Since A can be embedded into a square which is weakly chainable by Corollary 2, we see that, unlike chainable continua, a subcontinuum of a weakly chainable continuum needs not be weakly chainable.

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS, THE POLISH ACADEMY OF SCIENCES

Reçu par la Rédaction le 23. 12. 1961

On a family of 2-dimensional AR-sets

by

K. Borsuk (Warszawa)

In the present note we construct a family consisting of $2^{\aleph_0}$ two-dimensional AR-sets (compact) such that none of them contains a 2-dimensional closed subset homeomorphic to a subset of any other set. We also give some applications of this family to the problem of existence of universal n-dimensional AR-sets and to the theory of r-neighbours.

1. Zone of a triangulation. Let Δ be a triangle lying in the Euclidean 3-space E^3 and let b_Δ denote the barycentre of Δ. For every positive ε, let us denote by $L(\Delta, \varepsilon)$ the segment perpendicular to the plane of the triangle Δ with length 2ε and centre b_Δ. By the *ε-zone* of the triangle Δ we understand the minimal convex subset of E^3 containing the sets Δ and $L(\Delta, \varepsilon)$. It will be denoted by $Z(\Delta, \varepsilon)$. Evidently $Z(\Delta, \varepsilon)$ is the union of two 3-dimensional simplexes having Δ as their common base and the endpoints of the segment $L(\Delta, \varepsilon)$ — as opposite vertices. The polytope $Z(\Delta, \varepsilon)$ is a neighbourhood of every point lying in the interior of the triangle Δ. The segment $L(\Delta, \varepsilon)$ is said to be the *axis* of the zone $Z(\Delta, \varepsilon)$.

Now let T be a triangulation of a polytope P. The union of all m-dimensional simplexes of T is said to be the *m-skeleton* of T. Evidently the polytope P is homogeneously n-dimensional if and only if it coincides with the n-skeleton of T. In this case we understand by the *boundary* of P the union $P^{\cdot}$ of all $(n-1)$-dimensional simplexes of T incident exactly to one n-dimensional simplex of T, and by the *edge* of P the set P^* of all points $x \in P$ such that no neighbourhood of x in P is homeomorphic to a subset of the Euclidean n-space E^n. Evidently $P^{\cdot}$ and P^* are unions of some simplexes of the triangulation T, but they do not depend on the choice of this triangulation.

Now let us consider a homogeneously 2-dimensional polytope $P \subset E^3$ with a triangulation T and let ε be a positive number. One easily sees that for ε sufficiently small the common part of the zones of different triangles of the triangulation T coincides with the common part of the boundaries of those triangles. A positive number ε satisfying this condition is said to be *suitable* for the triangulation T.

Let ε be a suitable number for the triangulation T. By the *ε-zone* of the triangulation T we shall understand the polytope

$$Z(T, \varepsilon) = \bigcup_{\Delta \in T} Z(\Delta, \varepsilon) .$$

Evidently the polytope P is a deformation retract of the zone $Z(T, \varepsilon)$. It follows, in particular, that P is an AR-set if and only if $Z(T, \varepsilon)$ is an AR-set.

2. Some geometrical constructions. Let P be a polytope in E^3 which is disk (i.e. a set homeomorphic to a triangle) and let T be a triangulation of P with the maximal diameter of simplexes $\leqslant 1$. Let us set:

$$P_1 = P , \quad T_1 = T ,$$
$$\varepsilon_1 = \text{a positive number suitable for the triangulation } T .$$

Given a sequence $\{n_k\}$ of natural numbers > 1, let us assume that for a natural k a polytope P_k, its triangulation T_k and a positive number ε_k are defined, satisfying the following conditions:

(1_k) *P_k is a homogeneously 2-dimensional polytope in E^3 which is an AR-set with the boundary $P_k^{\cdot} = P^{\cdot}$.*

(2_k) *The edge P_k^* of P_k is a subset of $P_k - P_k^{\cdot}$ and its components are rectilinear segments.*

(3_k) *T_k is a triangulation of P_k with the maximal diameter of simplexes $\leqslant 1/k$.*

(4_k) *For every point $x \in P_k - P_k^*$ the union of all triangles of T_k containing x is a disk.*

(5_k) *ε_k is suitable for the triangulation T_k and $\varepsilon_k \leqslant 1/k$.*

It follows by (5_k) that P_k is a deformation retract of the ε_k-zone $Z(T_k, \varepsilon_k)$ and consequently $Z(T_k, \varepsilon_k)$ is an AR-set. Moreover, $Z(T_k, \varepsilon_k)$ is a neighbourhood (in E^3) of all barycenters b_Δ of triangles $\Delta \in T_k$. We infer that there exists a positive number ε such that every point $x \in E^3$ lying at a distance $< \varepsilon$ from the barycentre b_Δ of a triangle $\Delta \in T_k$ belongs to $Z(T_k, \varepsilon_k)$.

Now let us consider, for each triangle $\Delta \in T_k$, a system consisting of n_k triangles $\Delta_1, \ldots, \Delta_{n_k}$ lying in the interioir of the triangle Δ and such that b_Δ is their common vertex and that $\Delta_i \cap \Delta_j = (b_\Delta)$ for $i \neq j$. Let a_Δ be one of the endpoints of the axis $L(\Delta, \varepsilon)$ of the zone $Z(\Delta, \varepsilon)$. Consider the system of $3n_k$ triangles $\Delta_1', \ldots, \Delta_{3n_k}'$ for which a_Δ is one of the vertices and the other two are vertices of one of the triangles $\Delta_1, \ldots, \Delta_{n_k}$. One easily sees that the polytope

$$R_\Delta = (\Delta - \bigcup_{i=1}^{n_k} \Delta_i) \cup (\bigcup_{j=1}^{3n_k} \Delta_j')$$

