_2, W_3 \) such that, for \( i = 1, 2, 3 \),

- \( W_i \) connects the two points \( a_i \) (\( j \neq i \));
- \( a_i \) is not a neighbour of \( W_i \).

Such a triple of points \( a, b, c \) is called an asteroidal triple.

**Definition 4.** Let \( G \) be a graph. Suppose that there exists a set \( \Gamma \) of open intervals on the real line such that the following properties hold:

- (i) there is a one-to-one correspondence between the points \( a, b, \ldots \) of \( G \) and the intervals \( a_i, \ldots \) of \( \Gamma \);
- (ii) two intervals \( a, b \) intersect if and only if the corresponding points \( a_i, b \) belong to the same copy of \( G \) and \( a_i \neq b \) is true in \( G \). We say that \( G \) is connected.

Finally, we wish to introduce the concept of duplication of a graph. Let \( H \) be a subgraph of \( G \). Then we form a new graph \( K \) by taking two copies of \( G \) and by identifying corresponding points of \( H \). In particular, if \( a, b \) are two points of \( E \setminus H \), then the relation \( a \neq b \) subsists if and only if \( a, b \) belong to the same copy of \( G \) and \( a_i \neq b \) is true in \( G \). We say that \( K \) is obtained by duplication of \( G \) with respect to \( H \).

The above construction will only be carried out in the case that \( H \) is a simple.

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3. Some lemmas.

**Lemma 1.** If \( G \) is an acyclic graph and \( a_1a_2 \ldots a_k \) is a cycle in \( G \), with \( k > 4 \), then we necessarily have

(i) \( a_i = a_j \) or \( a_i \neq a_j \) or

(ii) \( a_i = a_j \) or \( a_i \neq a_j \) for some \( i < 4 \leq k \).

**Proof.** For \( k = 4 \) the lemma is true, as the cycle is irreducible, by definition. Now let \( k > 4 \) and suppose that for no \( i > 4 \) we have \( a_i = a_j \) or \( a_i \neq a_j \). Then, since the cycle is reducible, there must be two indices \( i', i'' \) with

\[
(a_i = a_{i''}) \text{ or } (a_i \neq a_{i''}), \quad i'' < i', \quad i' \neq 2.
\]

Then we can make a shorter cycle, in which \( a_1, a_2, a_3 \) all occur and in which \( a_i, a_i' \) occur as one point or as successive points. Assuming the lemma to be true for this cycle, we must have \( a_i = a_j \) or \( a_i \neq a_j \). Hence, the lemma follows by induction on \( k \).

**Lemma 2.** Each path \( a_1a_2 \ldots a_i \), with \( a_1 \neq a_i \), contains an irreducible path with the same endpoints.

**Proof.** Put \( i_i = 1 \). Take the maximal index \( i_i \) with \( a_i \neq a_{i+1} \), thereafter the maximal index \( i_j \) with \( a_j \neq a_{j+1} \), etc. Then the path \( a_1a_2 \ldots a_i \) is irreducible.

**Lemma 3.** Let \( G \) be an acyclic graph. If \( ca_1a_2 \ldots a_k \) (\( c \neq a_i \) for \( i = 1, 2, \ldots, k \)) is a cycle in \( G \) and \( a_1a_2 \ldots a_k \) is an irreducible path, then we have \( c \neq a_i \) for \( i = 1, 2, \ldots, k \).

**Proof.** By Lemma 1, with \( c \) instead of \( a_1 \), we have \( c \neq a_i \) for some \( i \) with \( 1 < i < k \). Then \( ca_1a_2 \ldots a_k \) is a cycle of the same type as the given one. The lemma now follows by induction.

**Lemma 4.** Let \( G \) be an acyclic graph, and let \( S \subseteq G \) be a simplex. Further, let \( K \) be the graph obtained by duplicating \( G \) with respect to \( S \). Then \( K \) is acyclic. Furthermore, each point \( a \in S \) which has a neighbour in \( G \) \((G \cap S) \), is not a simplicial point of \( K \).

**Proof.** A cycle \( C \) in \( K \) which has a point in each of the two copies of \( G \) is clearly not irreducible. From this the first assertion follows. The second assertion is also clear.

4. Existence of simplicial points. We begin by proving the following fundamental

**Theorem 1.** Each finite acyclic graph contains a simplicial point.

**Proof.** We use induction on the number of points. Let \( G \) be an acyclic graph with \( n > 1 \) points, and suppose that the theorem is true for graphs with less than \( n \) points. Then we shall prove that \( G \) has a simplicial point.

(1) The theorem is no longer true for infinite graphs, as is seen from the example \( G = (a_1a_2 \ldots a_n) \) with neighbour relations \( a_i a_{i+1} \) (\( k = 0, \ldots, n \), \( n \leq 2 \)).
Let $b$ be an arbitrary point of $G$ and let $a$ be a simplicial point of $G \backslash \{b\}$. Put $G_1 = G \backslash \{b\}$ and $S_1(a) = [S(a) \backslash \{b\}]_a$.

First, we dispose of some trivial cases. If we do not have $a \in E$, then $a$ is also a simplicial point of $G$. More generally, if some point $a \in S(a)$ has no neighbour in $G \backslash S(a)$, then $S(a) = S_1(a)$, and so $a$ is a simplicial point of $G$. If, on the other hand, $b \in E$ for each point $a \in S(a)$, then $S(a) = S_1(a)$ is a simplex, so that $a$ is a simplicial point of $G$. Henceforth, we may restrict ourselves to the case that the following three properties hold:

(i) $a \in E$;
(ii) each point $a \neq e$ of $S_1(a)$ has at least one neighbour in $G \backslash S_1(a)$;
(iii) there is a point $c \neq e$ in $S_1(a)$, such that not $b \in E$.

