Since $A$ can be embedded into a square which is weakly chainable by Corollary 2, we see that, unlike chainable continua, a subcontinuum of a weakly chainable continuum needs not be weakly chainable.

References


In the present note we construct a family consisting of $2^n$ two-dimensional AR-sets (compact) such that none of them contains a 2-dimensional closed subset homeomorphic to a subset of any other set. We also give some applications of this family to the problem of existence of universal $n$-dimensional AR-sets and to the theory of $r$-neighbours.

1. Zone of a triangulation. Let $\triangle$ be a triangle lying in the Euclidean 3-space $E^3$ and let $b_\triangle$ denote the barycentre of $\triangle$. For every positive $\varepsilon$, let us denote by $L(\triangle, \varepsilon)$ the segment perpendicular to the plane of the triangle $\triangle$ with length $2\varepsilon$ and centre $b_\triangle$. By the $\varepsilon$-zone of the triangle $\triangle$ we understand the minimal convex subset of $E^3$ containing the sets $\triangle$ and $L(\triangle, \varepsilon)$. It will be denoted by $Z(\triangle, \varepsilon)$. Evidently $Z(\triangle, \varepsilon)$ is the union of two 3-dimensional simplexes having $\triangle$ as their common base and the endpoints of the segment $L(\triangle, \varepsilon)$ — as opposite vertices. The polytope $Z(\triangle, \varepsilon)$ is a neighbourhood of every point lying in the interior of the triangle $\triangle$. The segment $L(\triangle, \varepsilon)$ is said to be the axis of the zone $Z(\triangle, \varepsilon)$.

Now let $T$ be a triangulation of a polytope $P$. The union of all $m$-dimensional simplexes of $T$ is said to be the $m$-skeleton of $T$. Evidently the polytope $P$ is homogeneously $n$-dimensional if and only if it coincides with the $n$-skeleton of $T$. In this case we understand by the boundary of $P$ the union $P^*$ of all $(n-1)$-dimensional simplexes of $T$ incident exactly to one $n$-dimensional simplex of $T$, and by the edge of $P$ the set $P^*$ of all points $x \in P$ such that no neighbourhood of $x$ in $P$ is homeomorphic to a subset of the Euclidean $n$-space $E^n$. Evidently $P^*$ and $P^*$ are unions of some simplexes of the triangulation $T$, but they do not depend on the choice of this triangulation.

Now let us consider a homogeneously 2-dimensional polytope $P \subset E^3$ with a triangulation $T$ and let $\varepsilon$ be a positive number. One easily sees that for $\varepsilon$ sufficiently small the common part of the zones of different triangles of the triangulation $T$ coincides with the common part of the boundaries of those triangles. A positive number $\varepsilon$ satisfying this condition is said to be suitable for the triangulation $T$.
Let $\varepsilon$ be a suitable number for the triangulation $T$. By the $\varepsilon$-zone of the triangulation $T$ we shall understand the polytope

$$Z(T, \varepsilon) = \bigcup_{\alpha \in T} Z(\alpha, \varepsilon).$$

Evidently the polytope $P$ is a deformation retract of the zone $Z(T, \varepsilon)$. It follows, in particular, that $P$ is an AR-set if and only if $Z(T, \varepsilon)$ is an AR-set.

2. Some geometrical constructions. Let $P$ be a polytope in $E^p$ which is disk (i.e. a set homeomorphic to a triangle) and let $T$ be a triangulation of $P$ with the maximal diameter of simplices $\leq 1$. Let us set:

$$P_T = P, \quad T_T = T,$$

$\varepsilon_T$ a positive number suitable for the triangulation $T_T$.

Given a sequence $\{a_n\}$ of natural numbers $\geq 1$, let us assume that for a natural $k$ a polytope $P_k$, its triangulation $T_k$ and a positive number $\varepsilon_k$ are defined, satisfying the following conditions:

1. $P_k$ is a homogeneously 2-dimensional polytope in $E^p$ which is an AR-set with the boundary $P_k = P_k'$.

2. The edge $P_k'$ of $P_k$ is a subset of $P_k - P_k'$ and its components are rectilinear segments.

3. $T_k$ is a triangulation of $P_k$ with the maximal diameter of simplices $\leq 1/k$.

4. For every point $x \in P_k - P_k'$ the union of all triangles of $T_k$ containing $x$ is a disk.

5. $\varepsilon_k$ is suitable for the triangulation $T_k$ and $\varepsilon_k \leq 1/k$.

It follows by (5) that $P_k$ is a deformation retract of the $\varepsilon_k$-zone $Z(T_k, \varepsilon_k)$ and consequently $Z(T_k, \varepsilon_k)$ is an AR-set. Moreover, $Z(T_k, \varepsilon_k)$ is a neighbourhood (in $E^p$) of all barycentres $b_k$ of triangles $\alpha \in T_k$. We infer that there exists a positive number $\varepsilon$ such that every point $x \in E^p$ lying at a distance $< \varepsilon$ from the barycentre $b_k$ of a triangle $\alpha \in T_k$ belongs to $Z(T_k, \varepsilon)$.