is a homogeneously 2-dimensional polytope which is a deformation retract of the zone $Z(\Delta, \varepsilon_k)$. Setting

$$P_{k+1} = \bigcup_{\Delta \in T_k} R_\Delta ,$$

we get a homogeneously 2-dimensional polytope which is a deformation retract of the zone $Z(T_k, \varepsilon_k)$. Since P_k is an AR-set, we infer that P_{k+1} is also an AR-set. Moreover, it is easy to see that P_{k+1} contains the 1-dimensional skeleton of T_k, the boundary $P_{k+1}^{\cdot}$ coincides with the boundary $P_k^{\cdot} = P^{\cdot}$, and the edge P_{k+1}^* coincides with the union of the edge P_k^* and of all segments $\overline{a_\Delta b_\Delta}$, with $\Delta \in T_k$. Since these segments are disjoint from one another, and also disjoint from P_k^*, we infer that the polytope P_{k+1} satisfies the conditions (1_{k+1}) and (2_{k+1}) which we get from conditions (1_k) and (2_k) replacing k by $k+1$. Moreover, one easily sees that every triangulation T_{k+1} of P_{k+1} with diameters of simplexes sufficiently small, satisfies conditions (3_{k+1}) and (4_{k+1}). We can find this triangulation in such a manner that every 1-dimensional simplex belonging to T_k is the union of some simplexes of T_{k+1}. Consequently the 1-skeleton of T_k is a subset of the 1-skeleton of T_{k+1}. Evidently the zone $Z(T_k, \varepsilon_k)$ is a neighbourhood for all barycentres b_Δ of triangles $\Delta \in T_{k+1}$. It follows that, if we fix the triangulation T_{k+1}, we can find a positive number ε_{k+1} such that condition (5_{k+1}) is satisfied and such that

(6_k) *If Δ is a triangle of T_k and $\hat{\Delta}$ a triangle of T_{k+1} included in $Z(\Delta, \varepsilon_k)$, then $Z(\hat{\Delta}, \varepsilon_{k+1}) \subsetneqq Z(\Delta, \varepsilon_k)$.*

Thus the sequences $\{P_k\}$, $\{T_k\}$ and $\{\varepsilon_k\}$ satisfying conditions (1_k)-(4_k) are defined. Let us observe that condition (6_k) implies the inclusion

$$(7_k) \qquad Z(T_{k+1}, \varepsilon_{k+1}) \neq Z(T_k, \varepsilon_k) \quad \textit{for} \quad k = 1, 2, \ldots ,$$

i.e. the sequence of polytopes $\{Z(T_k, \varepsilon_k)\}$ is decreasing.

Moreover, our construction implies that, for every $m = 1, 2, \ldots$, the components of the edge P_m^* coincide with the segments $\overline{a_\Delta b_\Delta}$, where Δ is a triangle of a triangulation T_k with $k < m$. Since $\overline{a_\Delta b_\Delta}$ is the common part of exactly $2n_k$ triangles among the triangles $\Delta_1', \ldots, \Delta_{3n_k}'$, we shall say that it is a *segment of ramification of order* $2n_k$.

Finally let us observe that if Δ is a triangle of T_k and if $\hat{T}$ denotes the system of all simplexes of T_{k+1} lying in $Z(\Delta, \varepsilon_k)$, then:

(8) *For every triangle $\hat{\Delta} \in \hat{T}$ the boundary $\Delta^{\cdot}$ of Δ is a retract of $[Z(\hat{T}, \varepsilon_{k+1}) - Z(\hat{\Delta}, \varepsilon_{k+1})] \cup \hat{\Delta}^{\cdot}$.*

In fact, it is evident that there exists in E^3 a straight line L, perpendicuarl to the plane of Δ, which intersects R_Δ at a single point belonging to the interior of $\hat{\Delta}$. Then $\Delta^{\cdot}$ is a retract of $E^3 - L$, and consequently also a retract of the set $(R_\Delta - \hat{\Delta}) \cup \hat{\Delta}^{\cdot} \subset E^3 - L$, and $(R_\Delta - \hat{\Delta}) \cup \hat{\Delta}^{\cdot}$ is a retract of the set $[Z(\hat{T}, \varepsilon_{k+1}) - Z(\hat{\Delta}, \varepsilon_{k+1})] \cup \hat{\Delta}^{\cdot}$.

3. Membranes. Every space X homeomorphic to the set

(9) $$P(\{n_k\}) = \bigcap_{k=1}^{\infty} Z(T_k, \varepsilon_k)$$

will be said to be a *membrane on P corresponding to the sequence $\{n_k\}$*. Since $Z(T_k, \varepsilon_k) = \bigcup_{\Delta \in T_k} Z(\Delta, \varepsilon_k)$ and the diameters of the zones $Z(\Delta, \varepsilon_k)$ are $\leqslant 2/k$, and since for different triangles $\Delta_1, \Delta_2 \in T_k$

(10) $$Z(\Delta_1, \varepsilon_k) \cap Z(\Delta_2, \varepsilon_k) \subset \Delta_1^{\cdot} \cap \Delta_2^{\cdot},$$

we infer that

(11) *Every membrane is a compactum of dimension $\leqslant 2$.*

It follows by the construction given in No. 2, that the simplexes $\hat{\Delta}$ of the triangulation T_{k+1} contained in $Z(\Delta, \varepsilon_k)$ constitute a triangulation $T_{\Delta,k+1}$ of the polytope R_Δ which is an AR-set. Since R_Δ is a deformation retract of $Z(T_{\Delta,k+1}, \varepsilon_{k+1})$, we infer that the polytope $Z(T_{\Delta,k+1}, \varepsilon_{k+1})$ is an AR-set. Consequently there exists a retraction r_Δ of $Z(\Delta, \varepsilon_k)$ to $Z(T_{\Delta,k+1}, \varepsilon_{k+1})$. By virtue of the inclusion $\Delta^{\cdot} \subset R_\Delta \subset Z(T_{\Delta,k+1}, \varepsilon_{k+1})$, we infer that