We now consider the graph $G \backslash S_1(a)$. It need not be connected. We denote by $C_1$ the component of $G \backslash S_1(a)$ which contains the point $b$, and put $C_1 = G \backslash (S_1(a) \cup C_1)$. We shall prove that $e \in S_1(a)$ implies that $e \in E$.

Let $e$ be a point in $S_1(a)$, with $e \in E$, and let $d_i$ be a neighbour of $e$ in $C_1$. If $e = a$, then there is nothing to prove. Hence we may suppose that $e \neq a$ and $d_i \neq b$. Then, since $d_i \in C_1$, $e \neq C_1$, and $e$ is connected, there is a path $d_i \ldots d_k b$ in $C_1$, with $b \neq d_i$ for $i = 1, ..., k$ and $k > 1$.

Now $b = d_i d_j b$ is a cycle. Further, we do not have $a = d_i$, nor $a \in d_i$, for any $i = 1, 2, ..., k$, as the only neighbours of $a$ are given by $b$ and the points $d_i \neq a$ of $S_1(a)$. Then lemma 1, with $(a, a, a) = (b, a, e)$, implies that $e \in E$.

It follows now from (iii) that there is a point $e \in S_1(a)$, which is not a neighbour of $C_1$. Then, by (ii), this point has a neighbour in $C_1$. This implies that $C_1$ is not empty. Now define

$S_1 = \text{graph of the points } e \in S_1(a)$ with $e \in E$, but not $e \in C_1$;
$S_2 = \text{graph of the points } e \in S_1(a)$ with $e \neq e$, $e \in C_1$;
$S_3 = \text{graph of the points } e \in S_1(a)$ with $e \neq e$, but not $e \in C_1$.

Then, by (iii), $S_1 \cup S_2 \cup S_3 = S_1(a)$. Further, the subsimplices $S_1$ and $S_2$ are not empty, as $a \in S_1$ and $a \neq S_1$. The sub simplex $S_3$ may be empty.

Next, we wish to duplicate a suitable part of the graph $G$ (see fig. 2, where each $S_1$ is represented by a single point). The subgraphs $C_1$ and $C_1 \cup S_1$ are disjoint and are separated by the simplex $S_1 \cup S_2$. More precisely, each point of $S_1 \cup S_2$ has a neighbour in $C_1$, by definition of $S_1$ and $S_2$, and also a neighbour in $C_1 \cup S_1$, because $S_1 \cup S_2 \cup S_3 = S_1(a)$ is a simplex. Similarly, the subgraphs $C_1 \cup S_1$ and $C_1$ are disjoint, whereas each point of $S_1 \cup S_3$ has a neighbour in both of them.

We shall now investigate whether there are more simplicial points in a given acyclic graph $G$.

A simple consequence of definition 1 is the following:

**Lemma 5.** Let $G$ be a graph. If $S$ is a connected subgraph of $G$ and each point of $S$ is a simplicial point of $G$, then $S$ is a simplex. All points of $S$ have the same star.

**Proof.** Let $a, b, c$ be three different points of $S$ with $a \in E$, $b \in E$. Then, since $b$ is simplicial, we necessarily have $a \in E$, and so $(a, b, c)$ is a simplex. It is clear that, by a repetition of the argument, we find that $S$ is a simplex.

Next, let $a, b$ be two distinct points of $S$. If $a \in E$ and $b \in E$, then $a \in E$, $b \in E$, as a simplex. It follows that any two points of $S$ have the same star.

Using the principle of duplication one easily deduces from theorem 1 that an acyclic graph with more than one point must have at least two
simplicial points. More generally, this principle leads to a proof of the following.

**Lemma 6.** Let \( G \) be an acyclic graph, which is not a simplex. Then \( G \) contains two non-neighbouring simplicial points.

**Proof.** We may, clearly, suppose that \( G \) is connected. Let \( a \) be a simplicial point of \( G \), and let \( S' \) denote the subsimplex of points \( \sigma \in S(a) \), which have no neighbour in \( G \setminus S(a) \). These points \( \sigma \) are just the points of \( S(a) \) which are simplicial points of \( G \).

We have \( S' \neq \emptyset \), as \( a \in S' \), and \( S' \neq S(a) \), as otherwise we should have \( G = S(a) \). Now we duplicate \( G \) with respect to \( S' \). In virtue of lemma 4 and by the choice of \( S' \), we then get a graph \( K \) which is acyclic and which has no simplicial point in \( S' \). Then \( K \) has a simplicial point outside \( S' \). This point corresponds with a simplicial point \( b \notin S' \) of \( G \). Then, by the choice of \( S' \), \( b \notin S(a) \). This proves the lemma.

There are various examples of acyclic graphs with exactly two simplicial points. For example the graphs which can be represented by a diagram of one of the following types:
1. A broken line (= an irreducible path);
2. A polygon with all diagonals through one given vertex;
3. A polygon with all diagonals except one;
4. Figures like fig. 3.

Such examples, as well as lemma 6 and its proof, suggest that the simplicial points are to be sought at the "extremities" of the graph (cf. also lemma 7 in section 5). We can give a more precise meaning to this statement by proving the following theorem 2 (which is easily seen to be a generalization of lemma 6).

**Theorem 2.** Let \( G \) be an acyclic graph. Let \( H \) be a connected subgraph of \( G \) and suppose that \( G \setminus S(H) \) is not empty. Then \( G \setminus S(H) \) contains a point \( s \) which is a simplicial point of \( G \).

**Proof.** Let \( m \) be the number of points of \( G \setminus S(H) \). We shall prove the theorem by induction on \( m \).