Now let us consider, for each triangle $\alpha \in T_k$, a system consisting of $3a_k$ triangles $\alpha_i, \ldots, \alpha_{3a_k}$ lying in the interior of the triangle $\alpha$ and such that $\alpha_i$ is their common vertex and that $\alpha_i \cap \alpha_j = (b_k)$ for $i \neq j$. Let $\alpha_{3a_k}$ be one of the endpoints of the axis $L(\alpha, \varepsilon)$ of the zone $Z(\alpha, \varepsilon)$. Consider the system of $3a_k$ triangles $\alpha_i, \ldots, \alpha_{3a_k}$ for which $\alpha_i$ is one of the vertices and the other two are vertices of one of the triangles $\alpha_1, \ldots, \alpha_{3a_k}$. One easily sees that the polytope

$$R_k = (\alpha - \bigcup_{i=1}^{3a_k} \alpha_i) \cup \bigcup_{j=1}^{3a_k} \alpha_j$$

is a homogeneously 2-dimensional polytope which is a deformation retract of the zone $Z(\alpha, \varepsilon_k)$. Setting

$$P_{k+1} = \bigcup_{\alpha \in T_k} R_k,$$

we get a homogeneously 2-dimensional polytope which is a deformation retract of the zone $Z(T_k, \varepsilon_k)$. Since $P_k$ is an AR-set, we infer that $P_{k+1}$ is also an AR-set. Moreover, it is easy to see that $P_{k+1}$ contains the 1-dimensional skeleton of $T_k$, the boundary $P_{k+1}'$, coincides with the boundary $P_k' = P_k'$, and the edge $P_{k+1}'$ coincides with the union of the edge $P_k'$ and of all segments $b_k \alpha_i$, with $\alpha_i \in T_k$. Since these segments are disjoint from one another, and also disjoint from $P_k'$, we infer that the polytope $P_{k+1}$ satisfies the conditions (1) and (2) which we get from conditions (1a) and (2a) replacing $k$ by $k+1$. Moreover, one easily sees that every triangulation $T_{k+1}$ of $P_{k+1}$ with diameters of simplices sufficiently small, satisfies conditions (3) and (4). We can find this triangulation in such a manner that every 1-dimensional simplex belonging to $T_k$ is the union of some simplices of $T_{k+1}$. Consequently the 1-skeleton of $T_k$ is a subset of the 1-skeleton of $T_{k+1}$. Evidently the zone $Z(T_k, \varepsilon_k)$ is a neighbourhood for all barycentres $b_k$ of triangles $\alpha \in T_{k+1}$. It follows that, if we fix the triangulation $T_{k+1}$, we can find a positive number $\varepsilon_{k+1}$ such that condition (5) is satisfied and such that

$$Z(T_{k+1}, \varepsilon_{k+1}) \neq Z(T_k, \varepsilon_k)$$

for $k = 1, 2, \ldots$ i.e. the sequence of polytopes $\{Z(T_k, \varepsilon_k)\}$ is decreasing.

Moreover, our construction implies that for every $m = 1, 2, \ldots$ the components of the edge $P_k'$ coincide with the segments $a_k b_k$, where $\alpha$ is a triangle of a triangulation $T_k$ with $k < m$. Since $x_k b_k$ is the common part of exactly $2a_k$ triangulations among the triangles $\alpha$, $a_k b_k$, we shall say that it is a segment of ramification of order $2a_k$.

Finally let us observe that if $\alpha$ is a triangle of $T_k$ and if $\hat{\alpha}$ denotes the system of all simplices of $T_{k+1}$ lying in $Z(\alpha, \varepsilon_k)$, then

$$Z(\hat{\alpha}, \varepsilon_{k+1}) = Z(\alpha, \varepsilon_k)$$

for every triangle $\alpha \in T_k$ the boundary $\hat{\alpha}'$ of $\hat{\alpha}$ is a retract of $Z(\hat{\alpha}, \varepsilon_{k+1}) = Z(\alpha, \varepsilon_k)$.
3. Membranes. Every space $X$ homeomorphic to the set

$$P(\{m\}) = \bigcap_{k=1}^{\infty} Z(T_k, e_k)$$

will be said to be a membrane on $P$ corresponding to the sequence $\{m\}$. Since $Z(T_k, e_k) = \bigcup Z(\Delta, e_k)$ and the diameters of the zones $Z(\Delta, e_k)$ are $\leq 2/k$, and since for different triangles $\Delta_1, \Delta_2 \in T_2$

$$Z(\Delta_1, e_k) \cap Z(\Delta_2, e_k) \subset \Delta_1 \cap \Delta_2,$$

we infer that

$$Z(\Delta_1, e_k) \cap Z(\Delta_2, e_k) \subset \Delta_1 \cap \Delta_2,$$

we infer that

(10) Every membrane is a compactum of dimension $\leq 2$.

It follows by the construction given in No. 2, that the simplexes $\Delta$ of the triangulation $T_{k+1}$ contained in $Z(\Delta, e_k)$ constitute a triangulation $T_{k+1}$ of the polytope $R_3$ which is an AR-set. Since $R_3$ is a deformation retract of $Z(T_{k+1}, e_{k+1})$, we infer that the polytope $Z(T_{k+1}, e_{k+1})$ is an AR-set. Consequently there exists a retract $r_2$ of $Z(\Delta, e_k)$ to $Z(T_{k+1}, e_{k+1})$. By virtue of the inclusion $\Delta \subset R_3 \subset Z(T_{k+1}, e_{k+1})$, we infer that

(11) $r_2(x) = e$ for every $x \in \Delta$. Setting

$$r_2(x) = r_2(\alpha) \text{ for every } x \in Z(\Delta, e_k) \text{ where } \Delta \in T_k,$$

we infer by (10) and (11) that $r_2$ is a retract of $Z(T_k, e_k)$ to $Z(T_{k+1}, e_{k+1})$ satisfying the condition

(12) $r_2[Z(\Delta, e_k)] = Z(T_{k+1}, e_{k+1})$ for every triangle $\Delta \in T_k$.

