On weakly chainable continua

by

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A finite sequence of sets $X_1, \ldots, X_m$ is said to be a *chain* provided that

$$(*) \quad X_i \cap X_j \neq 0 \quad \text{if and only if} \quad |i-j| \leqslant 1 \quad (i, j = 1, \ldots, m).$$

A continuum C (i.e. a compact connected metric space) is said to be *chainable* provided that there are, for every $\varepsilon > 0$, open sets $G_1, \ldots, G_m$ in C such that $C = G_1 \cup \ldots \cup G_m$, the diameter $\delta(G_i)$ of G_i is less than ε for $i = 1, \ldots, m$ and the sequence of sets $G_1, \ldots, G_m$ is a chain. Let us remark that the chain $G_1, \ldots, G_m$ can then easily be improved so that the sequence of closures $\bar{G}_1, \ldots, \bar{G}_m$ in C is also a chain. Obviously, each subcontinuum of a chainable continuum is chainable.

In 1922 Knaster constructed a hereditarily indecomposable chainable plane continuum K. In 1948 Moise showed that there existed an indecomposable plane continuum M which, like any arc, was homeomorphic to each of its nondegenerate subcontinua. Then M was hereditarily indecomposable. Bing proved, also in 1948, that there existed hereditarily indecomposable and topologically homogeneous chainable plane continua, and in 1951—that each chainable continuum was homeomorphic to a subset of the plane and each hereditarily indecomposable chainable continuum was homeomorphic to K. So K had all properties of M, and at that time it began to be called a *pseudo-arc*. This term became usual also for denoting every hereditarily indecomposable chainable continuum, i.e. every continuum homeomorphic to K [1].

If C is a chainable continuum, then there evidently exists an infinite sequence $\boldsymbol{G}_1, \boldsymbol{G}_2, \ldots$ of finite open covers of C such that each $\boldsymbol{G}_n$ is a chain, each element of $\boldsymbol{G}_n$ has diameter less than $1/n$ and each element of $\boldsymbol{G}_{n+1}$ is contained in some element of $\boldsymbol{G}_n$ $(n = 1, 2, \ldots)$. But if each element G_i of a chain $\boldsymbol{G} = (G_1, \ldots, G_m)$ is contained in some element

[1] The reader can find other historical remarks and references in nearly all of numerous papers concerning these things; for instance, among the recent papers, in one of Lida K. Barrett, *The structure of decomposable snakelike continua*, Duke Math. J. 28 (1961), pp. 515-522.

G'_{k_i} of a chain $G' = (G'_1, \ldots, G'_{m'})$, then $|i-j| \leqslant 1$ implies $0 \neq G_i \cap G_j \subset G'_{k_i} \cap G'_{k_j}$, which gives $|k_i - k_j| \leqslant 1$, according to $(*)$, for $i, j = 1, \ldots, m$. This leads to the notion of weakly chainable continua.

Namely, a finite sequence of sets $X_1, \ldots, X_m$ is said to be a *weak chain* provided that

$$(**) \qquad X_i \cap X_j \neq 0 \quad \text{if} \quad |i-j| \leqslant 1 \quad (i, j = 1, \ldots, m).$$

Since $(*)$ yields $(**)$, every chain is a weak chain.

A weak chain $\boldsymbol{X} = (X_1, \ldots, X_m)$ is said to be a *refinement* of a weak chain $\boldsymbol{X}' = (X'_1, \ldots, X'_{m'})$, symbolically $\boldsymbol{X} \prec \boldsymbol{X}'$, provided that each element X_i of $\boldsymbol{X}$ is contained in some element X'_{k_i} of $\boldsymbol{X}'$ such that

$$({*}{*}{*}) \qquad |k_i - k_j| \leqslant 1 \quad \text{if} \quad |i-j| \leqslant 1 \quad (i, j = 1, \ldots, m).$$

The relation $\prec$ is transitive, i.e. if $\boldsymbol{X}$, $\boldsymbol{X}'$ and $\boldsymbol{X}''$ are weak chains with $\boldsymbol{X} \prec \boldsymbol{X}'$ and $\boldsymbol{X}' \prec \boldsymbol{X}''$, then $\boldsymbol{X} \prec \boldsymbol{X}''$.

A continuum C is said to be *weakly chainable* provided that there exists an infinite sequence $\boldsymbol{G}_1, \boldsymbol{G}_2, \ldots$ of finite open covers of C such that each $\boldsymbol{G}_n$ is a weak chain, each element of $\boldsymbol{G}_n$ has diameter less than $1/n$ and $\boldsymbol{G}_{n+1}$ is a refinement of $\boldsymbol{G}_n$ $(n = 1, 2, \ldots)$. Every chainable continuum is, of course, weakly chainable.

If a continuum C is weakly chainable, then the weak chains $\boldsymbol{G}_n$ which exist by the definition can be modified so that each $\boldsymbol{G}_{n+1}$, besides being a refinement of $\boldsymbol{G}_n$, has its first and last elements in the first and last elements of $\boldsymbol{G}_n$, respectively.

More precisely, a weak chain $\boldsymbol{X} = (X_1, \ldots, X_m)$ is said to be an *exact refinement* of a weak chain $\boldsymbol{X}' = (X'_1, \ldots, X'_{m'})$, symbolically $\boldsymbol{X} \prec_e \boldsymbol{X}'$, provided that each element X_i of $\boldsymbol{X}$ is contained in some element X'_{k_i} of $\boldsymbol{X}'$ such that $k_1 = 1$, $k_m = m'$ and $({*}{*}{*})$ holds.

Then *if C is a weakly chainable continuum, there exists an infinite sequence $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots$ of finite open covers of C such that each $\boldsymbol{I}_n$ is a weak chain* ([2]), *each element of $\boldsymbol{I}_n$ has diameter less than $1/n$ and $\boldsymbol{I}_{n+1}$ is an exact refinement of $\boldsymbol{I}_n$* $(n = 1, 2, \ldots)$.