First, let \( m = 1 \). Then \( G \setminus S(H) \) consists of one point, \( a \), say. Let \( b_1, b_2 \) be two distinct neighbours of \( a \). Then these points belong to \( S(H) \), but not to \( H \), as \( a \notin S(H) \). Hence, there exists a path \( W = a_1 \ldots a_n (n \geq 1) \) in \( H \), such that

\[ b_1 \neq a_1, \quad b_2 \neq a_n. \]

Now consider the cycle \( a_1 a_2 a_3 \ldots a_n b_1 a_1 \). Since \( a \) is not a neighbour of \( W \), it follows from lemma 1 that we must have \( b_1 \neq b_2 \). It follows that \( S(a) \) is a simplex, i.e. that \( a \) is a simplicial point of \( G \).

Now, let \( m > 1 \), and suppose that the theorem holds true if \( G \setminus S(H) \) has less than \( m \) points.

Let \( b \) be an arbitrary point of \( G \setminus S(H) \) and put \( G_1 = G \setminus \{b\}, \quad S_1(a) = \{a_1 \ldots a_n \}\). Let \( a \) be a simplicial point of \( G_1 \) in \( G \setminus S(H) \). If not \( a \neq b \), then \( a \) is a simplicial point of \( G \). Hence we may suppose \( \neq b \).

We consider the components of \( G \setminus S_1(a) \). Let \( C_1 \) be a component containing some point of \( H \). Now \( H \) is connected and \( H \cap S_1(a) \) is empty, as \( a \notin S(H) \). Hence \( C_1 \) contains the whole graph \( H \). Then it also contains \( S(H) \setminus S_1(a) \). We now distinguish two cases.

**Case 1.** \( G \setminus S_1(a) \) is connected, i.e. \( G \setminus S_1(a) = C_1 \). In particular \( b \notin C_1 \).

Then, as in the proof of theorem 1, it is true that \( c \notin S_1(a), \quad c \neq C_1 \) implies \( c \neq C_1 \). There are two possibilities.

**Case 1a.** \( c \notin C_1 \) for each point \( c \in S_1(a) \). Then \( a \) is a simplicial point of \( G \).

**Case 1b.** There is a point \( c \neq a \) in \( S_1(a) \), such that \( a \neq C_1 \).

Then, \( a \neq C_1 \) is not a neighbour of \( H \), and so \( a \notin S(H) \). Further, \( S_1(c) = S_1(a) \). Hence \( a \) is a simplicial point of \( G \).

**Case 2.** \( G \setminus S_1(a) \) is not connected. Let \( C_2 \) be a second component of \( G \setminus S_1(a) \) and let \( D = C_1 \cup C_2 \).

The subgraph \( D \) is either a simplex not contained in \( S_1(a) \) or else, by lemma 6, it has two non-neighbouring simplicial points, which cannot both belong to \( S_1(a) \). Hence, there is a point \( s \in C_2 \) which is a simplicial...
point of $D$. It does not belong to $S(H)$, because $S(H)$ is contained in $G \cup S(H)$. Further, $[S(s)]_D = [S(s)]_D$, so $s$ is a simplicial point of $G$.

So, in all cases there is a simplicial point of $G$ in $G \setminus S(H)$. This proves the theorem.

5. **Representability of graphs.** In this section we wish to derive a criterion for representability. Here the notion of asteroidal graph will come in. We shall further have to consider two types of simplicial points. First, we define

**Definition 5.** A simplicial point $a$ of a graph $G$ is called **strongly simplicial** if $G \setminus S(a)$ is connected, and **weakly simplicial** if $G \setminus S(a)$ is not connected. Further, an acyclic graph $G$ is called extremal if it is connected and if all its simplicial points are strongly simplicial.

If a graph $G$ formed by points $a, b, \ldots$ is representable, then we shall denote the corresponding intervals in a model $I'$ by corresponding Greek letters $\alpha, \beta, \ldots$. The left-hand and right-hand end points of an interval $a$ will be denoted by $I(a), I(a)$ respectively. Then, by an **end-interval** of a model $I'$ we shall mean an interval $a \in I'$ such that either

(i) $I(\beta) > I(\alpha)$ for each interval $\beta \in I'$, or

(ii) $I(\beta) < I(\alpha)$ for each interval $\beta \in I'$.

In these cases, $a$ is a left-hand or a right-hand endinterval respectively.

**Lemma 7.** If $G$ is representable and $a$ is a strongly simplicial point of $G$, then, in each model $I'$ of $G$, $a$ is an endinterval.

**Proof.** First observe that any model of a connected graph is connected. Now consider the simplex $S(a)$. If $G = S(a)$, then the assertion is trivial. If not, then take in $I'$ the submodel of $G \setminus S(a)$. It is connected, and no interval meets $a$. From this and the definition of $S(a)$ it follows that $a$ is an endinterval.

We now come to the main result of this section.

**Theorem 3.** A graph $G$ is representable if and only if it is acyclic and not asteroidal.

**Proof.** The proof of the "only if" part is easy. Indeed, let there be a model $I'$ of $G$. First, suppose that $G$ contains an irreducible cycle $a_0, a_0, \ldots, a_k$, with $k > 0$. Then, in $I'$, the intervals $a_0, a_0$ are disjoint. The interval $a_0$ meets both $a_0$ and $a_0$, but no interval $a_i$, with $i > 3$, while these intervals $a_i$ connect $a_0$ and $a_0$. This is impossible. Hence, $G$ is acyclic.

Next, suppose that $G$ contains an asteroidal triple $(a_0, a_1, a_2)$. In $I'$, the intervals $a_0, a_1, a_2$ are mutually disjoint. So, without loss of generality, we may suppose that $a_0$ separates $a_0$ and $a_0$. Then $a_0$ meets the image in $I'$ of each path $a_0, a_0, a_0, a_0$, and so $a_0 \lor a_0, a_0, a_0, a_0$ for each choice of $W$. This contradicts the definition of asteroidal triple. Hence, $G$ is not asteroidal.