Now let us set

(13) $r_2(x) = r_{2+1} \cdots r_{2+k}(x)$ for every $x \in Z(T_k, e_k)$.

Applying (13), we easily see that $r_2$ is a retract of the zone $Z(T_k, e_k)$ to the zone $Z(T_{k+1}, e_{k+1})$. Moreover, we infer by (13), (14) and (6) that, if $x \in Z(T_k, e_k)$ and if $\Delta$ is a triangle of the triangulation $T_{k+1}$ such that $r_2(x) \in Z(\Delta, e_{k+1})$, then each of the points $r_{k+1}(x) = r_{2+1}r_2(x), \ldots, r_{k+1}(x) = r_{2+k}(x)$ belongs to $Z(\Delta, e_{k+1})$.

Since the diameter of the zone $Z(\Delta, e_{k+1})$ is $\leq 2/(k+1)$, we conclude that $d(r_2(x), r_{k+1}(x)) \leq 2/(k+1)$ for every point $x \in Z(T_k, e_k)$ and every $k = 1, 2, \ldots$. It follows that the sequence $(r_2(x))$ converges uniformly to a map $r$ of $Z(T_k, e_k)$ into $P(m)$. Since for every point $x \in P(m)$ we have $x \in Z(T_{k+1}, e_{k+1})$ and consequently $r_2(x) = x$ for every $k = 1, 2, \ldots$, we infer that $r$ is a retract of $Z(T_k, e_k)$ to $P(m)$.

Thus we have shown that

(15) Every membrane is an AR-set.

The construction of the triangulation $T_{k+1}$ implies that the 1-skeleton of $T_k$ is included in the $1$-skeleton of $T_{k+1}$. We infer by (9) that $P(sk)$ contains the 1-skeleton of $T_2$ for every $k = 1, 2, \ldots$. In particular, the boundaries $P_k$ of all polytopes $P_k$ and also their edges $P'_k$ are all included in $P(m)$. In particular, the boundary of the disk $P = P$ is a subset of $P(m)$. By virtue of (11) and (15), we infer that

(16) Every membrane is a 2-dimensional AR-set.

If $h$ maps $P(m)$ homeomorphically onto a membrane $X$, then the simple closed curve $h(P) \subset X$ will be said to be the boundary of the membrane $X$ and will be denoted by $X$.

As we have seen at the end of No. 2, the components of the edge $P'_k$ coincide with the segments $\Sigma e_\Delta$ of ramification for triangles $\Delta \in T_k$ with $k \leq m$. Since $P'_k \subset P(m)$, the homeomorphism $h$ maps $\Sigma e_\Delta$ into a simple arc lying in $X$. This arc $h(\Sigma e_\Delta)$ will be said to be an arc of ramification of order $2n_2$.

Since, for every triangle $\Delta \in T_k$, the segment $\Sigma e_\Delta$ is a subset of the order $2n_2$.

Let $X$ be a membrane corresponding to a sequence $\{m\}$, and let $\{m\}$ be a subsequence of $\{m\}$. Then the union of all arcs of ramification of orders $n_2$ is dense in $X$.

Let us point out, however, that the definition of the boundary and of the area of ramification of the membrane $X = h(P(m))$ given here depends on the geometrical construction of $P(m)$ and also on the choice of the homeomorphism $h$.

4. Bits of a membrane. By a bit of a membrane $X = P(m)$ we understand a membrane $Y$ (corresponding to an arbitrary sequence $\{m\}$ of naturals $\geq 2$) such that $Y \subset X$ and that $Y \sim X$. In particular, if $Q$ is a disk which is the union of some triangles of the triangulation $Q_0$ of $P_0$ and if $T$ denotes the triangulation of $Q$ included in $T_0$, then by the same construction as in No. 2, but applied only to simplexes of the triangulations $T_0, \ldots, T_k$, lying in $Z(T, e_0), \ldots, Z(T, e_k)$, we get the set

$$X_0 = P(\{m\}) \subset Z(T, e_0),$$

which is a membrane on $Q$ corresponding to the sequence $\{m_0, \ldots, m_d\}$. Evidently the boundary $X_0$ of $X_0$ coincides with $Q'$ and we have

$$X_0 \sim X - X_0 \subset Q'.

Thus $X_0$ is a bit of $X$. In particular, if $Q = \Delta$ is a triangle of the triangulation $T_0$, then $X_0 = X_0 \subset Z(\Delta, e_0)$ and, since the diameter of $Z(\Delta, e_0)$ is $\leq 2/m_0$, we have

(18) $d(X_0) \leq 2/m_0$ for every $\Delta \in T_0$. 
Moreover,
\[ X = \bigcup_{m \geq 1} \{X_m\} \text{ for every } m = 1, 2, \ldots \]
(19)

Using the notion of bit, let us prove the following

Lemma 1. If \( Y \) is a closed proper subset of a membrane \( X \), then the boundary \( \partial X \) of \( X \) is a retract of the set \( X \setminus Y \).

Proof. We can assume that \( X = P\left(\{x_k\}\right) \) and that \( X^* = P^* \). By virtue of (18) and (19), there exists in the triangulation \( T_m \), for \( m \) sufficiently large, a triangle \( \Delta \) such that the bit \( X_\Delta \) lies in \( X \setminus Y \). In order to prove our lemma, it suffices to show that \( X_\Delta \) is a retract of the set \( (X_\Delta - X_\Delta) \cup \Delta' \). We shall do it by induction.