([2]) Let us observe that, in the case where C is chainable, weak chains cannot be here replaced by chains. In fact, let C_1 be the well-known indecomposable chainable plane continuum, which is the union of all demi-circumferences D with the centre $(\frac{1}{2}, 0)$ such that D joins the points of the Cantor ternary set T and lies in the demi-plane $y \geqslant 0$, and of all demi-circumferences D' with centres $(5/2 \cdot 3^n, 0)$ such that D' joins the points of T contained in the segment $2/3^n \leqslant x \leqslant 1/3^{n-1}$ and lies in the demi-plane $y \leqslant 0$ for $n = 1, 2, \ldots$ Let C_2 be any set homeomorphic to C_1 so that the point $(0, 0)$ is a fixed point of this homeomorphism and the unique common point of C_1 and its image C_2. Then the union $C = C_1 \cup C_2$ is a chainable continuum and there exists no infinite sequence $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots$ of finite open covers of C such that each $\boldsymbol{I}_n$ is a chain, each element of $\boldsymbol{I}_n$ has diameter less that $1/n$ and $\boldsymbol{I}_{n+1}$ is an exact refinement of $\boldsymbol{I}_n (n = 1, 2, \ldots)$

To prove this, it is convenient to consider weak chains $\boldsymbol{X}$ whose elements have not indices from 1 to m. Namely, $\boldsymbol{X}$ is only assumed to be a finite ordered sequence (possibly with repetitions) of m sets which, after being provided with successive indices from 1 to m, form a weak chain $\boldsymbol{X}'$ in the sense described above. In the sequel we shall identify $\boldsymbol{X}$ with $\boldsymbol{X}'$.

For any weak chain $\boldsymbol{X} = (X_1, X_2, \ldots, X_m)$ and positive integers a and b such that $1 \leqslant a \leqslant b \leqslant m$, we denote by $\boldsymbol{X}(a, b)$ the weak chain

$$\boldsymbol{X}(a, b) = (X_a, X_{a+1}, \ldots, X_b).$$

For any two weak chains $\boldsymbol{X} = (X_a, X_{a+1}, \ldots, X_b)$ and $\boldsymbol{Y} = (Y_c, Y_{c+1}, \ldots, Y_d)$ such that $X_b = Y_c$, we define a weak chain $\boldsymbol{X}+\boldsymbol{Y}$ by the formula

$$\boldsymbol{X}+\boldsymbol{Y} = (X_a, X_{a+1}, \ldots, X_b, Y_{c+1}, \ldots, Y_d).$$

Further, for any weak chain $\boldsymbol{X} = (X_a, \ldots, X_{b-1}, X_b)$, we denote by $-\boldsymbol{X}$ the weak chain

$$-\boldsymbol{X} = (X_b, X_{b-1}, \ldots, X_a),$$

and also assume the notation $(-1)\boldsymbol{X} = -\boldsymbol{X}$.

Now let $\boldsymbol{G}_n$ be weak chains given by the definition of the weakly chainable continuum C. Denote by m_n the number of elements of $\boldsymbol{G}_n$ $(n = 1, 2, \ldots)$. As $\boldsymbol{G}_{n+1} \prec \boldsymbol{G}_n$, there is a function k_n mapping the set $\{1, \ldots, m_{n+1}\}$ into the set $\{1, \ldots, m_n\}$ so that the i-th element of $\boldsymbol{G}_{n+1}$ is contained in the $k_n(i)$-th element of $\boldsymbol{G}_n$ for $i = 1, \ldots, m_{n+1}$, and condition $({*}{*}{*})$ holds for $k_i = k_n(i)$ and $m = m_{n+1}$. It follows that the function k_n has the *Darboux property*, i.e. if integers $a \leqslant b$ are the values under k_n of a' and b', respectively, then each integer l satisfying $a \leqslant l \leqslant b$ is the value under k_n of an integer in the segment with end-points a' and b'.

For every $n = 1, 2, \ldots$ and $i = 0, 1, \ldots$, let a_n^i and b_n^i denote the minimum and the maximum of the function $k_n k_{n+1} \ldots k_{n+i}(j)$, where $j = 1, \ldots, m_{n+i+1}$, respectively. So we have

$$1 \leqslant a_n^0 \leqslant a_n^1 \leqslant \ldots \leqslant b_n^1 \leqslant b_n^0 \leqslant m_n$$

for every $n = 1, 2, \ldots$, and therefore there exist positive integers i_n, a_n and b_n such that $1 \leqslant a_n \leqslant b_n \leqslant m_n$, $a_n^i = a_n$ and $b_n^i = b_n$ for $i \geqslant i_n$ $(n = 1, 2, \ldots)$. Then, for every $n = 1, 2, \ldots$, the function k_n maps the segment $a_{n+1} \leqslant j \leqslant b_{n+1}$ of integers j into the segment $a_n \leqslant j \leqslant b_n$, and the extrema a_n and b_n must be the values under k_n of some integers in the segment $a_{n+1} \leqslant j \leqslant b_{n+1}$, i.e. there are positive integers c_n and d_n such that $a_{n+1} \leqslant c_n \leqslant b_{n+1}$, $a_{n+1} \leqslant d_n \leqslant b_{n+1}$, $k_n(c_n) = a_n$ and $k_n(d_n) = b_n$.

Consequently, each element of the weak chain

$$\boldsymbol{H}_n = \boldsymbol{G}_n(a_n, b_n)$$

has diameter less than $1/n$ and $\boldsymbol{H}_{n+1} \prec \boldsymbol{H}_n$ for $n = 1, 2, \ldots$ Moreover, we have $\boldsymbol{G}_{n+i_n+1} \prec \boldsymbol{H}_n$, whence each $\boldsymbol{H}_n$ is an open cover of C.