Conversely, let $G$ be an acyclic graph which is not asteroidal. Then we shall prove, by using theorem 1, that $G$ is representable. We distinguish two cases.

Case 1. $G$ is extremal (this implies that $G$ is connected). By theorem 1, it has a simplicial point. By lemma 6, if $G$ is not a simplex, it even has two non-neighbouring simplicial points. But it cannot have three simplicial points $a_1, a_2, a_3$, no two of which are non-neighbouring points. For, by hypothesis, these points would be strongly simplicial, and so $a_1, a_2$ could be connected by a path not meeting $S(a_0)$ (by any permutation of $(1, 2, 3)$), so that $G$ would be asteroidal, against hypothesis.

Consequently, it suffices to prove the following.

**Assertion.** A graph $G$ which is acyclic and extremal and which does not contain three non-neighbouring simplicial points, is representable.

We shall do this by using induction on the number of points, say $n$. If $n = 1$, then the assertion is trivially true. Now take $n > 1$ and suppose that the assertion holds for graphs with less than $n$ points.

Let $a$ be a simplicial point of $G$ and let $G_1$ be the subgraph consisting of those points of $S(a)$ which are simplicial points of $G$.

Then $G_1 = S(a) \setminus S_a$, $G_1 = G \setminus S(a) = G \setminus S_a$.

Thus, $G_1$ is extremal, connected and acyclic. We investigate its simplicial points. First, let $b \in G_1$ be a simplicial point of $G_1$. Then, since not $b \lor S_a$, $[S(b)]_G = [S(b)]_{G_1}$. Consequently, $b$ is a simplicial point of $G$. Now take two arbitrary points $a_1, a_2$ in $G \setminus S(b)$. Since $b$ is a strongly simplicial point of $G_1$, there exists a path $W$ in $G \setminus S(b)$ connecting the points $a_1, a_2$. By lemma 2, there is an irreducible path $W'$ which is a subgraph of $W$ and which connects $a_1, a_2$. This path cannot contain a point of $S_a$.

Hence, it is contained in $G \setminus S(b)$. It follows that $b$ is a strongly simplicial point of $G_1$.

On the other hand, a simplicial point of $G$ which belongs to $G_1$ is also a simplicial point of $G$.

Next, let a point $d \in S_a$ be a simplicial point of $G_1$. Write $S_a(d) = (S(d))_a$. Let $G_1 \setminus S_a(d)$ have $k$ components $C_1, \ldots, C_k$ ($k > 0$). For each $C_i$, the graph $G_1 \setminus S_a(d)$ is not a simplex (because of $G_1 \neq G$) and so, by lemma 6, $G_1 \setminus S_a(d)$ has a simplicial point $e_i \in S_a(d)$. Then $e_i \in S_a(d)$. It is easy to see that $[S(e_i)]_{G_1 \setminus S_a(d)} = [S(e_i)]_G$.

Hence, $e_i$ is a simplicial point of $G$ (i = 1, 2, ..., k). Then, by our hypotheses, we must have $k = 0$ or 1, i.e. $d$ is a strongly simplicial point of $G_1$ (note that it may happen that $k = 0$, i.e. $G_1 = S_a(d)$).
Combining the results obtained so far, we see that $G_1$ is extremal. It is also acyclic. Further, it has simplicial points outside $S_1$; moreover, the simplicial points of $G_1$ belonging to $G_1 \backslash S_1$ form a simplex. It follows from lemma 6 that $G_1$ has a simplicial point $d \in S_1(\delta)$. Also, by our induction hypothesis, there is a model $\Gamma_1$ of $G_1$. In this model, $d$ is represented by an endinterval $\delta$ on account of lemma 7. Let $\delta$ be a left-hand endinterval. Since $S_1$ is contained in $S_1(\delta)$ and $S_1(\delta)$ is a simplex, we can produce to the left, in $\Gamma_1$, the intervals corresponding with $S_1$; this does not give rise to new overlappings. We can do this and add an interval $e$ in such a way that $e$ meets exactly the intervals of $S_1$. Representing every point of $S_1$ by such an interval $e$ we get a model of $G$.

Case 2. $G$ is not extremal. We may suppose that each proper subgraph of $G$ is representable. Let $e$ be a weakly simplicial point of $G$. Put $S = S(e(\delta))(\delta)$ and denote the components of $S, S(\delta)$ by $G_1, \ldots, G_k \ (k \geq 2)$. It is convenient to call a point $c \in G \setminus S$ a full neighbour of $S$ and to write $c \sim S$, if we have $c \sim b$ for each point $b \in S$.

We shall apply the induction hypothesis in two different ways.

We first consider some trivial cases. Let $\Gamma_1$ be a model of $G \setminus (\delta)$ and let $\delta$ be the intersection of the intervals corresponding with points of $S$. If no $G_1$ contains a point $e$ with $e \sim S$, then $\delta$ is not met by other intervals, and so a model $\Gamma_1$ is obtained by adding to $\Gamma_1$ an interval $e = \delta$. If, on the other hand, there is an index $i$, such that each point $c \in G_i$ is a full neighbour of $S$, then we argue as follows.

In $\Gamma_1$, each interval $G$ of $G_i$ meets $\delta$. Further, these intervals form a connected model, and they do not intersect other intervals of $G \setminus S$. We can diminish arbitrarily the dimensions of the submodel of $G_i$ in $\Gamma_1$. We can do this and add an interval $e$ to $\Gamma_1$ in such a way that we obtain a model of $G$.