(12) $$r_\Delta(x) = x \quad \text{for every} \quad x \in \Delta^{\cdot}.$$

Setting

$$r_k(x) = r_\Delta(x) \quad \text{for every} \quad x \in Z(\Delta, \varepsilon_k) \quad \text{where} \quad \Delta \in T_k,$$

we infer by (10) and (12) that r_k is a retraction of $Z(T_k, \varepsilon_k)$ to $Z(T_{k+1}, \varepsilon_{k+1})$ satisfying the condition

(13) $$r_k(Z(\Delta, \varepsilon_k)) = Z(T_{\Delta,k+1}, \varepsilon_{k+1}) \quad \text{for every triangle} \quad \Delta \in T_k.$$

Now let us set

(14) $$\hat{r}_k(x) = r_k r_{k-1} \ldots r_2 r_1(x) \quad \text{for every} \quad x \in Z(T_1, \varepsilon_1).$$

Applying (7_k), we easily see that $\hat{r}_k$ is a retraction of the zone $Z(T_1, \varepsilon_1)$ to the zone $Z(T_{k+1}, \varepsilon_{k+1})$. Moreover, we infer by (13), (14) and (6_k) that, if $x \in Z(T_1, \varepsilon_1)$ and if $\hat{\Delta}$ is a triangle of the triangulation T_{k+1} such that $\hat{r}_k(x) \in Z(\hat{\Delta}, \varepsilon_{k+1})$, then each of the points $\hat{r}_{k+1}(x) = r_{k+1}\hat{r}_k(x), \ldots, \hat{r}_{k+l}(x) = r_{k+l}r_{k+l-1}\ldots r_{k+1}\hat{r}_k(x)$ belongs to $Z(\hat{\Delta}, \varepsilon_{k+1})$.

Since the diameter of the zone $Z(\hat{\Delta}, \varepsilon_{k+1})$ is $\leqslant 2/(k+1)$, we conclude that $\varrho(\hat{r}_k(x), \hat{r}_{k+l}(x)) \leqslant 2/(k+1)$ for every point $x \in Z(T_1, \varepsilon_1)$ and every $k, l = 1, 2, \ldots$ It follows that the sequence $\{\hat{r}_k\}$ converges uniformly to a map r of $Z(T_1, \varepsilon_1)$ into $P(\{n_k\})$. Since for every point $x \in P(\{n_k\})$ we have $x \in Z(T_{k+1}, \varepsilon_{k+1})$ and consequently $\hat{r}_k(x) = x$ for every $k = 1, 2, \ldots$, we infer that r is a retraction of $Z(T_1, \varepsilon_1)$ to $P(\{n_k\})$.

Thus we have shown that

(15) *Every membrane is an AR-set.*

The construction of the triangulation T_{k+1} implies that the 1-skeleton of T_k is included in the 1-skeleton of T_{k+1}. We infer by (9) that $P(\{n_k\})$ contains the 1-skeleton of T_k for every $k = 1, 2, \ldots$ In particular, the boundaries $P_k^{\cdot}$ of all polytopes P_k and also their edges P_k^* are all included in $P(\{n_k\})$. In particular, the boundary of the disk $P_1 = P$ is a subset of $P(\{n_k\})$. By virtue of (11) and (15), we infer that

(16) *Every membrane is a 2-dimensional AR-set.*

If h maps $P(\{n_k\})$ homeomorphically onto a membrane X, then the simple closed curve $h(P^{\cdot}) \subset X$ will be said to be the *boundary of the membrane X* and will be denoted by $X^{\cdot}$.

As we have seen at the end of No. 2, the components of the edge P_m^* coincide with the segments $\overline{a_\Delta b_\Delta}$ of ramification for triangles $\Delta \in T_k$ with $k < m$. Since $P_m^* \subset P(\{n_k\})$, the homeomorphism h maps $\overline{a_\Delta b_\Delta}$ into a simple arc lying in X. This arc $h(\overline{a_\Delta b_\Delta})$ will be said to be an *arc of ramification of order $2n_k$*.

Since, for every triangle $\Delta \in T_k$, the segment $\overline{a_\Delta b_\Delta}$ is a subset of the zone $Z(\Delta, \varepsilon_k)$, we immediately see the following:

(17) *Let X be a membrane corresponding to a sequence $\{n_k\}$ and let $\{m_k\}$ be a subsequence of $\{n_k\}$. Then the union of all arcs of ramification of orders m_k is dense in X.*

Let us point out, however, that the definition of the boundary and of the arcs of ramification of the membrane $X = h(P(\{n_k\}))$ given here depends on the geometrical construction of $P(\{n_k\})$ and also on the choice of the homeomorphism h.

