Generalized topologies for statistical metric spaces *

by

E. O. Thorp (University Park, N. M.)

1. Introduction. Statistical metric spaces, introduced by K. Menger in [5], are a generalization of metric spaces in which distances are given by distribution functions rather than by numbers (1). Just as with metric spaces, there is no a priori topology. But whereas metric spaces have a single natural topology, there are many structures, satisfying some or all of the axioms of a topology, that may be associated with a statistical metric space in a natural way. One such structure for statistical metric spaces was introduced by B. Schweizer and A. Sklar in [7] (2).

Our aims in this paper are as follows: (1) to generalize this topological structure; (2) to study various properties of and relations among the class of (generalized) topologies which are so obtained (3); (3) to establish sufficient conditions for the metrization of a subclass of these topologies, and, in so doing, to extend the metrization theorem of B. Schweizer, A. Sklar and E. Thorp [8]; and lastly, (4) to introduce a different method of obtaining topologies for SM spaces.

2. Statistical metric spaces.

DEFINITION 2.1. A *statistical metric space* (briefly, an SM space) is an ordered pair (S, F), where S is a set and F is a mapping from $S\times S$ into the set of distribution functions (i.e., real-valued functions of a real variable which are everywhere defined, nondecreasing, left-continuous and have inf 0 and sup 1). The distribution function $F(p, q)$ associated with a pair of points (p, q) in S is denoted by F_{pq}. It represents the probability

* This research was supported in part by the United States Air Force under contract No. AF-49(638)-42, monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

(1) This concept was further developed by B. Schweizer and A. Sklar in [6] and [7], by B. Schweizer, A. Sklar and E. Thorp in [8], and by E. Thorp in [9].

(2) The reader is referred to this paper for an introduction to the theory of statistical metric spaces, references, and background material. The author wishes to thank Professors Schweizer and Sklar for helpful conversations and useful suggestions.

(3) The author wishes to thank Professor R. Arens for suggesting the possibility of generalized topologies as a tool for treating SM spaces.

that the "distance" between p and q is less than x. The functions F_{pq} are assumed to satisfy:

(SM-I) $F_{pq}(x) = 1$ *for all* $x > 0$ *iff* $p = q$.

(SM-II) $F_{pq}(0) = 0$.

(SM-III) $F_{pq} = F_{qp}$.

(SM-IV) *If* $F_{pq}(x) = 1$ *and* $F_{qr}(y) = 1$, *then* $F_{pr}(x+y) = 1$.

Hereafter, when we use the term "distribution function", we mean only those distribution functions F that satisfy $F(0) = 0$, i.e. that are possible distribution functions for some SM space. It will be convenient for the sequel to define the distribution function H by $H(0) = 0$, $H(x) = 1$, $x > 0$.

We will have occasion to refer to the following particularly "simple" kind of SM space.

DEFINITION 2.2. A simple space [7] is an SM space which can be generated from a metric space (S, d) and a single distribution function G by the correspondence $F_{pq}(x) = G(x/d(p, q))$ when $p \neq q$ and $F_{pp}(x) = H(x)$.

DEFINITION 2.3. A real-valued function T, whose domain is the set of real number pairs (x, y) such that $0 \leqslant x,\ y \leqslant 1$, is called a *t-norm* ([4]) if it satisfies:

(T-I) $T(a, 1) = a$, $T(0, 0) = 0$.

(T-II) $T(c, d) \geqslant T(a, b)$ if $c \geqslant a$, $d \geqslant b$ (monotonicity).

(T-III) $T(a, b) = T(b, a)$ (commutativity).

(T-IV) $T[T(a, b), c] = T[a, T(b, c)]$ (associativity).

DEFINITION 2.4. The t-norm T_1 is *weaker* than the t-norm T_2, and we write $T_1 \leqslant T_2$, if $T_1(x, y) \leqslant T_2(x, y)$ for all x and y such that $0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1$; T_1 is *strictly weaker* than T_2 if $T_1 \leqslant T_2$ and if $T_1(x, y) < T_2(x, y)$ for at least one pair (x, y).