If \( m = 1 \), then \( \Delta \) is one of the triangles of the triangulation \( T_1 \) of the disk \( D_1 \), and we see at once that the set \( (X - X_\Delta) \) is a subset of the union \( \bigcup_{\Delta \in T_1} X_\Delta \) with \( \Delta' \in T_1 - \{\Delta\} \) and that \( P^* \) is a retract of \( W \cap \Delta' \). Now we assume that the statement holds for some \( m \geq 1 \) and that \( \Delta \in T_m \). Evidently there exists a retraction \( r_m' \) of the set \( (X_\Delta - X_\Delta) \cup \Delta' \) to \( P^* \). Moreover, we infer by (8), No. 2, that, if \( \hat{P} \) denotes the system of all triangles of \( T_{m+1} \) lying in \( (X_\Delta - X_\Delta) \cup \Delta' \), then \( \Delta'' \) is a retract of the set \( \hat{P} \).

Since \( (X_\Delta - X_\Delta) \cup \Delta' \) contains \( \Delta'' \) and it is a subset of \( \hat{P} \), we infer that there exists a retraction \( r'' \) of the set \( (X_\Delta - X_\Delta) \cup \Delta' \) to \( \Delta'' \).

Setting \( r'''(x) = \frac{r''(x)}{x} \) for \( x \in (X_\Delta - X_\Delta) \cup \Delta' \),

we get a retraction \( r''' \) of the set \( (X - X_\Delta) \cup \Delta' \) to the set \( (X - X_\Delta) \cup \Delta'' \).

Setting \( r(x) = x \) for every \( x \in (X - X_\Delta) \cup \Delta' \),

we get the demanded retraction of the set \( (X - X_\Delta) \cup \Delta' \) to \( P^* \).

Lemma 2. A closed subset \( Y \) of a membrane \( X \) is 2-dimensional if and only if \( X \) contains at least one bit of \( Y \).

Proof. If \( Y \) contains a bit \( X_\Delta \) of \( X \) then by (16) we have \( 2 = \dim X_\Delta \leq \dim Y \leq \dim X = 2 \),

and consequently \( \dim Y = 2 \).

Now let us assume that \( Y \) does not contain any bit of the membrane \( X \). We can assume that \( X = P\left(\{x_k\}\right) \). Given a positive \( \varepsilon \), let us consider a natural \( m \) such that \( 2/m < \varepsilon \). Then

\[ X = \bigcup_{\Delta \in T_m} X_\Delta, \]

where the diameter of \( X_\Delta \) is \( < \varepsilon \) for every triangle \( \Delta \in T_m \), and there exists a point \( x \in X_\Delta \). Applying lemma 1 we infer that there exists a retraction \( r_\Delta \) of the set \( (X_\Delta - X_\Delta) \cup \Delta' \) to \( \Delta' \). Setting

\[ f(y) = r_\Delta(y) \text{ for every } y \in Y, \]

we get a continuous map \( f \) of \( Y \) into the 1-skeleton of \( T_m \) and this map satisfies the condition

\[ e(f(y), y) < \varepsilon \text{ for every } y \in Y. \]

But this implies that \( \dim Y < 2 \).

5. Boundary of a Membrane. Let us prove the following

Lemma 3. A point \( x \in X = P\left(\{x_k\}\right) \) belongs to \( X \) if and only if there exists for every \( \varepsilon > 0 \) an open neighborhood \( U \) of \( x \) in \( X \) with diameter \( < \varepsilon \) and such that \( X - U \) is a retract of \( X \).

Proof. If \( x \in X \) then let us consider an index \( k \) such that \( 1/k < \varepsilon/2 \). By (3a), (3b) and (4b), the union of all triangles of the triangulation \( T_k \) containing \( x \) is a disk \( Q \) with diameter \( < \varepsilon \). Evidently

\[ X_\Delta \cap X \setminus X_\Delta \subset X_\Delta = Q, \]

and there exists a simple arc \( L \) such that

\[ X_\Delta \cap X \setminus X_\Delta \subset L \subset X - (x). \]

Setting \( U = X_\Delta \setminus L \), we get an open neighbourhood \( U \) of \( x \) with diameter \( < \varepsilon \). Let \( r_L \) be a retraction of \( X_\Delta \) to \( L \). Setting

\[ r(x) = \begin{cases} r_L(x) & \text{for every } x \in X_\Delta, \\ x & \text{for every } x \in X - X_\Delta, \end{cases} \]

we get a retraction \( r \) of \( X \) to \( X - U \).

On the other hand, if \( x \in X \setminus X_\Delta \) then \( \varepsilon = \varphi(x, X_\Delta) > 0 \) and, if \( U \) is an open neighbourhood of \( x \) with diameter \( < \varepsilon \), then \( X_\Delta \subset X - U \). It follows, by lemma 1, that the simple closed curve \( X_\Delta \) is a retract of \( X - U \). Since \( X \) is an AR-set, we infer that \( X - U \) is not a retract of \( X \).