4. Bits of a membrane. By a *bit* of a membrane $X = P(\{n_k\})$ we understand a membrane Y (corresponding to an arbitrary sequence $\{m_k\}$ of naturals $\geqslant 2$) such that $Y \subset X$ and that $Y \cap \overline{X - Y} \subset Y^{\cdot}$. In particular, if Q is a disk which is the union of some triangles of the triangulation T_m of P_m and if $\hat{T}$ denotes the triangulation of Q included in T_m, then by the same construction as in No. 2, but applied only to simplexes of the triangulations T_{m+k-1} $(k = 1, 2, \ldots)$ lying in $Z(\hat{T}, \varepsilon_m)$, we get the set

$$X_Q = P(\{n_k\}) \cap Z(\hat{T}, \varepsilon_m),$$

which is a membrane on Q corresponding to the sequence $\{n_{m+k-1}\}$. Evidently the boundary $X_Q^{\cdot}$ of X_Q coincides with $Q^{\cdot}$ and we have

$$X_Q \cap \overline{X - X_Q} \subset Q^{\cdot}.$$

Thus X_Q is a bit of X. In particular, if $Q = \Delta$ is a triangle of the triangulation T_m then $X_Q = X_\Delta \subset Z(\Delta, \varepsilon_m)$ and, since the diameter of $Z(\Delta, \varepsilon_m)$ is $\leqslant 2/m$, we have

(18) $$\delta(X_\Delta) \leqslant 2/m \quad \text{for every} \quad \Delta \in T_m.$$

Moreover,

(19) $$X = \bigcup_{\Delta \in T_m} X_\Delta \quad \text{for every} \quad m = 1, 2, \ldots$$

Using the notion of bit, let us prove the following

LEMMA 1. *If Y is a closed proper subset of a membrane X, then the boundary $X^\bullet$ of X is a retract of the set $Y \cup X^\bullet$.*

Proof. We can assume that $X = P(\{n_k\})$ and that $X^\bullet = P^\bullet$. By virtue of (18) and (19), there exists in the triangulation T_m, for m sufficiently large, a triangle Δ such that the bit X_Δ lies in $X-Y$. In order to prove our lemma, it suffices to show that $X^\bullet$ is a retract of the set $(X-X_\Delta) \cup \Delta^\bullet$. We shall do it by induction.

If $m = 1$, then Δ is one of the triangles of the triangulation T_1 of the disk P, and we see at once that the set $X-X_\Delta$ is a subset of the union W of all zones $Z(\Delta', \varepsilon_1)$ with $\Delta' \in T_1-(\Delta)$ and that $P^\bullet = X^\bullet$ is a retract of $W \cup \Delta^\bullet$. Now we assume that the statement holds for some $m \geqslant 1$ and that $\Delta \in T_{m+1}$. Evidently there exists in the triangulation T_m a triangle Δ' such that $\Delta \subset R_{\Delta'}$ (with $R_{\Delta'}$ defined as in No. 2). Then $X_\Delta \subset X_{\Delta'}$ and we infer (by the hypothesis of induction) that there exists a retraction r' of the set $(X-X_{\Delta'}) \cup \Delta'^\bullet$ to $P^\bullet$. Moreover, we infer by (8), No. 2, that, if $\hat{T}$ denotes the system of all triangles of T_{m+1} lying in $Z(\Delta', \varepsilon_m)$, then $\Delta'^\bullet$ is a retract of the set

$$[Z(\hat{T}, \varepsilon_{m+1}) - Z(\Delta, \varepsilon_{m+1})] \cup \Delta^\bullet .$$

Since $(X_{\Delta'}-X_\Delta) \cup \Delta^\bullet$ contains $\Delta'^\bullet$ and it is a subset of

$$[Z(\hat{T}, \varepsilon_{m+1}) - Z(\Delta, \varepsilon_{m+1})] \cup \Delta^\bullet ,$$

we infer that there exists a retraction r'' of the set $(X_{\Delta'}-X_\Delta) \cup \Delta^\bullet$ to $\Delta'^\bullet$. Setting

$$r'''(x) = \begin{cases} r''(x) & \text{for} \quad x \in (X_{\Delta'}-X_\Delta) \cup \Delta^\bullet , \\ x & \text{for} \quad x \in (X-X_{\Delta'}) \cup \Delta'^\bullet \end{cases}$$

we get a retraction r''' of the set $(X-X_\Delta) \cup \Delta^\bullet$ to the set $(X-X_{\Delta'}) \cup \Delta'^\bullet$. Setting

$$r(x) = r'r'''(x) \quad \text{for every} \quad x \in (X-X_\Delta) \cup \Delta^\bullet$$

we get the demanded retraction of the set $(X-X_\Delta) \cup \Delta^\bullet$ to $P^\bullet$.

LEMMA 2. *A closed subset Y of a membrane X is 2-dimensional if and only if Y contains at least one bit of X.*

Proof. If Y contains a bit X_Q of X then by (16) we have

$$2 = \dim X_Q \leqslant \dim Y \leqslant \dim X = 2 ,$$

and consequently $\dim Y = 2$.

Now let us assume that Y does not contain any bit of the membrane X. We can assume that $X = P(\{n_k\})$. Given a positive ε, let us consider a natural m such that $2/m < \varepsilon$. Then

$$X = \bigcup_{\Delta \in T_m} X_\Delta ,$$

where the diameter of X_Δ is $< \varepsilon$ for every triangle $\Delta \in T_m$, and there exists a point $c_\Delta \in X_\Delta - Y$. Applying lemma 1 we infer that there exists a retraction r_Δ of the set $(X_\Delta \cap Y) \cup \Delta^\bullet$ to $\Delta^\bullet$. Setting

$$f(y) = r_\Delta(y) \quad \text{for every} \quad y \in Y ,$$

we get a continuous map f of Y into the 1-skeleton of T_m and this map satisfies the condition

$$\varrho(f(y), y) < \varepsilon \quad \text{for every} \quad y \in Y .$$

But this implies that $\dim Y < 2$.