We define the required weak chains $\boldsymbol{I}_n$ inductively, so that they are all of the form

$$\boldsymbol{I}_n = (-1)^{v_n}[-\boldsymbol{H}_n(a_n, s_n) + \boldsymbol{H}_n - \boldsymbol{H}_n(t_n, b_n)],$$

where s_n, t_n and v_n are integers, $a_n \leqslant s_n \leqslant b_n$ and $a_n \leqslant t_n \leqslant b_n$. Namely, we first put $s_1 = a_1$, $t_1 = b_1$ and $v_1 = 0$, that is $\boldsymbol{I}_1 = \boldsymbol{H}_1$. Suppose now that the weak chain $\boldsymbol{I}_n$ is defined. To define $\boldsymbol{I}_{n+1}$, it is sufficient to choose suitable integers s_{n+1}, t_{n+1} and v_{n+1}. We shall do it so that the condition $\boldsymbol{I}_{n+1} \prec_e \boldsymbol{I}_n$ will be satisfied.

Indeed, we have $c_n \leqslant d_n$ or $d_n \leqslant c_n$. In the case where both inequalities hold (then $a_n = b_n$), we consider only one of them. The function k_n having the Darboux property, the integers s_n and t_n are the values under k_n of integers in the segment with end-points c_n and d_n. If $c_n \leqslant d_n$ (or $d_n \leqslant c_n$), we define s_{n+1} as the minimum of integers j satisfying $c_n \leqslant j \leqslant d_n$ and $k_n(j) = s_n$ (or $d_n \leqslant j \leqslant c_n$ and $k_n(j) = t_n$)—t_{n+1} as the maximum of integers j satisfying $c_n \leqslant j \leqslant d_n$ and $k_n(j) = t_n$ (or $d_n \leqslant j \leqslant c_n$ and $k_n(j) = s_n$), and finally put $v_{n+1} = v_n$ (or $v_{n+1} = v_n + 1$, respectively).

THEOREM. *A continuum is weakly chainable if and only if it is a continuous image of the pseudo-arc* [3].

Proof. If a continuum C is a continuous image of the pseudo-arc K under a mapping f, then for every $n = 1, 2, \ldots$ there exists a number $\varepsilon_n > 0$ such that $\delta(A) < \varepsilon_n$ implies $\delta[f(A)] < 1/3n$ for each $A \subset K$. Since K is chainable, there is an infinite sequence of chains $\boldsymbol{D}_1, \boldsymbol{D}_2, \ldots$ such that every $\boldsymbol{D}_n$ is a finite open cover of K, $\boldsymbol{D}_{n+1} \prec \boldsymbol{D}_n$ and $\delta(D) < \varepsilon_n$ for $D \in \boldsymbol{D}_n$ $(n = 1, 2, \ldots)$. Suppose that $\boldsymbol{D}_n = (D_1, \ldots, D_m)$ for an arbitrary $n = 1, 2, \ldots$ Consider $\boldsymbol{G}_n = (G_1, \ldots, G_m)$, where G_i is the set of points of C whose distances from some points of $f(D_i)$ are less than $1/3n$ $(i = 1, \ldots, m)$. Then $\boldsymbol{G}_n$ is a finite open cover of C and $\delta(G_i) < 1/n$ for $i = 1, \ldots, m$. Further, $|i-j| \leqslant 1$ implies $0 \neq f(D_i \cap D_j) \subset f(D_i) \cap f(D_j) \subset G_i \cap G_j$ $(i, j = 1, \ldots, m)$, whence $\boldsymbol{G}_n$ is a weak chain. Moreover, $\boldsymbol{D}_{n+1} \prec \boldsymbol{D}_n$ gives $\boldsymbol{G}_{n+1} \prec \boldsymbol{G}_n$ $(n = 1, 2, \ldots)$, and so C is weakly chainable.

Conversely, let us suppose that C is weakly chainable and denote by $\boldsymbol{I}_1, \boldsymbol{I}_2, \ldots$ weak chains constituting the already modified sequence of covers of C. So each element of $\boldsymbol{I}_n$ has diameter less than $1/n$ and $\boldsymbol{I}_{n+1}$ is an exact refinement of $\boldsymbol{I}_n$ $(n = 1, 2, \ldots)$. Let m_n be the number of elements of $\boldsymbol{I}_n$ $(n = 1, 2, \ldots)$. As K is the pseudo-arc, there exist points p and q and an infinite sequence $\boldsymbol{D}_1, \boldsymbol{D}_2, \ldots$ of finite open covers of K such that each $\boldsymbol{D}_n$ is a chain from p to q, each element of $\boldsymbol{D}_n$ has diameter less than $1/n$, the closure of each element of $\boldsymbol{D}_{n+1}$ is contained in some element

[3] Note that this theorem gives an answer to a question proposed by Fort (see [1], p. 541, Question 3).

of $\boldsymbol{D}_n$ and $\boldsymbol{D}_{n+1}$ is crooked in $\boldsymbol{D}_n$ (see [2], p. 370 and 374). Then the number of elements of $\boldsymbol{D}_n$ tends to infinity with n, and thus there is a consolidation $\boldsymbol{E}_1$ of some $\boldsymbol{D}_{n_1}$ (ibidem) that consists of m_1 elements. This is the beginning of the definition, by induction, of an infinite sequence $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots$ of finite open covers of K.