It follows that we may restrict ourselves to the case that there is an index $i$, such that some points of $G_i$ are full neighbours of $S$ and some points are not. We put

$$G_1 = G \setminus (\delta), \quad G_2 = S(e(\delta)) \cup G_1.$$

By induction, there are models $\Gamma_1, \Gamma_2$ of $G_1, G_2$ respectively. First, consider $\Gamma_1$. Since $e$ is a strongly simplicial point of $G_1$, $e$ is an endinterval of $\Gamma_1$, say a right-hand endinterval. There is an interval $\gamma$ in $\gamma$ which does not meet $\delta$ as there is a point in $G_i$ which is not a full neighbour of $S_1$, we choose one for which $r(\gamma)$ is minimal. Let $\Sigma$ be the set of intervals which correspond with points of $S$ and which meet $\gamma$ ($\Sigma$ may be empty), and let $S(\delta)$ be the submodel of $S(\delta)$ in $\Gamma_1$. Then each interval set $\Gamma_1$ obtained from $\Gamma_1$ by producing arbitrarily to the left one or more intervals of $\Sigma_1$ and arbitrarily to the right one or more intervals of $\Sigma(\delta)$, is again a model of $G_i$. For this does not cause new overlappings, because $\gamma$ and $\gamma$ are endintervals.

Next, consider $\Gamma_1$. It contains some model of $G \setminus (\delta)$; let $\gamma', \delta'$ be the intervals in $\Gamma_1$ corresponding with the intervals $\gamma, \delta$, respectively, in $\Gamma_1$. Then $\gamma', \delta'$ do not meet, without loss of generality we may suppose that $r(\gamma') < l(\gamma')$. On the real line, where $\Gamma_1$ is situated, we choose an interval $\xi$ such that each interval of $G_i$ is wholly contained in $\xi$ and that each interval of $G_j \ (j \neq i)$ falls outside $\xi$. Then $r(\xi)$ and $l(\xi)$ only belong to intervals of $S(e(\delta))$, and each interval of $S$ intersects $\xi$, as there is a point in $G_i$ which is a full neighbour of $S$. We prove that $l(\xi)$ can only belong to intervals of $S(\delta)$ (corresponding with $\Sigma_1$).

Let $b \in S(e(\delta)) \backslash S_1$ and let $\beta, \beta'$ be the corresponding intervals in $\Gamma_1, \Gamma_1$ respectively. Then $\beta$ does not meet $\gamma$. Hence, $\beta'$ does not meet $\gamma'$. But it meets $\delta'$. Hence, we have $l(\beta') > r(\gamma')$ and so $l(\xi) \notin \beta'$.

We can now construct a model of $G$ in the following way. Take $\Gamma_1$, remove the part $\Gamma_1 \cap \xi$ and then insert the model $\Gamma_1$; produce to the left those intervals of $\Sigma$ which in $\Gamma_1$ contain $r(\xi)$.

This proves the assertion and thus completes the proof of the theorem.

6. Structure of non-representable graphs. In this section we follow the original idea of Professor de Groot of determining a minimal set of graphs with the property that any graph is representable if and only if it does not contain a graph of this set. It turned out that a complete set with this property is given by figure 5; there, in each diagram, except IIIa, we have indicated the three points which constitute an asteroidal triple.

In other words, we have the following

**Theorem 4.** A graph $G$ is representable, if and only if it does not contain a subgraph which is one of the graphs I, II, IIIa, IVa, Va (1).

**Theorem 4** gives a less elegant characterization of representable graphs than theorem 3. But it lies deeper, as in this theorem the various types of non-representable graphs are analysed. Actually, the proof of theorem 4 will be based on theorem 3.

**Proof of theorem 4.** We leave it to the reader to check that the graphs I, II, IVa, Va are all asteroidal (2), and hence also, that the condition is necessary. It remains to show that if $G$ is not representable,

(1) Of course, it is understood that no junctions are present which are indicated in the diagrams.
(2) The cycle IIIa is asteroidal for $n \geq 6$. 
then $G$ contains one of the subgraphs listed above. So by theorem 3 the proof of the theorem will be completed if we can deduce the following assertion. Let $G$ be a graph with the following properties:

1. $G$ is acyclic;
2. $G$ is asteroidal;
3. $G$ is minimal, i.e., no proper subgraph is asteroidal.

Then $G$ is one of the graphs $I$, $II$, $IV_n$, $V_n$.

![Graphs I, II, IV, V](image)

**Fig. 5**

Let $G$ have the properties (1)-(3). Let $(a_i, a_j, a_k)$ be an asteroidal triple and let $W_t, W_s, W_r$ be three paths such that

a. $W_t$ connects the two points $a_i (i \neq j)$

b. $a_t$ is not a neighbour of $W_t$.

In virtue of lemma 2, we may suppose that the paths $W_t$ are irreducible. Further, it follows from (3) that we have

$$W_s \cup W_t \cup W_r = G .$$

If $i \neq j$, then $W_t$ contains only one point $a_t$ of $S(a_i)$ as $W_t$ is irreducible. Hence, $S(a_i)$ contains at most two points $a_t$ if there are two, say $a_t'$ and $a_t''$, then we must have $a_t' \neq a_t''$. For the three paths $W_t$ constitute a cycle, in which $a_t', a_t, a_t''$ are successive points, $a_t \neq a_t'$ and $a_t$ has no other neighbours than $a_t', a_t''$. Then an application of lemma 1 learns that $a_t' \neq a_t''$.

We now distinguish some cases.

**Case 1.** Each $a_t$ has two neighbours. Let the $W_t$ be given by

$$W_s = c_0 \ldots c_{k-1} \ldots c_k a_t, \quad W_r = a_0 \ldots a_{k-1} a_t, \quad W_r = a_0 \ldots a_{k-1} c_k .$$

We do not exclude that $b_i = c_0$ or $b_i = c_k$. If this occurs then the corresponding path $W_t$ is called short.

We cannot have $b_i = c_0$ as otherwise the third point of $S(a_t)$ would not occur in any $W_t$. Hence, $b_i \neq c_0$. Similarly, $b_i \neq c_0$ and $b_i \neq c_k$. Further, the points $b_i$ are mutually distinct, and also the points $a_t$, because of $b_i, c_0 \in W_t (i = 1, 2, 3)$. So we have the situation of figure 6. Note that the paths $W_t$ may have interior points in common.