5. Boundary of a membrane. Let us prove the following

LEMMA 3. *A point $x \in X = P(\{m_k\})$ belongs to $X^\bullet$ if and only if there exists for every $\varepsilon > 0$ an open neighbourhood U of x in X with diameter $< \varepsilon$ and such that $X-U$ is a retract of X.*

Proof. If $x \in X^\bullet$ then let us consider an index k such that $1/k < \varepsilon/2$. By (2_k), (3_k) and (4_k), the union of all triangles of the triangulation T_k containing x is a disk Q with diameter $< \varepsilon$. Evidently

$$X_Q \cap \overline{X-X_Q} \subset X_Q^\bullet = Q^\bullet ,$$

and there exists a simple arc L such that

$$X_Q \cap \overline{X-X_Q} \subset L \subset X_Q^\bullet - (x).$$

Setting $U = X_Q - L$, we get an open neighbourhood U of x with diameter $< \varepsilon$. Let r_L be a retraction of X_Q to L. Setting

$$r(x) = \begin{cases} r_L(x) & \text{for every} \quad x \in X_Q , \\ x & \text{for every} \quad x \in X - X_Q , \end{cases}$$

we get a retraction r of X to $X-U$.

On the other hand, if $x \in X - X^\bullet$ then $\varepsilon = \varrho(x, X^\bullet) > 0$ and, if U is an open neighbourhood of x with diameter $< \varepsilon$, then $X^\bullet \subset X-U$. It follows, by lemma 1, that the simple closed curve $X^\bullet$ is a retract of $X-U$. Since X is an AR-set, we infer that $X-U$ is not a retract of X.

Let us observe that lemma 3 implies that the notion of the boundary X of a membrane is topological.

6. n-membranes. Let n be a natural number $\geqslant 2$. By an *n-membrane* we shall understand a set Y which is the union of n membranes $X_1, X_2, \ldots, X_n$ such that there exists a simple arc L satisfying the conditions:

$$X_i \cap X_j = X_i^{\cdot} \cap X_j^{\cdot} = L \quad \text{for} \quad i \neq j\,.$$

The arc L will be said to be the *edge* of the n-membrane Y and it will be denoted by Y^*. By Y° we shall denote the interior of the edge Y^*. The membranes $X_1, X_2, \ldots, X_n$ will be called the *wings* of the n-membrane X. By the *boundary* $Y^{\cdot}$ of the n-membrane Y we understand the union of n simple arcs $X_i^{\cdot} - Y^\circ$, with common endpoints. It is clear that n-membrane Y is an AR-set, but its boundary $Y^{\cdot}$ is not an AR-set.

LEMMA 4. *Let $X_1, \ldots, X_n$ be the wings of an n-membrane Y and let M be a closed subset of Y such that $X_i - M \neq 0$ for $i = 2, \ldots, n$. Then $Y^{\cdot}$ is a retract of the set $M \cup Y^{\cdot}$.*

Proof. By lemma 1, there exists a retraction r_i of the set $(M \cap X_i) \cup X_i^{\cdot}$ to $X_i^{\cdot}$, for every $i = 2, \ldots, n$. Moreover, there exists a retraction r_1 of the set X_1 to the simple arc $X_1^{\cdot} - Y^\circ$. Setting

$$r_0(x) = \begin{cases} x & \text{for every} \quad x \in Y^{\cdot}\,, \\ r_1(x) & \text{fore very} \quad x \in X_1 \end{cases}$$

we get a retraction r_0 of the set $X_1 \cup Y^{\cdot}$ to $Y^{\cdot}$. In order to obtain a retraction r of $M \cup Y^{\cdot}$ to $Y^{\cdot}$, it suffices to set

$$r(x) = \begin{cases} r_1(x) & \text{for} \quad x \in M \cap X_1\,, \\ r_0 r_i(x) & \text{for} \quad x \in M \cap X_i,\ i = 2, \ldots, n\,, \\ x & \text{for} \quad x \in Y^{\cdot}\,. \end{cases}$$

7. Topological classification of points of a membrane. Let n be a natural number $\geqslant 2$. By an *n-bit* of a membrane X we shall understand a subset Y of X satisfying the following two conditions:

(i) Y *is an n-membrane.*

(ii) $Y^{\cdot} \supset Y \cap \overline{X - Y}$.

Condition (ii) implies that $Y - Y^{\cdot}$ is an open subset of X.

Now we consider the following subsets of the membrane X:

X_{I}, consisting of all points $x \in X$ such that for every $\varepsilon > 0$ there exists a bit Y of X with diameter $< \varepsilon$ such that Y is a neighbourhood of x in X and $x \in Y^{\cdot}$. The points of X_{I} will be said to be *frontier points of* X.

X_{II}, consisting of all points $x \in X - X_{\mathrm{I}}$ such that for every $\varepsilon > 0$ there exists a bit Y of X with diameter $< \varepsilon$ such that $x \in Y - Y^{\cdot}$. The points of X_{II} will be said to be *regular points of* X.

X_{III}^n, (where $n > 2$) consisting of all points $x \in X - X_{\mathrm{I}} - X_{\mathrm{II}}$ such that for for every $\varepsilon > 0$ there exists an n-bit Y of X with diameter $< \varepsilon$ such that $x \in Y^\circ$. The points of X_{III}^n will be said to be *points of the ramification of order n of the membrane X.*

$X_{\mathrm{IV}} = X - X_{\mathrm{I}} - X_{\mathrm{II}} - \bigcup_{n=3}^{\infty} X_{\mathrm{III}}^n$. The points of X_{IV} will be said to be *singular points of* X.

Since by a homeomorphic map of the membrane X onto another membrane X' to every bit (resp. to every n-bit) Y of X corresponds a bit (resp. an n-bit) Y' of X', and to the boundary $Y^{\cdot}$ of Y corresponds the boundary $Y'^{\cdot}$ of Y' and to the set Y° corresponds the set Y'°, we infer, by lemma 3 of No. 5, that the sets X_{I}, X_{II}, X_{III}^n and X_{IV} are topologically invariant. Evidently

(20) $$X = X_{\mathrm{I}} \cup X_{\mathrm{II}} \cup \Big(\bigcup_{n=3}^{\infty} X_{\mathrm{III}}^n\Big) \cup X_{\mathrm{IV}}\,,$$

and

(21) *the sets* X_{I}, X_{II}, $\bigcup_{n=3}^{\infty} X_{\mathrm{III}}^n$ *and* X_{IV} *are disjoint*.