Namely, suppose a consolidation $\boldsymbol{E}_l$ of $\boldsymbol{D}_{n_l}$ is defined so that $\boldsymbol{E}_l$ consists of m_l elements, for instance $\boldsymbol{E}_l = (E'_1, \ldots, E'_{m_l})$. Then $\boldsymbol{D}_{n_l+1}$ is crooked in $\boldsymbol{E}_l$ (ibidem, Lemma 1). Let us write

$$\boldsymbol{I}_{l+1} = (I_1, \ldots, I_{m_{l+1}}), \quad \boldsymbol{I}_l = (I'_1, \ldots, I'_{m_l})$$

and take, by virtue of the relation $\boldsymbol{I}_{l+1} \prec_e \boldsymbol{I}_l$, positive integers

$$1 = k_1, k_2, \ldots, k_{m_{l+1}} = m_l$$

such that $I_i \subset I'_{k_i}$ for $i = 1, \ldots, m_{l+1}$ and condition (**) holds for $m = m_{l+1}$. According to the Bing theorem (ibidem, Theorem 1), there exist an integer $n_{l+1} > n_l$ and a consolidation $\boldsymbol{E}_{l+1} = (E_1, \ldots, E_{m_{l+1}})$ of $\boldsymbol{D}_{n_{l+1}}$ such that $\bar{E}_i \subset E'_{k_i}$ for $i = 1, \ldots, m_{l+1}$.

Using the chains $\boldsymbol{E}_1, \boldsymbol{E}_2, \ldots$ we define a mapping f of K onto C as follows. For any x belonging to K, let $J_n(x)$ denote the union of elements I_i of $\boldsymbol{I}_n$ such that x belongs to elements E_i of $\boldsymbol{E}_n$, having the same indices. Since $x \in E_i \cap E_j$ implies $|i-j| \leqslant 1$, by (*), and thus $I_i \cap I_j \neq 0$, by (**), the open subset $J_n(x)$ of C is the union of at most two elements of $\boldsymbol{I}_n$, which intersect. It follows that the diameter of $J_n(x)$ is less than $2/n$ for every $x \in K$ and $n = 1, 2, \ldots$ Moreover, if I_i is one of elements of $\boldsymbol{I}_{l+1}$ that form $J_{l+1}(x)$, then $x \in E_i \subset E'_{k_i}$, whence $I'_{k_i} \subset J_l(x)$, and so $I_i \subset J_l(x)$ as $I_i \subset I'_{k_i}$. Consequently, $J_{l+1}(x) \subset J_l(x)$ for every $x \in K$ and $l = 1, 2, \ldots$ Therefore $\bar{J}_1(x), \bar{J}_2(x), \ldots$ is a decreasing sequence of compact subsets of C whose diameters tend to zero. We define

$$f(x) = \bigcap_{n=1}^{\infty} \bar{J}_n(x).$$

Since every $y \in C$ belongs to some elements of each $\boldsymbol{I}_n$, the closures of the unions of elements of $\boldsymbol{E}_n$, having the same indices, as previously form a decreasing sequence of compact subsets of K (whose diameters need not tend to zero now). Hence they have at least one point x in common. We obtain $y \in J_n(x)$ for $n = 1, 2, \ldots$, which gives $y = f(x)$; the equality $C = f(K)$ follows.

Finally, for an arbitrary $\varepsilon > 0$, let us choose a positive integer n such that $4/n < \varepsilon$, and denote by $\eta > 0$ the Lebesgue number of the cover $\boldsymbol{E}_n$ of K. Then, for every points $x, x' \in K$ whose distance is less than η, the sets $J_n(x)$ and $J_n(x')$ intersect, whence their union has diameter less than or equal to $\delta[J_n(x)] + \delta[J_n(x')] < 4/n < \varepsilon$. But, as $f(x) \in \bar{J}_n(x)$ and $f(x') \in \bar{J}_n(x')$, the distance between $f(x)$ and $f(x')$ is less than ε. Consequently, f is a continuous mapping.

Corollary 1. *Every continuous image of a weakly chainable continuum is weakly chainable.*

Corollary 2. *Every locally connected continuum is weakly chainable.*

Corollary 3. *Every chainable continuum is a continuous image of the pseudo-arc* [4].

In 1930 Whyburn constructed a non-degenerate plane continuum W whose every non-degenerate subcontinuum cut the plane. Now we prove that there exists a weakly chainable plane continuum X like continuum W [5]. According to the theorem, the pseudo-arc K can therefore be mapped onto X, though no subcontinuum of K cuts the plane [6]. Moreover, X then contains only degenerate chainable continua [7].

Example 1. *A weakly chainable non-degenerate plane continuum whose every non-degenerate subcontinuum cuts the plane.*

The description of Example 1 requires introducing some notions of geometrical character. In construction we shall use special plane weak chains which will be shortly called *d-chains*. Namely, we first assume that a d-chain $\boldsymbol{D}$ is a weak chain whose every element, called a *link* of $\boldsymbol{D}$, is an open disk lying on the plane, i.e. a bounded subset of the plane, homeomorphic to the set of interior points of the circle. It is also assumed that every two disjoint links of a d-chain $\boldsymbol{D} = (D_1, \ldots, D_m)$ have disjoint closures, D_1 meets only D_2, D_m meets only D_{m-1}, and no three different links of $\boldsymbol{D}$ (links are different if their indices are different) have a point in common, i.e. the nerve of $\boldsymbol{D}$ is 1-dimensional. Further, we require that every link of $\boldsymbol{D}$ intersects at most three other links of $\boldsymbol{D}$, i.e. the nerve of $\boldsymbol{D}$ has ramification at most 3, and that the union of closures of all links of $\boldsymbol{D}$, different than some D_i, does not cut D_i. Any sequence $\boldsymbol{L} = (D_i, \ldots, D_j)$, where $i+1 < j$ and $D_i \cap D_j \neq 0$, is called a *loop* of the d-chain $\boldsymbol{D}$. Then the links D_i, D_j are called *end* links of the loop $\boldsymbol{L}$, and the links D_{i-1}, D_{j+1}—*adjacent* links of the loop $\boldsymbol{L}$. We assume on each pair $\boldsymbol{L}, \boldsymbol{L}'$ of different loops in a d-chain $\boldsymbol{D}$ that either one of them is contained in the other, or they are disjoint, i.e. no link of $\boldsymbol{L}$ meets any link of $\boldsymbol{L}'$. Moreover, we fix an orientation of the plane and assume that every loop $\boldsymbol{L}$ of a d-chain $\boldsymbol{D}$ must be "on the right side" of $\boldsymbol{D}$, which means that the sequence of links of $\boldsymbol{L}$, ordered by their indices in $\boldsymbol{D}$ and

(4) This has recently been proved in another way by Mioduszewski (this volume, p. 179).