Let us observe that lemma 3 implies that the notion of the boundary \( Y \) of a membrane is topological.
6. \textit{\(n\)-membranes}. Let \(n\) be a natural number \(\geq 2\). By an \(n\)-membrane we shall understand a set \(Y\) which is the union of \(n\) membranes \(X_1, X_2, \ldots, X_n\) such that there exists an arc \(L\) satisfying the conditions:

\[ X_i \cap X_j = X_i^* \cap X_j^* = L \quad \text{for} \quad i \neq j. \]

The arc \(L\) will be said to be the edge of the \(n\)-membrane \(Y\) and it will be denoted by \(Y^*\). By \(Y^*\) we shall denote the interior of the edge \(Y^*\).

The membranes \(X_1, X_2, \ldots, X_n\) will be called the wings of the \(n\)-membrane \(X\). By the boundary \(Y^*\) of the \(n\)-membrane \(Y\) we understand the union of \(n\) simple arcs \(X_i^* - Y^*\) with common endpoints. It is clear that \(n\)-membrane \(Y\) is an \(\mathcal{A}^n\)-set, but its boundary \(Y^*\) is not an \(\mathcal{A}^n\)-set.

**Lemma 4.** Let \(X_1, \ldots, X_n\) be the wings of an \(n\)-membrane \(Y\) and let \(M\) be a closed subset of \(Y\) such that \(X_i \cap M = \emptyset\) for \(i = 2, \ldots, n\). Then \(Y^*\) is a retract of the set \(M \cup Y\).

**Proof.** By lemma 1, there exists a retraction \(r_i\) of the set \((M \cap X_i) \cup X_i^*\) to \(X_i^*\) for every \(i = 2, \ldots, n\). Moreover, there exists a retraction \(r_1\) of the set \(X_1\) to the simple arc \(X_1 - Y^*\). Setting \(r(x) = \begin{cases} x & \text{for every } x \in Y^*, \\ r_1(x) & \text{for every } x \in X_1 \end{cases}\)

we get a retraction \(r\) of the set \(X_1 \cup Y^*\) to \(Y\). In order to obtain a retraction \(r\) of \(M \cup Y^*\) to \(Y^*\), it suffices to set \(r(x) = \begin{cases} r_1(x) & \text{for } x \in M \cap X_1, \\ r_i(x) & \text{for } x \in M \cap X_i, \quad i = 2, \ldots, n, \\ x & \text{for } x \in Y^*. \end{cases}\)

7. **Topological classification of points of a membrane.** Let \(n\) be a natural number \(\geq 2\). By an \(n\)-membrane \(X\) we shall understand a subset \(Y\) of \(X\) satisfying the following two conditions:

(i) \(Y\) is an \(n\)-membrane.

(ii) \(Y^* \supset Y \cap X - Y\).

Condition (ii) implies that \(Y - Y^*\) is an open subset of \(X\).

Now we consider the following subsets of the membrane \(X\):

- \(X_{11}\) consisting of all points \(x \in X\) such that for every \(\varepsilon > 0\) there exists a \(Y\) of \(X\) with diameter \(\varepsilon\) such that \(x \in \text{neighbourhood of } Y\) and \(x \in Y^*\). The points of \(X_{11}\) will be said to be **frontier points** of \(X\).

- \(X_{111}\) (where \(n > 2\)) consisting of all points \(x \in X - X_{11}\) such that for every \(\varepsilon > 0\) there exists an \(n\)-membrane \(Y\) of \(X\) with diameter \(\varepsilon\) such that \(x \in Y\).

such that \(x \in Y^*\). The points of \(X_{11}\) will be said to be **points of the ramification of order \(n\)** of the membrane \(X\).

\[ X_{1V} = X - X_{1} - X_{11} - \bigcup_{k=2}^{\infty} X_{111}. \]

The points of \(X_{1V}\) will be said to be **singular points** of \(X\).

Since by a homeomorphic map of the membrane \(X\) onto another membrane \(X'\) to every bit (resp. to every \(n\)-bit) \(Y\) of \(X\) corresponds a bit (resp. an \(n\)-bit) \(Y'\) of \(X'\), and to the boundary \(Y^*\) of \(Y\) corresponds the boundary \(Y^*\) of \(Y'\) and to the set \(Y^*\), we infer, by lemma 3 of No. 5, that the sets \(X_1, X_{11}, X_{111}\) and \(X_{1V}\) are topologically invariant. Evidently

\[ X = X_1 \cup X_{11} \cup \bigcup_{k=2}^{\infty} X_{111} \cup X_{1V}, \]

and

\[ \text{the sets } X_1, X_{11}, \bigcup_{k=2}^{\infty} X_{111} \text{ and } X_{1V} \text{ are disjoint}. \]

Let us observe that

\[ X_1 = X^*. \]

In fact, if \(x \in X^*\), where \(X = P([a, b])\), then we infer by \((3a)\) and \((4a)\) that for every \(\varepsilon > 0\) there exists a bit \(Y\) of \(X\) with diameter \(\varepsilon\) such that \(x \in Y^*\) and \(Y\) is a neighbourhood of \(x\). Consequently \(x \in X_1\).

On the other hand, if \(x \in X_1\) then there exists a neighbourhood \(Y\) of \(x\) which is a bit containing \(x\) in its boundary \(Y^*\). Applying lemma 3 of No. 5, we infer that for every \(\varepsilon > 0\) there exists an open neighbourhood \(U \subset Y - X^*\) of \(x\) with diameter \(\varepsilon\) and a retraction \(r\) of \(Y\) to \(Y^* - U\). Setting

\[ r(y) = \begin{cases} r_1(y) & \text{for every } y \in Y, \\ y & \text{for every } y \in X^* - U, \end{cases}\]

we get a retraction of \(X\) to \(X^* - U\). It follows, by lemma 3, that \(x \in X^*\).