Let us observe that

(22) $$X_{\mathrm{I}} = X^{\cdot}\,.$$

In fact, if $x \in X^{\cdot}$, where $X = P(\{n_k\})$, then we infer by (3_k) and (4_k) that for every $\varepsilon > 0$ there exists a bit Y of X with diameter $< \varepsilon$ such that $x \in Y^{\cdot}$ and that Y is a neighbourhood of x. Consequently $x \in X_{\mathrm{I}}$.

On the other hand, if $x \in X_{\mathrm{I}}$ then there exists a neighbourhood Y of x which is a bit containing x in its boundary $Y^{\cdot}$. Applying lemma 3 of No. 5, we infer that for every $\varepsilon > 0$ there exists an open neighbourhood $U \subset Y - \overline{X - Y}$ of x with diameter $< \varepsilon$ and a retraction r' of Y to $Y - U$. Setting

$$r(y) = \begin{cases} r'(y) & \text{for every} \quad y \in Y\,, \\ y & \text{for every} \quad y \in X - U\,, \end{cases}$$

we get a retraction of X to $X - U$. It follows, by lemma 3, that $x \in X^{\cdot}$.

According to (22), there is exactly one of three cases for each point $x \in X - X_{\mathrm{I}}$. In fact:

1. There exists a natural m such that x belongs to the 1-skeleton of the triangulation T_m but x does not belong either to $X^{\cdot} = P_m^{\cdot}$ or to P_m^*.

2. For every $m = 1, 2, \ldots$ the point x belongs to the set

$$\bigcup_{\Delta \in T_m} [Z(\Delta, \varepsilon_m) - \Delta^{\cdot}]\,.$$

3. There exist a natural m such that $x \in P_m^*$.

In case 1, one easily sees, by condition (4_k) with sufficiently large k, that x belongs to the interior of arbitrarily small ordinary bits of X, and thus $x \in X_{\mathrm{II}}$.

In case 2, there exists every $m=1,2,\ldots$, a triangle $\Delta \in T_m$ such that $x \in X \cap Z(\Delta, \varepsilon_m) - \Delta^{\cdot}$. But $X \cap Z(\Delta, \varepsilon_m)$ is an ordinary bit of X containing x in its interior and its diameter is $\leqslant 2/m$. Thus also in this case $x \in X_{\mathrm{II}}$.

It follows by (21) that only in case 3 the point x can belong to the set $\bigcup\limits_{n=3}^{\infty} X_{\mathrm{III}}^n \cup X_{\mathrm{IV}}$, i.e.

(23) *Every point of ramification and every singular point of X belongs to one of the segments of ramification $\overline{a_\Delta b_\Delta}$.*

Finally, let us observe that if x belongs to the interior of $\overline{a_\Delta b_\Delta}$, with $\Delta \in T_k$, then there exist $2n_k$ triangles among the triangles $\Delta_1', \ldots, \Delta_{3n_k}'$ containing $\overline{a_\Delta b_\Delta}$ on their boundaries. Let us denote them by $\Delta_1'', \ldots, \Delta_{2n_k}''$. Applying condition (4_k) we infer that for every $l=1,2,\ldots$ the union of all triangles of the triangulation T_{k+l} lying on Δ_ν'' (where ν is one of the numbers $1,2,\ldots,2n_k$) and containing the point x is a disk Q_ν, and we see at once that the union of corresponding membranes $Q_\nu(\{n_{k+l}\})$ is a $2n_k$-bit Y of X with diameter $\leqslant 4/(k+l)$ and the point x belongs to Y°. It follows that no point x lying in the interior of the segment $\overline{a_\Delta b_\Delta}$ is singular. Consequently only the endpoints of segments of ramification $\overline{a_\Delta b_\Delta}$ can be singular.

8. Points of ramification. Now let us prove the following

LEMMA 5. *Let Y by an n-bit $(n>2)$ of a membrane X and let x be a point of Y°. Then x does not belong to any of the sets X_{I}, X_{II} and X_{III}^m with $m \neq n$.*

Proof. Let $Y_1, \ldots, Y_n$ be the wings of Y. First let us suppose that $x \in X_{\mathrm{II}}$. Then there exists a bit Y_0 of X such that

$$x \in Y_0 - Y_0^{\cdot} \quad \text{and} \quad Y_0 \Subset Y .$$

Since $Y_0 - Y_0^{\cdot}$ is open in X, we infer that the set

$$G = Y_0 \cup Y_1 - (Y_0^{\cdot} \cup Y_1^{\cdot})$$

is not empty and that it is open in Y_0. By lemma 1, there exists a retraction r_0 of the set $Y_0 - G$ to Y_0. Setting

$$r_1(x) = \begin{cases} r_0(x) & \text{for} \quad x \in Y_0 - G, \\ x & \text{for} \quad x \in Y - Y_0, \end{cases}$$

we get a retraction r_1 of the set $Y-G$ to $(Y-Y_0) \cup Y_0^{\cdot}$. But Y_0 is a neighbourhood of x in X, and consequently $Y_i - [(Y-Y_0) \cup Y_0^{\cdot}] \neq 0$ for

$i=1,2,\ldots n$. It follows, by lemma 4, that there exists a retraction r_2 of the set $(Y-Y_0) \cup Y_0^{\cdot} \cup Y^{\cdot} = (Y-Y_0) \cup Y_0^{\cdot}$ to $Y^{\cdot}$. Finally, it is evident that there exists a retraction r_3 of the $Y^{\cdot}$ to the simple closed curve $C = Y_2^{\cdot} \cup Y_3^{\cdot} - Y^\circ$. Setting

$$r(x) = r_3 r_2 r_1(x) \quad \text{for every} \quad x \in Y_2 \cup Y_3 ,$$

we get a retraction of the set $Y_2 \cup Y_3$ to the simple closed curve C. But this is impossible, because $Y_2 \cup Y_3$ is an AR-set (since Y_2, Y_3 and $Y_2 \cap Y_3$ are AR-sets). Thus the supposition that $x \in X_{\mathrm{II}}$ leads to a contradiction.