(5) Perhaps, already Whyburn's original continuum W (see [3], p. 319) is weakly chainable, but to show this it seems necessary to have another definition of W, similar to the definition of X which is given below.

(6) A problem raised by Knaster (see Colloq. Math. 8 (1961), p. 139, Problem 324) is thus settled.

(7) This answers in the negative some of my question (ibidem 7 (1960), p. 109, Problem 289).

corresponding to the irreducible cycle in the nerve of $\boldsymbol{D}$ to which the end links of $\boldsymbol{L}$ belong, induces the positive orientation of the plane (see fig. 1, where a d-chain with two loops is presented). This completes the definition of d-chains.

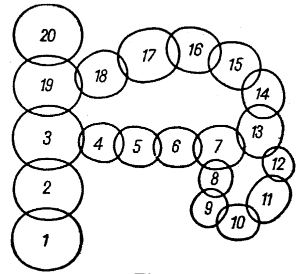

Fig. 1

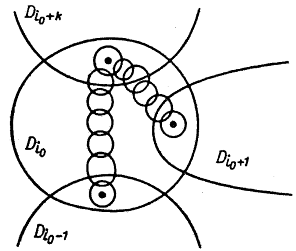

Fig. 2

Now we establish some operations on d-chains, which are needed in the construction of Example 1. Let $\boldsymbol{D} = (D_1, \ldots, D_m)$ be a d-chain; let us take an arbitrary point p_{ij} in the set $D_i \cap D_j$ for every pair of links D_i and D_j having a point in common. A *simple refinement* of $\boldsymbol{D}$ is understood to mean any refinement $\boldsymbol{D}'$ of $\boldsymbol{D}$ which is a d-chain and is obtained from $\boldsymbol{D}$ by replacing every link D_{i_0} of $\boldsymbol{D}$ by a chain in D_{i_0} (see fig. 2), successively joining the points $p_{i_0 j}$ (j may assume at most three values), so that the nerve of $\boldsymbol{D}'$ is a simplicial subdivision of the nerve of $\boldsymbol{D}$ the closure of every link of $\boldsymbol{D}'$ lies in a link and meets no 3 links of $\boldsymbol{D}$.

According to the Janiszewski theorem, the union of links of any loop $\boldsymbol{L}$ of a d-chain cuts the plane between some points p and q. Then we shall shortly write that *the loop* $\boldsymbol{L}$ *cuts* the plane between p and q. Similarly, we shall write that a set Y *is contained in the loop* $\boldsymbol{L}$ instead of that Y is contained in the union of links of $\boldsymbol{L}$. Suppose $\boldsymbol{D}$ and $\boldsymbol{D}'$ are d-chains with loops $\boldsymbol{L}$ and $\boldsymbol{L}'$, respectively, and $\boldsymbol{D}' \prec \boldsymbol{D}$. The loop $\boldsymbol{L}'$ is said to be *related* to the loop $\boldsymbol{L}$ if the end links of $\boldsymbol{L}'$ are contained in the end links of $\boldsymbol{L}$, respectively, and $\boldsymbol{L}'$ cuts the plane between every pair of points p and q such that $\boldsymbol{L}$ cuts the plane between p and q. We see that if $\boldsymbol{D}'$ is a simple refinement of $\boldsymbol{D}$, the relationship is a 1-1 correspondence between loops of $\boldsymbol{D}'$ and those of $\boldsymbol{D}$. In this case any loop of $\boldsymbol{D}'$ is, moreover, contained in the loop of $\boldsymbol{D}$ to which it is related.

Let $\boldsymbol{D}$ be a d-chain with loop $\boldsymbol{L} = (D_i, \ldots, D_j)$ and let n be a positive integer. We define a refinement $\Omega(\boldsymbol{D}, \boldsymbol{L}, n)$ of $\boldsymbol{D}$ as follows. Take a simple refinement $\boldsymbol{D}'$ of $\boldsymbol{D}$ such that the union of every two adjacent links of $\boldsymbol{D}'$ is contained in a link of $\boldsymbol{D}$ and has diameter less than $1/n$ and less than the distance between the sets $\overline{D}_{i-1}$ and $\overline{D}_j$. Then there exists a link

D'_h of $\boldsymbol{D}'$ such that $D'_h \subset D_i - (D_{i-1} \cup D_j)$ (see fig. 3). The d-chain $\Omega(\boldsymbol{D}, \boldsymbol{L}, n)$, being a refinement of $\boldsymbol{D}$, is obtained from $\boldsymbol{D}'$ by replacing the link D'_h by a chain $\boldsymbol{C}$ joining the links D'_{h-1} and D'_{h+1} so that the closure of the union of every two adjacent links of $\boldsymbol{C}$ is contained in