According to \((2a)\), there is exactly one of three cases for each point \(x \in X - X_1\). In fact:

1. There exists a natural \(m\) such that \(x\) belongs to the \((1)\)-skeleton of the triangulation \(T_m\) but \(x\) does not belong to either to \(X = P_m\) or to \(P_m\).

2. For every \(m = 1, 2, \ldots\) the point \(x\) belongs to the set \(\bigcup_{\Delta \in T_m} [\Delta(\Delta, \varepsilon_m - \Delta)]\).

3. There exists a natural \(m\) such that \(x \in P_m\).

In case 1, one easily sees, by condition \((4a)\) with sufficiently large \(\varepsilon\), that \(x\) belongs to the interior of arbitrarily small ordinary bits of \(X\), and thus \(x \in X_1\).
In case 2, there exists every \( m = 1, 2, \ldots \), a triangle \( a \in T \) such that \( x \in X \cap Z(a, t_a) = d \). But \( X \cap Z(a, t_a) \) is an ordinary bit of \( X \) containing \( x \) in its interior and its diameter is \( \leq 2/n \). Thus also in this case \( x \in X \).

It follows by (21) that only in case 3 the point \( x \) can belong to the set \( \bigcup_{n=3}^{\infty} X_{3n} \cup X_{3r} \), i.e.

(23) Every point of ramification and every singular point of \( X \) belongs to one of the segments of ramification \( a \partial d \).

Finally, let us observe that if \( x \) belongs to the interior of \( a \partial d \), with \( d \in T \), then there exist \( 2n \) triangles among the triangles \( d_1, \ldots, d_{2n} \) containing \( a \partial d \) on their boundaries. Let us denote them by \( d_1', \ldots, d_{2n}' \). Applying condition (4) we infer that for every \( l = 1, 2, \ldots \) the union of all triangles of the triangulation \( T_{l+1} \) lying on \( d_1' \) (where \( x \) is one of the numbers \( 1, 2, \ldots, 2n \)) and containing the point \( x \) in a disk \( G \), and we see at once that the union of corresponding membranes \( Q_l(n_{l+1}) \) is a 2n-bit of \( X \) with diameter \( \leq 2/(k + l) \) and the point \( x \) belongs to \( Y \).

It follows that no point \( x \) lying in the interior of the segment \( a \partial d \) is singular. Consequently only the endpoints of segments of ramification \( a \partial d \) can be singular.

8. Points of ramification. Now let us prove the following lemma.

**Lemma 5.** Let \( X \) be an \( n \)-bit (\( n > 2 \)) of a membrane \( X \) and let \( x \) be a point of \( Y \). Then \( x \) does not belong to any of the sets \( X_3, X_{3r} \) and \( X_{3r} \) with \( m \neq n \).

**Proof.** Let \( X_1, \ldots, X_n \) be the wings of \( Y \). First let us suppose that \( x \in X \). Then there exists a bit of \( X \) such that

\[
x \in X_3 \quad \text{and} \quad X_3 \subseteq Y.
\]

Since \( X_3 \subseteq Y \), \( X_3 \subseteq \overline{X} \), we infer that the set

\[
X = Y_3 \cup X_3 - (Y_3 \cup Y_1)
\]

is not empty and that it is open in \( Y \). By lemma 1, there exists a retraction \( r \) of the set \( Y \) to \( G \) in \( Y \). Setting

\[
r(x) = \begin{cases} r(x) & \text{for } x \in Y_3 - G, \\ x & \text{for } x \in Y_1 - Y_3.
\end{cases}
\]

we get a retraction \( r \) of the set \( Y \) to \( (Y - Y_3) \cup Y_3 \). But \( Y_3 \) is a neighbourhood of \( x \) in \( X \), and consequently \( Y_1 - (Y - Y_2) \cup Y_1 \neq 0 \) for

\[i = 1, 2, \ldots, n.
\]

It follows by (21) that there exists a retraction \( r \) of the set \( (Y - Y_3) \cup Y_3 \) to \( Y_3 \). Finally, it is evident that there exists a retraction \( r \) of \( Y_3 \) to the simple closed curve \( C = Y_3 \cup Y_3 - Y_3 \). Setting

\[
r(x) = r(x)^2(x) \quad \text{for every } x \in Y_3 \cup Y_3,
\]

we get a retraction of the set \( Y_3 \cup Y_3 \) to the simple closed curve \( C \). But this is impossible, because \( Y_3 \cup Y_3 \) is an \( AR \)-set (since \( Y_3 \), \( Y_3 \), and \( Y_3 \cup Y_3 \) are \( AR \)-sets). Thus the supposition that \( x \in X \) leads to a contradiction.