Suppose now that $x \in X_{\mathrm{I}}$ or $x \in X_{\mathrm{III}}^m$, with $2 < m \neq n$. We can consider both these cases simultaneously, setting $X_{\mathrm{III}}^1 = X_{\mathrm{I}}$ and supposing that $x \in X_{\mathrm{III}}^m$ with $2 \neq m \neq n$. Since the hypotheses concerning m and n are symmetric, it suffices to consider the case where

$$(24) \qquad 2 \neq m < n .$$

Suppose that $x \in X_{\mathrm{III}}^m \cap X_{\mathrm{III}}^n$. Then there exists a system of m membranes $W_1, \ldots, W_m \subset X$ and there exists a simple arc M such that

$$M = W_i \cap W_j = W_i^{\cdot} \cap W_j^{\cdot} \quad \text{for} \quad i \neq j ,$$

$$x \text{ belongs to the interior of } M.$$

$$W = W_1 \cup W_2 \cup \ldots \cup W_m \text{ is an } m\text{-bit of } X$$

(if $m=1$, W is a bit of X),

$$W \subset Y - Y^{\cdot},$$
$$W^{\cdot} \supset W \cap \overline{X-W} .$$

Since W_i is 2-dimensional, there exists a point

$$a_i \in W_i - W_i^{\cdot} - \bigcup_{\nu=1}^{n} Y_\nu^{\cdot} \quad \text{for} \quad i=2,3,\ldots,m .$$

The sets $Y_\nu - Y_\nu^{\cdot}$ being disjoint, there exists for every $i=2,\ldots,m$ exactly one index k_i such that $a_i \in Y_{k_i} - Y_{k_i}^{\cdot}$. It follows by (24) that among the indices $1,2,\ldots,n$ at least two are distinct from all indices k_i; we can assume that

$$(25) \qquad 1 \neq k_i \neq 2 \quad \text{for every} \quad i=2,3,\ldots,m .$$

Now let us assign to every $i=2,3,\ldots,m$ an open subset G_i of X such that

$$a_i \in G_i \subset W_i \cap Y_{k_i} - W_i^{\cdot} - Y_{k_i}^{\cdot} .$$

If $m > 2$, then, applying lemma 4, we infer that there exists a retraction r_0 of the set $W - \bigcup_{i=2}^{m} G_i$ to the set $W^{\cdot} \supset W \cap \overline{X-W}$. An analogous statement is true also in the case of $m = 2$, if we replace W by a simple arc lying in the boundary $W^{\cdot}$ and containing the set $W \cap \overline{X-W}$. Setting

$$r_1(x) = \begin{cases} r_0(x) & \text{for} \quad x \in W - \bigcup_{i=2}^{m} G_i, \\ x & \text{for} \quad x \in Y - W, \end{cases}$$

we get a retraction r_1 of the set $Y - \bigcup_{i=1}^{m} G_i$ to the set $(Y-W) \cup W^{\cdot}$. Since W is a neighbourhood of the point $x \in Y^{\cdot}$, it follows that $Y_j - \overline{Y-W} \neq 0$ for every $j = 1, 2, \ldots, m$. Since $\overline{Y-W} \supset Y^{\cdot}$, we infer by lemma 4 that there exists a retraction r_2 of the set $\overline{Y-W}$ to $Y^{\cdot}$. Finally, it is evident that there exists a retraction r_3 of the set Y to the simple closed curve $C = Y_1^{\cdot} \cup Y_2^{\cdot} - Y^{\circ}$. Setting

$$r(x) = r_3 r_2 r_1(x) \quad \text{for every point} \quad x \in Y_1 \cup Y_2,$$

we get, by (28), a retraction r of the set $Y_1 \cup Y_2$ to C. But this is impossible, because $Y_1 \cup Y_2$ is an AR-set.

Thus the proof of lemma 5 is concluded.

In the case of $X = P(\{n_k\})$ we infer by this lemma and by formula (20) that every point x lying in the interior of a segment of ramification $\overline{a_\Delta b_\Delta}$, where Δ is a triangle of the triangulation T_k, belongs to $X_{\mathrm{III}}^{2n_k}$. It follows, by (17), that:

(26) *For every subsequence $\{m_k\}$ of the sequence $\{n_k\}$ and for every open subset G of X the set $G \cap (\bigcup_{k=1}^{\infty} X_{\mathrm{III}}^{2m_k})$ is of power $2^{\aleph_0}$.*

On the other hand, only the endpoints of segments of ramification may be singular points or points of ramification of orders which do not belong to the sequence $\{2n_k\}$. Consequently (23) implies that:

(27) *If N is the set of all natural numbers > 2 which do not belong to the sequence $\{2n_k\}$ then the set $\bigcup_{n \in N} X_{\mathrm{III}}^{n} \cup X_{\mathrm{IV}}$ is at most countable.*

9. Main theorem. The fundamental result of this note is the following

THEOREM. *There exists a function Φ assigning to every real number t a membrane $\Phi(t) \subset E^3$ in such a manner that for $t \neq t'$ no 2-dimensional closed subset of $\Phi(t)$ is homeomorphic to any subset of $\Phi(t')$.*