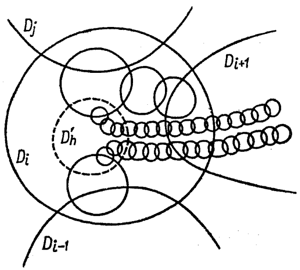

Fig. 3

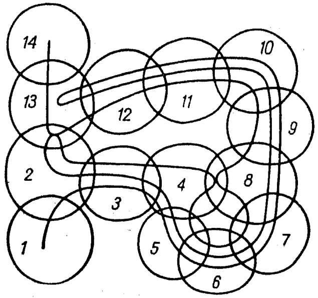

Fig. 4

a link of $\boldsymbol{L}$ and has diameter less than $1/n$, every link of $\boldsymbol{L}$ contains a link of $\boldsymbol{C}$, and the nerve of $\Omega(\boldsymbol{D}, \boldsymbol{L}, n)$ is a simplicial subdivision of the nerve of $\boldsymbol{D}'$ (more precisely, only the star of the vertex corresponding to D'_h is divided). Let us observe that if the loop $\boldsymbol{L}$ is *maximal* in $\boldsymbol{D}$, i.e. $\boldsymbol{L}$ is no proper subloop of any loop of $\boldsymbol{D}$, then each continuum contained in the union of links of $\Omega(\boldsymbol{D}, \boldsymbol{L}, n)$ and joining the adjacent links of $\boldsymbol{L}$ must intersect every link of $\boldsymbol{L}$ and both adjacent links of the loop of $\Omega(\boldsymbol{D}, \boldsymbol{L}, n)$, related to $\boldsymbol{L}$ (see fig. 4, where $\boldsymbol{D}$ has 14 links and $\Omega(\boldsymbol{D}, \boldsymbol{L}, n)$ is presented as a line). Moreover, the relationship of loops is a 1-1 correspondence between loops of $\Omega(\boldsymbol{D}, \boldsymbol{L}, n)$ and those of $\boldsymbol{D}$.

Denoting by $\boldsymbol{L}_1, \ldots, \boldsymbol{L}_m$ all the loops of $\boldsymbol{D}$, we define a refinement $\Omega(\boldsymbol{D}, n)$ of $\boldsymbol{D}$ by finite induction. Namely, let $\Omega_1 = \Omega(\boldsymbol{D}, \boldsymbol{L}_1, n)$ and suppose that Ω_k is defined. Let $\boldsymbol{L}'_{k+1}$ be the loop of Ω_k related to $\boldsymbol{L}_{k+1}$. We put $\Omega_{k+1} = \Omega(\Omega_k, \boldsymbol{L}'_{k+1}, n)$. Let $\Omega(\boldsymbol{D}, n) = \Omega_m$. So $\Omega(\boldsymbol{D}, n) \prec \boldsymbol{D}$ and $\Omega(\boldsymbol{D}, n)$ is a d-chain. Since each link of $\Omega(\boldsymbol{D}, n)$ is contained in a link of Ω_1 whose closure lies in a link of $\boldsymbol{D}$, the closure of the union of links of $\Omega(\boldsymbol{D}, n)$ is contained in the union of links of $\boldsymbol{D}$. Further, every link of $\Omega(\boldsymbol{D}, n)$ has diameter less than $1/n$ and every two adjacent links of $\Omega(\boldsymbol{D}, n)$ are contained in the same link of $\boldsymbol{D}$. Moreover, the relationship of loops is here also a 1-1 correspondence between loops of $\Omega(\boldsymbol{D}, n)$ and those of $\boldsymbol{D}$.

Except the operation Ω, just defined on d-chains having loops, another operation Π on d-chains is needed. $\boldsymbol{D}$ being a d-chain, we choose a point p_{ij} in $D_i \cap D_j$ for every pair of links D_i and D_j of $\boldsymbol{D}$ which have a point in common. The d-chain $\Pi(\boldsymbol{D}) \prec \boldsymbol{D}$ is obtained from $\boldsymbol{D}$ by

replacing every link D_i of $\boldsymbol{D}$ which intersects only D_{i-1} and D_{i+1} by a d-chain $\boldsymbol{D}_i$ joining the points $p_{i,i-1}$ and $p_{i,i+1}$ so that $\boldsymbol{D}_i$ has exactly one loop $\boldsymbol{L}_i$, every link of $\boldsymbol{L}_i$ is contained in $D_i - (D_{i-1} \cup D_{i+1})$ (see fig. 5) and any loop of the whole $\Pi(\boldsymbol{D})$ either is related to a loop of $\boldsymbol{D}$ or is one of the loops $\boldsymbol{L}_i$.

We now define the continuum X constituting Example 1. Let $\boldsymbol{E}_0$ be any d-chain with a loop. We define d-chains $\boldsymbol{E}_n$ for $n = 1, 2, \ldots$ inductively by putting

$$\boldsymbol{E}_n = \Pi[\Omega(\boldsymbol{E}_{n-1}, n)].$$

Hence every link of $\boldsymbol{E}_n$ has diameter less than $1/n$, $\boldsymbol{E}_n \prec \boldsymbol{E}_{n-1}$ and the closure F_n of the union of links of $\boldsymbol{E}_n$ is contained in $\boldsymbol{E}_{n-1}$ $(n = 1, 2, \ldots)$.

Obviously each F_n is a continuum and so

$$X = \bigcap_{n=1}^{\infty} F_n$$

is a non-degenerate plane continuum. Since $\boldsymbol{E}_{n+1}$ is a refinement of $\boldsymbol{E}_n$ $(n = 1, 2, \ldots)$, X is weakly chainable. It is thus sufficient to show that every non-degenerate subcontinuum C of X cuts the plane.

Let us choose a positive integer n_0 such that $1/n_0 < \delta(C)$. Then the continuum C is contained in none of the links of $\boldsymbol{E}_n$ for $n \geqslant n_0$. Moreover, if C is contained in a loop $\boldsymbol{L}$ of $\boldsymbol{E}_n$, where $n > n_0$, then $\boldsymbol{L}$ is related to a loop of $\boldsymbol{E}_{n_0}$, because if it were not so, $\boldsymbol{L}$ would be contained in a link of $\boldsymbol{E}_{n-1}$ and hence in a link of $\boldsymbol{E}_{n_0}$.