![Graphs 6 and 7](image)

**Fig. 6**

**Fig. 7**

We now prove that at least two paths $W_t$ are short. Suppose that e.g. $W_s$ and $W_r$ are not short, and consider the point $c_0$. It is a point of the cycle $c_0 \ldots c_k \ldots c_0$.

If $c_0 \in W_s$, then we may replace $W_s$ by $W_s = a_0 a_1 a_t a_2 a_3$. If $c_0 \in W_r$ for some interior point $d \neq c_0$, then we may replace $W_r$ by $W_r = a_0 a_1 d a_2 a_3$.

In both cases, $G$ would not be minimal. Similarly, if $b_i \in W_s$ for some interior point $e \neq c_0$ of $W_s$ or $W_r$, $G$ would not be minimal.

Hence,

$$S(a_t) \cap (W_s \cup W_r) = b_i$$

and

$$S(b_i) \cap (W_s \cup W_r) = c_0 .$$

It is now easily verified that in the cycle $c_0 \ldots c_k \ldots c_0$ none of the implications of lemma 1 holds. This contradiction proves that at least two paths are short.

So we may suppose that $W_r$ and $W_s$ are both short (fig. 7). Suppose that we do not have $b_i \in W_s$. Then $c_0 \neq b_i$, and then $(a_t, a_0, c_0)$ is an asteroidal triple in $G \setminus \{a_t\}$. Hence, by (3), we must have $b_i \in W_t$. Then, by lemma 3, we have $b_i \in W_t$ for each point $c$ of the irreducible path $b_i \ldots c$. Similarly, $b_i \in W_t$ for each such point $c$.

Then $G$ is of the form $V_n$ (the case $n = 1$ occurs if $b_i = c_0$).

**Case 2.** One of the points $a_t$ has only one neighbour. Let $a_t$ be such a point and let $b_i$ be its neighbour. Then $W_t$ and $W_s$ necessarily contain
the point \( b \). If not \( b \not\in W_1 \), then \( (b, a_2, a_1) \) is an asteroidal triple of \( G \setminus \{a_1\} \). So we have \( b \not\in W_1 \). We now have to distinguish some subcases.

Case 2.1. \( b \) has \( k \geq 2 \) neighbours on \( W_1 \). Let \( c, c' \) be the first and the last neighbour respectively. To \( b \) and the part \( e \cdots e' \) of the (irreducible) path \( W_1 \), we can apply lemma 3. We can also say that \( a_1, a_2 \) are not neighbours of \( b \). It follows that \( G \) contains a graph \( IV_2 \). Hence, \( G \) is actually of the form \( IV_2 \).

Case 2.2. \( b \) has only one neighbour \( e \not\in a_1 \). Then \( e \not\in W_1 \). Also, necessarily, \( e \not\in W_2, e \in W_3 \). Then we do not have \( a_2 \not\in e \) or \( a_1 \not\in e \). It follows that \( G \) is of the form I.

Case 2.3. \( b \) has exactly one neighbour \( c_1 \not\in W_1 \) and at least one neighbour \( d_i \not\in W_1 \). We write \( W_1 = c_{n-1} \cdots c_{1}, c_1, \ldots, c_{n-2} = a_1, a_2, a_3 \). We suppose that \( d_i \not\in W_1 \).

We have the form \( W_3 = a_2 d_i \cdots c_{p-1}, c_{p} \) where the point preceding \( c_{p} \) is the last point of \( W_3 \) not belonging to \( W_1 \). Then \( p > 0 \) as \( W_3 = a_2 d_i \ldots \) is irreducible and so \( a_1 \not\in W_3 \) (see fig. 8).

Next, we show that \( W_3 \) does not contain a point \( d \) with \( d \not\in c_{i} \) for some \( i > 0 \) (then \( W_3 \) does not contain a point \( c_{i} \), \( i > 0 \), either). Suppose that there was such an index \( i \). Then \( d \not\in b \). Then, replacing \( W_1, W_2 \) successively by

\[
W_1 = c_{n-1} \cdots c_{p-1}, a_1, a_2, \ldots, a_i \quad \text{and} \quad W_2 = a_2 d_i \cdots c_{p}, c_{i} \quad \text{(which do not contain \( c_{i} \)),}
\]

we see that \( (a_1, a_2, a_3) \) would be an asteroidal triple of \( G \setminus \{c_{i}\} \). This contradicts the requirement (3).

Having reached this point, let us consider the case that the part \( c_{1} \cdots q_{a} \) of \( W_1 \) has no neighbour \( e \not\in W_1 \). If \( l \geq 2 \), then we have case 2.2, with \( a_2 \) replaced by \( a_1 \). If \( l = 1 \), then we apply lemma 1 to the cycle \( a_1 d_i \cdots c_{p-1}, c_{p} \). Since not \( b \not\in c_{p} \) or \( b \not\in c_{p-1}, c_{p} \not\in c_{p-1} \) (\( i > 1 \)) and \( c_{p} \) has no neighbour \( b \), \( b \not\in \), we find that \( b \) has at least two neighbours \( c_{p} \) and \( d_i \) on \( W_2 \). Then we have case 2.1, with \( a_1 \) replaced by \( a_2 \).

Consequently, we may suppose that some \( c_{r}, q > 0 \), has a neighbour on \( W_1 \). Then \( \not\in W_1 \). Hence \( \not\in W_2 \). Then \( c_{r} \not\in W_2 \). Otherwise, \( G \) would be asteroidal. Then \( W_2 \) has the form \( W_2 = a_2 d_i c_{p} \cdots c_{r} \) and it does contain neither a point \( c_{r} \) with \( i > 0 \), nor a neighbour of such a point. Further, we do not have \( d' = c_{r} \) or \( d' \not\in \) if \( d' \not\in W_2 \). Hence \( W_2 \) is not asteroidal, and \( \not\in a_1 \) (in the contrary case, \( G \) would be asteroidal).