Proof. Let $w_1, w_2, \ldots, w_n, \ldots$ be an enumeration of all rational numbers. Let us assign to every real number t an increasing sequence $\{n_k(t)\}$ consisting of all natural numbers n such that $w_n < t$. It is clear that for $t' < t$ the sequence $\{n_k(t)\}$ contains, besides all numbers $n_k(t')$, also all natural numbers n for which $t' \leqslant w_n < t$. Therefore

(28) *$t' < t$ implies that in the sequence $\{n_k(t)\}$ there exist an infinity of terms which do not belong to $\{n_k(t')\}$.*

Now let P be a triangle in E^3 and let us put

$$\Phi(t) = P(\{n_k(t)\}) \quad \text{for every real } t.$$

It remains to show that if there exists a homeomorphism h mapping a 2-dimensional closed subset A of $\Phi(t)$ onto a subset B of $\Phi(t')$, then $t = t'$.

Suppose to the contrary that $t \neq t'$. Since $\dim A = \dim B = 2$ and since the inverse homeomorphism h^{-1} maps B onto A, we see that the hypotheses concerning t and t' are symmetric, and thus we can suppose that $t' < t$. It follows by (28) that the sequence $\{n_k(t)\}$ contains a subsequence $\{m_k\}$ consisting of natural numbers which do not belong to $\{n_k(t')\}$.

It follows by (26) that the points of ramification of orders $2m_k$ lying in an arbitrarily given open subset $G \neq 0$ of the membrane $\Phi(t)$ constitute a subset of G of the power $2^{\aleph_0}$. On the other hand, we infer by (27) that the subset of $\Phi(t')$ consisting of all points of ramification of orders $2m_k$ and of singular points is at most countable. By lemma 2, the set A contains a bit X_0 of $P(\{n_k(t)\})$. Since $X_0 - X_0^{\cdot}$ is open in $\Phi(t)$, we infer from (27) that $X_0 - X_0^{\cdot}$ contains a dense subset R consisting of points of ramification of orders $2m_k$ of the membrane $\Phi(t)$ such that any point of $h(R)$ is neither a singular point nor a point of ramification of order $2m_k$ of the membrane $\Phi(t')$. If $h(R) \subset \overline{\Phi(t') - h(X_0)}$ then $h(X_0) = \overline{h(R)} \subset \overline{\Phi(t') - h(X_0)}$. But this is impossible because $h(X_0)$ contains (by lemma 2) a bit of $\Phi(t')$. Thus we see that there exists a point $a \in R$ such that $h(a)$ belongs to the interior of $h(X_0)$. Consequently h maps every neighbourhood of the point a in the set $A \subset \Phi(t)$ onto a neighbourhood of the point $h(a)$ in the space $\Phi(t')$. However, this is also impossible, by lemma 5 and formula (20), because a is a point of ramification of order $2m_k$ of the membrane $\Phi(t)$, and $h(a)$ is neither a singular point nor a point of ramification of the order $2m_k$ of the membrane $\Phi(t')$.

Thus the proof of our theorem is concluded.

10. The non-existence of a universal 2-dimensional AR-set. The problem of the existence of a universal n-dimensional AR-set, i.e. of an n-dimensional AR-set which topologically contains every other n-dimensional AR-set, is old. Only in the case of $n=1$ it is solved positively since the 1-dimensional AR-sets coincide with dendrites and it is known ([1]) that there exists a universal dendrite.

Mr. Sieklucki has recently remarked that the existence of the function Φ satisfying the theorem of No. 9, allows us to solve the problem of the existence of a universal 2-dimensional AR-set in the negative sense if we recall the following theorem ([2]):

In an n-dimensional ANR*-set every family of n-dimensional subsets which are* ANR*-sets with the common part of any two of them at most $(n-1)$-dimensional is necessarily at most countable.*

In fact, using this theorem, we infer from the theorem of No. 9, that no 2-dimensional ANR-set contains topologically all membranes $\Phi(t)$, because the common part of two sets homeomorphic to $\Phi(t)$ and to $\Phi(t')$ with $t \neq t$ is necessarily of dimension $\leqslant 1$.

11. An application to the theory of r-neighbours. A space X is said to be *r-smaller* than a space Y (or Y is *r-greater* than X) provided X is homeomorphic to a retract of Y but Y is not homeomorphic to any retract of X. If X is r-smaller than Y but there exists no space Z which is r-smaller than Y and r-greater than X, then X is said to be an *r-neighbour of Y on the left* (see [3]).

It is clear that every space which is r-smaller than an AR-set is also an AR-set. Evidently all membranes are r-smaller than the Euclidean cube Q^3, because each of them is topologically included in E^3, and consequently it is homeomorphic to a subset of Q^3 and this subset, being an AR-set, is a retract of Q^3.

Now let us assume that X is an r-neighbour of Q^3 on the left. We can assume that $X \subset Q^3$. Since X is r-smaller than Q^3, no open subset of Q^3 is included in X and consequently $\dim X \leqslant 2$. As we have already seen, it follows that there exists a real number t such that the membrane $\Phi(t)$ is not included topologically in X. Now let us consider a Euclidean ball $B \subset Q^3$ such that:

The interior of B is a subset of $Q^3 - X$,
There exists a point $x_0 \in X$ lying on the boundary of B.

Evidently there exists a set $F \subset B$ homeomorphic to the membrane $\Phi(t)$ such that $F \cap X = (x_0)$. We see at once that the set $Y = X \cup F$ is an AR-set which is r-smaller than Q^3 but r-greater than X. However, this is incompatible with the supposition that X is an r-neighbour of Q^3 on the left.

Thus we have shown that the 3-dimensional cube has no r-neighbours on the left.

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Reçu par la Rédaction le 22. 1. 1962