Suppose now that \( x \in X \) or \( x \in X_{3n} \), with \( 2 < m \neq n \). We can consider both cases simultaneously, setting \( X_{3n} = X \) and supposing that \( x \in X_{3n} \) with \( 2 < m \neq n \). Since the hypotheses concerning \( m \) and \( n \) are symmetric, it suffices to consider the case where

\[2 \neq m < n.
\]

Suppose that \( x \in X_{3n} \cap X_{3n} \). Then there exists a system of \( m \)-membranes \( W_1, \ldots, W_m \subseteq X \) and there exists a simple \( M \) such that

\[
M = W_i - W_j - W_i \cup W_j \quad \text{for } i \neq j,
\]

\( x \) belongs to the interior of \( M \).

\[
W = W_1 - W_2 \cup \ldots \cup W_m \quad \text{is an } m \text{-bit of } X
\]

(if \( m = 1 \), \( W \) is a bit of \( X \)).

\[
W \subseteq Y - Y,
\]

\[
W \subseteq \overline{W} \subseteq \overline{X}.
\]

Since \( W \) is a \( 2 \)-dimensional, there exists a point

\[
a_i \in W_i - W_j - \bigcup_{j=1}^{n} W_j \quad \text{for } i = 2, 3, \ldots, n.
\]

The sets \( Y_i - Y_i \) being disjoint, there exists for every \( i = 2, \ldots, m \) exactly one index \( k_i \) such that \( a_i \in Y_k - Y_i \). It follows by (24) that among the indices \( 1, 2, \ldots, m \) at least two are distinct from all indices \( k_i \); we can assume that

\[i = k_i = 2 \quad \text{for every } i = 2, 3, \ldots, m.
\]

Now let us assign to every \( i = 2, 3, \ldots, m \) an open subset \( G_i \) of \( X \) such that

\[
a_i \in G_i \subseteq W_i - X_k - X_k - Y_k.
\]
If \( m > 2 \), then, applying lemma 4, we infer that there exists a retraction \( r_2 \) of the set \( \text{W} - \bigcup_{i=1}^m G_i \) to the set \( \text{W} \). An analogous statement is true also in the case of \( m = 2 \). If we replace \( \text{W} \) by a simple arc lying in the boundary \( \text{W} \) and containing the set \( \text{W} \), we get a retraction \( r_3 \) of the set \( \text{W} - \bigcup_{i=1}^m G_i \) to the set \( \text{W} \). Furthermore, this is true also in the case of \( m = 2 \). If we replace \( \text{W} \) by a simple arc lying in the boundary \( \text{W} \) and containing the set \( \text{W} \), we get a retraction \( r_3 \) of the set \( \text{W} - G \) to the set \( \text{W} \). Setting

\[
r_3(x) = \begin{cases} 
  r_2(x) & \text{for } x \in \bigcup_{i=1}^m G_i, \\
  x & \text{for } x \in \text{W},
\end{cases}
\]

we get a retraction \( r \) of the set \( Y - \bigcup_{i=1}^m G_i \) to the set \( \text{W} \). Since \( \text{W} \) is a neighbourhood of the point \( x \in Y \), it follows that \( Y - Y - \text{W} \neq 0 \) for every \( j = 1, 2, ..., m \). Since \( Y - \text{W} \cap Y \), we infer by lemma 4 that there exists a retraction \( r_3 \) of the set \( Y - \text{W} \) to \( Y \). Finally, it is evident that there exists a retraction \( r_2 \) of the set \( Y \) to the simple closed curve \( C = Y \cup \partial Y \). Setting

\[
r_2(x) = r_3r_1r_2(x) \quad \text{for every point } \quad x \in Y, \quad Y, \quad Y.
\]

we get, by (28), a retraction \( r \) of the set \( Y, Y, Y \) to \( C \). But this is impossible, because \( Y, Y, Y \) is an AR-set.

Thus the proof of lemma 5 is concluded.

In the case of \( \text{X} = \text{P}(\text{A}) \) we infer by this lemma and by formula (20) that every point \( x \) lying in the interior of a segment of ramification \( \text{A}, \text{B}, \text{C} \), where \( \text{A} \) is a triangle of the triangulation \( T_1 \), belongs to \( \text{X} \). It follows, by (17), that:

\[
(26) \quad \text{For every subsequence } (m_0) \text{ of the sequence } (m_2) \text{ and for every open subset } G \text{ of } \text{X} \text{ the set } G - \bigcup_{i=1}^{m_0} \text{X} \text{ is of power } 2^\mathbb{R}.
\]

On the other hand, only the endpoints of segments of ramification may be singular points or points of ramification of orders which do not belong to the sequence \( (2m_2) \). Consequently (23) implies that:

\[
(27) \quad \text{If } \text{X} \text{ is the set of all natural numbers } > 2 \text{ which do not belong to the sequence } (2m_2) \text{ then the set } \bigcup_{i=1}^{m_0} \text{X} \text{ in at most countable.}
\]

9. Main theorem. The fundamental result of this note is the following

Theorem. There exists a function \( \Phi \) assigning to every real number \( t \) a membrane \( \Phi(t) \subseteq \mathbb{R} \) in such a manner that for \( t \neq t' \) no 2-dimensional closed subset of \( \Phi(t) \) is homeomorphic to any subset of \( \Phi(t') \).

Proof. Let \( w_1, w_2, ..., w_\infty \) be an enumeration of all rational numbers. Let us assign to every real number \( t \) an increasing sequence \( (n(t)) \) consisting of all natural numbers \( n \) such that \( w_n < t \). It is clear that for \( t < t' \) the sequence \( (n(t)) \) contains, besides all numbers \( n(t') \), also all natural numbers \( n \) for which \( t \leq w_n < t' \). Therefore

\[
(28) \quad t < t' \text{ implies that in the sequence } (n(t)) \text{ there exist an infinity of terms which do not belong to } (n(t')).
\]

Now let \( P \) be a triangle in \( \mathbb{R} \) and let us put \( \Phi(t) = \text{P}(\{n(t)\}) \) for every real \( t \).