We now present the following lemma 1 to the cycle \( a_2 d_i \cdots c_{p-1}, c_{p} \). The consequence is that \( c_{r} \not\in d_i \) and \( c_{r} \not\in \), whence, on account of (3), \( G \) must be of the form \( \Pi \).

This proves the assertion and completes the proof of theorem 4.

A simple consequence of theorem 4 is the following.

**Corollary.** An acyclic graph with not more than five points is always representable.

7. Numerical devices. In our final section we shall deal with a practical method by which we can decide whether a given graph \( G \) is representable. This method will be based on theorem 3. The treatment naturally splits up into two parts: we have to decide whether or not there are irreducible cycles in \( G \) and whether or not there are asteroidal triples in \( G \).

**A. Examination of cycles.** We begin with a definition and a theorem.

**Definition 6.** Let \( G \) be a graph and let \( a \in \mathcal{G} \) be arbitrary. Let \( C_1, C_2, \ldots, C_k \) \( (k \equiv k(a) \equiv 1) \) be the components of \( G \setminus \{a\} \). Then, for each \( C_i \), we denote by \( S_i(a) \) the graph of points \( b \) with

\[
b \not\in a, \quad b \not\in S_i(a), \quad b \not\in C_i,
\]

and call \( S_i(a) \) a substar of \( S(a) \).

**Theorem 5.** A graph \( G \) is acyclic if and only if for each point \( a \in G \) all substars \( S_i(a) \) are simplices.

**Proof.** First suppose that \( G \) is acyclic. Take any substar \( S_i(a) \), and let \( b, b_1 \) be two distinct points of \( S_i(a) \). Then there are points \( a, b \) in \( C_i \) with \( a, b \) and \( b_1 \) are connected in \( G \). Further, there is a path \( C_i \) connecting \( a \) and \( b_1 \). Then, by the definition of \( S_i(a) \), we do not have \( a \not\in W \). Applying lemma 1 to the cycle \( a b_1 \cdots a_1 a \), we find that \( b_1 \not\in b \). It follows that \( S_i(a) \) is a simplex.

Conversely, suppose that there is an irreducible cycle \( c_{r} \cdots c_{a} \) \( (k > 1) \) in \( G \). Put \( a \not\in c_{r} \) and let \( C_i \) be the component of \( G \) containing the point \( a \). Then \( S_i(a) \) contains \( c_{r} \) and \( a \), and it is a simplex.

Below we shall apply the following slightly different and less elegant proposition, the proof of which offers no difficulties.

**Proposition.** A graph \( G \) is acyclic if, for some point \( a \in G \), the substars \( S_i(a) \) are simplices and the graph \( G \setminus \{a\} \) is acyclic.
In order to find out whether \( G \) is acyclic, one could now proceed along the following lines.

a) Choose arbitrarily \( a \in G \) and determine the neighbours of \( a \).

b) Determine the components \( C_1, \ldots, C_n \) in the following way. Take any point \( a \in G \cap S(a) \). Determine the neighbours of \( a \) in \( G \cap S(a) \), say \( a_1, \ldots, a_k \). Then take the neighbours of \( a \) in \( G \setminus S(a) \) which do not belong to the set \( \{a_1, \ldots, a_k\} \), say \( a_{k+1}, \ldots, a_{k+\ell} \). Then take \( a \) and repeat the process until no new points are found. Then one component \( C_1 \) has been found. If \( G \setminus S(a) \) contains a point \( a \in C_1 \), then determine in the same way a second component \( C_2 \) of \( G \setminus S(a) \) containing \( a \). Repeat this process until \( G \setminus S(a) \) is exhausted.

c) For each component \( C_i \) determine the star \( S(a) \) by taking the points \( b \in S(a) \) which have at least one neighbour in \( C_i \).

d) Check whether \( S(a) \) is a simplex.

e) Omit \( a \) and examine in the same way \( G \setminus S(a) \), etc.

Let \( G \) have \( n \), \( m \), \( n \) points respectively \( (i = 1, \ldots, k) \). Then the points a)-d) require at most \( n \cdot \sum (n-m), \sum (m-1)m, \frac{1}{2} m^3 \) operations successively. The sum of these numbers is

\[
\sum (n-m)^3 + m(n-m) + \frac{1}{2} m^3(n-m) \leq \frac{1}{2} n^4 + \frac{1}{2} n^3 + O(n),
\]

the expression on the left attaining its maximum for \( m \approx \frac{1}{2} n - 1 \). So the examination requires in the aggregate not more than about \( \frac{1}{2} n^4 + 10 n^3 \) operations.

B. Examination of triples. First, we prove

**Theorem 6.** If \( G \) is acyclic and a asteroidal, then it contains an asteroidal triple of simplicial points.

**Proof.** Let \( (a_1, a_2, a_3) \) be an asteroidal triple and let \( W_2 \) be a path in \( G \cap S(a_1) \) connecting \( a_1 \) and \( a_2 \). We shall apply theorem 2, with \( H = W_2 \). We have \( a_3 \in S(H) \), so that \( G \cap S(H) \) is not empty. Let \( C \) be the component of \( G \cap S(H) \) which contains the point \( a_3 \).

By theorem 2, \( C \) contains a point \( a_4 \) which is a simplicial point of \( G \setminus S(H) \). It is also a simplicial point of \( G \). (Confer the end of the proof of theorem 2). Further, \( a_4 \in S(H) \). (Confer the end of the proof of theorem 2). Hence \( (a_1, a_2, a_3, a_4) \) is an asteroidal triple in \( G \).