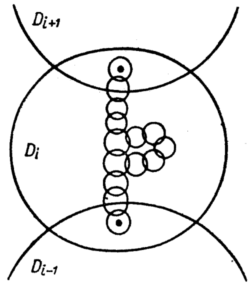

Fig. 5

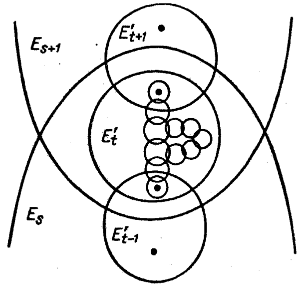

Fig. 6

We consider two cases:

Case 1. There exists an integer $n_1 \geqslant n_0$ such that the continuum C is contained in none of the loops of $\boldsymbol{E}_{n_1}$. Then there is a link E_s of $\boldsymbol{E}_{n_1}$ such that E_s belongs to no loop of $\boldsymbol{E}_{n_1}$ and C contains a point of E_s which belongs to no link of $\boldsymbol{E}_{n_1}$ different from E_s. The link E_s meets therefore at most two other links of $\boldsymbol{E}_{n_1}$, namely E_{s-1} and E_{s+1}. Since C is not contained in E_s, it has a point which belongs to one of the links E_{s-1} and E_{s+1}, to E_{s+1} say, and does not belong to E_s. Thus C meets $E_s - E_{s+1}$ and $E_{s+1} - E_s$. But, according to the definition of Ω, the operation Ω applied to the d-chain $\boldsymbol{E}_{n_1}$ transforms it into a d-chain differing from a simple refinement of $\boldsymbol{E}_{n_1}$ only inside the loops of $\boldsymbol{E}_{n_1}$. Hence the part of

$$\boldsymbol{E}' = \Omega(\boldsymbol{E}_{n_1}, n_1+1)$$

which lies in the link E_s coincides with that of a simple refinement, E_s belonging to no loop of $\boldsymbol{E}_{n_1}$. It follows that the part of C lying in $E_s - E_{s+1}$ is joined to the part of C lying in $E_{s+1} - E_s$ with some links of $\boldsymbol{E}'$ that form a chain. On the other hand, every two adjacent links of $\boldsymbol{E}'$ are contained in some link of $\boldsymbol{E}_{n_1}$. Hence there is a link E'_t of $\boldsymbol{E}'$ such that $E'_t \subset E_s \cap E_{s+1}$, E'_t intersects only the links E'_{t-1} and E'_{t+1} of $\boldsymbol{E}'$, adjacent to E'_t, E'_t belongs to no loop of $\boldsymbol{E}'$ and C meets both E'_{t-1} and E'_{t+1} (see fig. 6, where only four points of C are presented). But we have $\boldsymbol{E}_{n_1+1} = \Pi(\boldsymbol{E}')$ and, according to the definition of Π, there is a loop $\boldsymbol{L}_1$ of $\boldsymbol{E}_{n_1+1}$ wholly contained in $E'_t - (E'_{t-1} \cup E'_{t+1})$ (see fig. 6). So the loop $\boldsymbol{L}_1$ is maximal in $\boldsymbol{E}_{n_1+1}$ and the continuum C meets both adjacent links of $\boldsymbol{L}_1$.

Let $\boldsymbol{L}_k$ be the loop of $\boldsymbol{E}_{n_1+k}$ related to $\boldsymbol{L}_1$ ($k = 1, 2, \ldots$). Each loop $\boldsymbol{L}_k$ is thus maximal in $\boldsymbol{E}_{n_1+k}$. Suppose that C meets both adjacent links of $\boldsymbol{L}_k$; then, since the continuum $C \subset X$ is contained in

$$\boldsymbol{E}'' = \Omega(\boldsymbol{E}_{n_1+k}, \boldsymbol{L}_k, n_1+k+1),$$

C must intersect every link of $\boldsymbol{L}_k$ and both adjacent links of the loop of $\boldsymbol{E}''$ related to $\boldsymbol{L}_k$ (compare p. 278). Hence C must also intersect both adjacent links of the loop $\boldsymbol{L}_{k+1}$ of $\boldsymbol{E}_{n_1+k+1}$ related to $\boldsymbol{L}_k$, as $\boldsymbol{L}_k$ is maximal in $\boldsymbol{E}_{n_1+k}$.

We have thus proved by induction that C intersects every link of $\boldsymbol{L}_k$ for $k = 1, 2, \ldots$ Let p and q be points between which $\boldsymbol{L}_1$ cuts the plane. It follows that every loop $\boldsymbol{L}_k$ cuts the plane between p and q ($k = 1, 2, \ldots$). Take an arbitrary $\varepsilon > 0$ and a continuum V joining p and q on the plane. Then for k sufficiently large each link of $\boldsymbol{L}_k$ has diameter less than ε. But since V meets some link of every $\boldsymbol{L}_k$, V must have a point whose distance to a point of C is less than ε. Hence C and V intersect and it is shown that C cuts the plane between p and q.