It remains to show that if there exists a homeomorphism \( \lambda \) mapping a 2-dimensional closed subset \( A \) of \( \Phi(t) \) onto a subset \( B \) of \( \Phi(t') \), then \( t = t' \).

Suppose to the contrary that \( t \neq t' \). Since \( \dim A = \dim B = 2 \) and since the inverse homeomorphism \( \lambda^{-1} \) maps \( B \) onto \( A \), we see that the hypotheses concerning \( t \) and \( t' \) are symmetric, and thus we can suppose that \( t < t' \). It follows by (28) that the sequence \( (n(t)) \) contains a subsequence \( (m(t)) \) consisting of natural numbers which do not belong to \( (n(t')) \).

It follows by (26) that the points of ramification of orders \( 2m_2 \) lying in an arbitrarily given open subset \( \Phi(t) \text{ of the membrane } \Phi(t) \text{ constitutes a subset of } \Phi(t) \text{ of power } 2^\mathbb{R}. \) On the other hand, we infer by (27) that the subset \( \Phi(t) \text{ of all points of ramification of orders } 2m_2 \) of, and of singular points is at most countable. By lemma 2, the set \( A \) contains a bit \( X_0 \) of \( \Phi(t) \). Since \( X_0 - X_0 \) is open in \( \Phi(t) \), we infer from (27) that \( x_0 - X_0 \) contains a dense subset \( E \) consisting of points of ramification of orders \( 2m_2 \) of the membrane \( \Phi(t) \) such that any one of \( \lambda(x) \) is neither a singular point nor a point of ramification of order \( 2m_2 \) of the membrane \( \Phi(t') \). If \( \lambda(x) \subseteq \Phi(t') \subseteq \Phi(t) \), then \( \lambda(X_0) = \lambda(E) \subseteq \Phi(t') \subseteq \Phi(t) \). But this is impossible because \( \lambda(x) \) contains, by lemma 2 and formula (20), a bit of \( \Phi(t) \). Thus we see that there exists a point \( a \in R \text{ such that } \lambda(a) \text{ belongs to the interior of } \lambda(X_0). \) Consequently \( \lambda \) maps every neighbourhood of the point \( a \) in the set \( \Phi(t) \) onto a neighbourhood of the point \( \lambda(a) \) in the space \( \Phi(t) \). However, this is also impossible, by lemma 5 and formula (20), because \( a \) is a point of ramification of order \( 2m_2 \) of the membrane \( \Phi(t) \), and \( \lambda(a) \) is neither a singular point nor a point of ramification of the order \( 2m_2 \) of the membrane \( \Phi(t) \).

Thus the proof of our theorem is concluded.


The problem of the existence of a universal \( n \)-dimensional AR-set, i.e. of an \( n \)-dimensional AR-set which topologically contains every
other $n$-dimensional AR-sets, is old. Only in the case of $n=1$ it is solved positively since the 1-dimensional AR-sets coincide with dendrites and it is known (1) that there exists a universal dendrite.

Mr. Sieklucki has recently remarked that the existence of the function $\Phi$ satisfying the theorem of No. 9, allows us to solve the problem of the existence of a universal 2-dimensional AR-set in the negative sense if we recall the following theorem ([2]):

In an $n$-dimensional ANR-set every family of $n$-dimensional subsets which are ANR-sets with the common part of any two of them at most $(n-1)$-dimensional is necessarily at most countable.

In fact, using this theorem, we infer from the theorem of No. 9, that no 2-dimensional ANR-set contains topologically all membranes $\Phi(t)$, because the common part of two sets homeomorphic to $\Phi(t)$ and $\Phi(t')$ with $t \neq t$ is necessarily of dimension $\leq 1$.

11. An application to the theory of $r$-neighbours. A space $X$ is said to be $r$-smaller than a space $Y$ (or $Y$ is $r$-greater than $X$) provided $X$ is homeomorphic to a retract of $Y$ but $Y$ is not homeomorphic to any retract of $X$. If $X$ is $r$-smaller than $Y$ but there exists no space $Z$ which is $r$-smaller than $Y$ and $r$-greater than $X$, then $X$ is said to be an $r$-neighbour of $Y$ on the left (see [3]).

It is clear that every space which is $r$-smaller than an AR-set is also an AR-set. Evidently all membranes are $r$-smaller than the Euclidean cube $Q^2$, because each of them is topologically included in $E^2$, and consequently it is homeomorphic to a subset of $Q^2$ and this subset, being an AR-set, is a retract of $Q^2$.

Now let us assume that $X$ is an $r$-neighbour of $Q^2$ on the left. We can assume that $X \subset Q^2$. Since $X$ is $r$-smaller than $Q^2$, no open subset of $Q^2$ is included in $X$ and consequently $\dim X \leq 2$. As we have already seen, it follows that there exists a real number $r$ such that the membrane $\Phi(t)$ is not included topologically in $X$. Now let us consider a Euclidean ball $B \subset Q^2$ such that:

The interior of $B$ is a subset of $Q^2 - X$.

There exists a point $x_0 \in X$ lying on the boundary of $B$.

Evidently there exists a set $F \subset B$ homeomorphic to the membrane $\Phi(t)$ such that $F \cup X = (x_0)$. We see at once that the set $Y = X \cup F$ is an AR-set which is $r$-smaller than $Q^2$ but $r$-greater than $X$. However, this is incompatible with the supposition that $X$ is an $r$-neighbour of $Q^2$ on the left.

Thus we have shown that the 3-dimensional cube has no $r$-neighbours on the left.