Repeating this procedure two times, we get an asteroidal triple \( (a_1, a_2, a_3, a_4) \) of simplicial points.

Let now \( G \) be an acyclic graph. Let \( \Sigma \) be the set of its simplicial points. They form a certain number of non-neighbouring simplicial points (i.e. no simplex contains a neighbour of another simplex). From each simplex we select arbitrarily one point. Let \( \Sigma \) be the set of selected simplicial points.

Our method for deciding whether \( G \) is asteroidal or not now consists of the following stages.

a) For each point \( a \in G \) examine whether \( S(a) \) is a simplex.

b) Determine the simplices of which the set of simplicial points consists, in the following way. Take any simplicial point \( a \) and determine its neighbours, say \( s_1, \ldots, s_k \), among the other simplicial points. Repeat this process with a simplicial point \( s \neq s_1, \ldots, s_k \). Then select a simplicial point in each component thus found; this gives a set \( \Sigma \).

c) Construct a matrix \( i(a, b) \) \( (a, b \in \Sigma) \) \( i(a, b) \) a suitable positive integer) as follows. Take \( a \in \Sigma \) arbitrarily. Determine the components \( C_1, \ldots, C_\ell \) of \( G \cap S(a) \) as in a, b). For each component \( C_i \) put \( i(a, b) = i \) for all \( b \in C_i \).\( \Sigma \).

d) Check whether for each triple \( (a, b, c) \) in \( \Sigma \) the following equations hold true:

\[
i(a, b) = i(a, c), \quad i(b, c) = i(b, a), \quad i(c, a) = i(c, b)
\]

(the graph \( G \) is asteroidal if and only if there is a triple \( (a, b, c) \) in \( \Sigma \), such that the above equations hold).

Let \( G \) and \( \Sigma \) consist of \( n \) and \( s \) points respectively. Then the total number of operations needed for the steps a)-d) is

\[
n \cdot \frac{1}{2} n^3 + \frac{1}{2} n^3 + (n^3 + 3n^3 + 3n^3 + \frac{1}{2} n^3) = O(n^3).
\]

Finally, we make the following remarks. In part B the restriction to the set \( \Sigma \) with the proviso that we know already that \( G \) is acyclic—enables us to suppress the dimensions of the matrix \( i(a, b) \) to be constructed. Further, in part A we can begin by omitting the points of \( \Sigma \) which have to be determined in B); for \( G \) is acyclic if \( G \setminus \Sigma \) is acyclic. Then the various stages have to be performed in the following order: B, A; B, A, b)-d). Then, apart from a term of order \( O(n^3) \), the total number of operations needed can be estimated by

\[
n \cdot \frac{1}{2} n^3 + \frac{1}{2} n^3 + n^3 + \frac{1}{2} n^3 = \frac{1}{2} n^3 + \frac{1}{2} n^3 + n^3 + \frac{1}{2} n^3,
\]

which is not larger than \( \frac{1}{2} n^3 + \frac{1}{2} n^3 \) (note that the derivative with respect to \( n \) is negative if \( n \geq 4n^3 \)).

We have the impression that in general the method exhibited here cannot be improved essentially.

(*) The numbering of the components \( C_i \) is immaterial.
Sur l’enfilage et la fixation des ensembles compacts
par
D. Zaremba (Wrocław)

§ 1. Relations générales. Étant un espace métrique, un
ensemble $X \subset E$ sera dit fixable dans $E$ ([1]) lorsque, pour tout $\varepsilon > 0$, il
existe dans $E$ une somme finie $F_1 = F_1 \cup F_2 \cup \ldots \cup F_{n0}$ d’ensembles
fermés tels que $\delta(F_i) < \varepsilon$ pour $i = 1, 2, \ldots, k(\varepsilon)$, $F_i \cap F_j = \emptyset$ pour $i \neq j$
et $F_i \cap C = \emptyset$ pour toute composante $C$ de $X$.

De plus, si pour $a_k \downarrow 0$, c’est-à-dire pour toute suite $(a_k)$ décroissante
et convergente vers 0, les $F_k$ qui fixent $X$ peuvent être choisis de manière qu’ils
forment une suite descendante, j’appelle la fixation de $X$ monotone.

Knaster appelle un ensemble $X \subset E$ enfilable dans $E$ lorsque $E$
contient un arc $L$ tel que $L \cap C = \emptyset$ pour toute composante $C$ de $X$.

J’appelle réduit de $X$ tout ensemble $B \subset X$ tel que $B \cap C = \emptyset$ pour
toute composante $C$ de $X$. En outre, j’appelle l’adduit de $X$ l’ensemble $A$
de tous les points $p$ de $E$ tels que $(p) = \lim_{\varepsilon \to 0} C_\varepsilon$ pour une suite $(C_\varepsilon)$ de
composantes de $X$. Ainsi défini, $A$ est donc l’ensemble de tous les points
de $E$ qui sont des points-limites des suites de points appartenant à des
composantes $C_\varepsilon$ de $X$ tels que $\delta(C_\varepsilon)$ tend à 0. On voit aussitôt qu’un
adduit est toujours fermé, donc compact, pour des $X$ compacts.

Ces énoncés définissent constamment l’espace euclidien de dimension $n \geq 1$.

Théorème 1. Les trois propriétés suivantes sont équivalentes pour les $X$
compacts dans $C^n$:

1) l’existence d’une fixation monotone de $X$,
2) l’existence dans $X$ d’un réduit $R$ compact de dimension 0,
3) l’existence d’un enfilage de $X$.

La démonstration de ce théorème se trouve dans mon travail [2],

Théorème 2. Si un $X$ (compact ou non) est fixable dans $C^n$, son
adduit $A$ est vide ou de dimension 0.

(1) cf. Knaster [3], où l’on trouve une définition équivalente de cette notion
par des ensembles ouverts.

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