Case 2. For every integer $n \geqslant n_0$ the continuum C is contained in a loop of $\boldsymbol{E}_n$. Let us denote by $\boldsymbol{L}_n$ the *minimal* loop of $\boldsymbol{E}_n$ containing C ($n \geqslant n_0$), i.e. such that C is contained in $\boldsymbol{L}_n$ but in no proper subloop of $\boldsymbol{L}_n$. Further, let $\boldsymbol{L}'_n$ denote the loop of $\boldsymbol{E}_{n_0}$ to which $\boldsymbol{L}_n$ is related ($n > n_0$). Since, obviously, each $\boldsymbol{L}'_n$ is a subloop of $\boldsymbol{L}'_{n+1}$ and $\boldsymbol{E}_{n_0}$ has only a finite number of loops, all loops $\boldsymbol{L}'_n$ coincide beginning at some integer $n_2 > n_0$. Therefore every loop $\boldsymbol{L}_n$ is related to $\boldsymbol{L}_{n_2}$ for $n \geqslant n_2$. Let us denote by $\boldsymbol{M}_n$ ($n \geqslant n_2$) the collection of the links of $\boldsymbol{L}_n$ corresponding to the vertices of the irreducible cycle in the nerve of $\boldsymbol{L}_n$ to which the end links of $\boldsymbol{L}_n$ belong. Then the union of links of $\boldsymbol{M}_{n_2}$ cuts the plane between some points p and q, and the union of links of $\boldsymbol{M}_n$ does the same for every $n \geqslant n_2$. Take an arbitrary $\varepsilon > 0$ and a continuum V joining p and q on the plane. Then for $n \geqslant n_2$ sufficiently large each link of $\boldsymbol{M}_n$ has diameter less than ε. Suppose that C intersects every link of $\boldsymbol{M}_n$ for $n \geqslant n_2$. Since V meets some link of every $\boldsymbol{M}_n$ ($n \geqslant n_2$), V must have a point whose distance to a point of C is less than ε. Consequently, C and V intersect, and so C cuts the plane between p and q.

We can, therefore, assume that there exists an integer $m \geqslant n_2$ and a link E_u of $\boldsymbol{L}_m$, belonging to $\boldsymbol{M}_m$, such that C and E_u are disjoint. Let E_v and E_w ($v+1 < w$) be end links of the loop $\boldsymbol{L}_m$. The continuum C is thus contained in the union of links of the d-chain

$$\boldsymbol{E} = (E_{u+1}, E_{u+2}, \ldots, E_w, E_v, E_{v+1}, \ldots, E_{u-1}),$$

but C is contained in none of the loops of $\boldsymbol{E}$, since $\boldsymbol{L}_m$ is the minimal loop of $\boldsymbol{E}_m$ containing C. In this way, Case 2 reduces to Case 1.

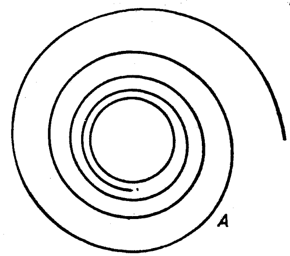

Fig. 7

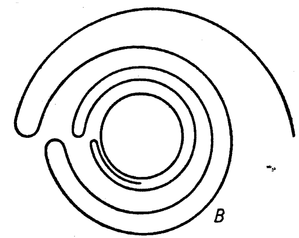

Fig. 8

Example 2. *A plane continuum that is not weakly chainable.*

As has been proved by Fort (see [1], p. 542), a plane continuum which does not cut the plane cannot be mapped onto the continuum Y which consists of two circumferences and of two spiral lines converging to them. More precisely, Y is the set of points having polar coordinates (r, θ) for which $r = 1$, $r = 2$ or $r = (2+e^\theta)/(1+e^\theta)$. The pseudo-arc being a plane continuum which does not cut the plane, it follows from the theorem (see p. 274) that Y is not weakly chainable.

Similarly, the continuum A (see fig. 7), which consists of a circumference and of a spiral line converging to it, is not weakly chainable. One can easily verify, however, that the continuum B (see fig. 8), which consists of a circumference and of another line converging to it, is weakly chainable.

Since A can be embedded into a square which is weakly chainable by Corollary 2, we see that, unlike chainable continua, a subcontinuum of a weakly chainable continuum needs not be weakly chainable.

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On a family of 2-dimensional AR-sets

by

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In the present note we construct a family consisting of $2^{\aleph_0}$ two-dimensional AR-sets (compact) such that none of them contains a 2-dimensional closed subset homeomorphic to a subset of any other set. We also give some applications of this family to the problem of existence of universal n-dimensional AR-sets and to the theory of r-neighbours.

1. Zone of a triangulation. Let Δ be a triangle lying in the Euclidean 3-space E^3 and let b_Δ denote the barycentre of Δ. For every positive ε, let us denote by $L(\Delta,\varepsilon)$ the segment perpendicular to the plane of the triangle Δ with length 2ε and centre b_Δ. By the *ε-zone* of the triangle Δ we understand the minimal convex subset of E^3 containing the sets Δ and $L(\Delta,\varepsilon)$. It will be denoted by $Z(\Delta,\varepsilon)$. Evidently $Z(\Delta,\varepsilon)$ is the union of two 3-dimensional simplexes having Δ as their common base and the endpoints of the segment $L(\Delta,\varepsilon)$ — as opposite vertices. The polytope $Z(\Delta,\varepsilon)$ is a neighbourhood of every point lying in the interior of the triangle Δ. The segment $L(\Delta,\varepsilon)$ is said to be the *axis* of the zone $Z(\Delta,\varepsilon)$.

Now let T be a triangulation of a polytope P. The union of all m-dimensional simplexes of T is said to be the *m-skeleton* of T. Evidently the polytope P is homogeneously n-dimensional if and only if it coincides with the n-skeleton of T. In this case we understand by the *boundary* of P the union $P^{\cdot}$ of all $(n-1)$-dimensional simplexes of T incident exactly to one n-dimensional simplex of T, and by the *edge* of P the set P^* of all points $x\in P$ such that no neighbourhood of x in P is homeomorphic to a subset of the Euclidean n-space E^n. Evidently $P^{\cdot}$ and P^* are unions of some simplexes of the triangulation T, but they do not depend on the choice of this triangulation.

Now let us consider a homogeneously 2-dimensional polytope $P\subset E^3$ with a triangulation T and let ε be a positive number. One easily sees that for ε sufficiently small the common part of the zones of different triangles of the triangulation T coincides with the common part of the boundaries of those triangles. A positive number ε satisfying this condition is said to be *suitable* for the triangulation T.