There is a weakest t-norm, which we denote by T_w. It is given by $T_w(1, x) = T_w(x, 1) = x$, $0 \leqslant x \leqslant 1$; $T_w(x, y) = 0$ if $0 \leqslant x < 1$ and $0 \leqslant y < 1$.

DEFINITION 2.5. A *Menger space* (S, F, T) is an SM space (S, F) and a t-norm T such that the triangle inequality,

(SM-IVm) $F_{pr}(x+y) \geqslant T(F_{pq}(x), F_{qr}(y))$,

holds for all points p, q, r in S and for all numbers $x, y \geqslant 0$.

We often find it convenient to work with the tails of the distribution functions rather than with these distribution functions themselves. The *tail* of F_{pq}, which we denote by G_{pq}, is defined by $G_{pq}(x) = 1 - F_{pq}(x)$, for each x.

([4]) Further information on t-norms is available in [9], where they are called t-functions.

3. A class of generalized topologies for SM spaces. The concept of neighborhood for SM spaces was introduced in [7], definition 7.1:

Let p be a point in the SM space (S, F) and let u and v be positive numbers. The (u, v) neighborhood of $N_p(u, v)$ is $\{q \text{ in } S : F_{pq}(u) > 1 - v\} = \{q \text{ in } S : G_{pq}(u) < v\}$.

We will refer to $N_p(u, v)$ as the (u, v) *sphere with center* p. For fixed positive u and v, we define $U(u, v) = \{(p, q) \text{ in } S \times S : G_{pq}(u) < v\}$. Note that with these definitions of $N_p(u, v)$ and $U(u, v)$, two points are "near" each other if the tail of their corresponding distribution is "small". For any set X of ordered pairs of positive numbers, set $\mathcal{N}(X) = \{N_p(u, v) : (u, v) \text{ in } X,\ p \text{ in } S\}$ and $\mathcal{U}(X) = \{U(u, v) : (u, v) \text{ in } X\}$. This relativizes the neighborhood concept introduced in [7]. The latter results when X is the positive quadrant.

Topological spaces and uniform structures are a natural setting for metric spaces [4]. Generalizations of these notions, as defined and studied by A. Appert and Ky Fan in [1] ([5]), turn out to be a natural tool for studying SM spaces. From this point on, whenever a familiar concept is replaced by a generalized one, we prefix the familiar term by "g-", e.g. "generalized topology" is written "g-topology".

A non-empty family $\{N_p\}$ of subsets N_p of a set S associated with a point p of S is a *family of neighborhoods* ([6]) for p if each N_p contains p. Let a family of neighborhoods be associated with each point p of a set S. The set S and the collection of neighborhoods is a *g-topological space of type V*. The *closure* of a subset E of S, written $\bar{E}$, is the set of points p such that each neighborhood of p intersects E. The *interior* of E is the complement of the closure of the complement of E. A g-topological space is *symmetric* if, for every pair of points p and q, p is in $\overline{\{q\}}$ iff q is in $\overline{\{p\}}$ ([1], p. 62). The table below shows how the various g-topological spaces which we use are related.

Type V — topological space: type V_D = N + N1 + N2; type V_α = N + N1.

N.

N1. For each point p and each neighborhood U_p of p, there is a neighborhood W_p of p such that, for each point q of W_p, there is a neighborhood U_q of q contained in U_p ([1], p. 17, condition α'').

N2. For each point p and each pair of neighborhoods U_p and W_p of p, there is a neighborhood of p contained in the intersection of U_p and W_p.

([5]) The structures that these authors call topological spaces are considerably more general than those structures, to which the term is usually applied. Since there is general agreement in other circles as to the use of the term topological space, we refer to the Appert-Ky Fan structure as a generalized topology.

([6]) This latter definition is identical with that of the neighborhoods defined in [7].

The (T_0), (T_1), and (T_2) separation axioms for generalized topologies are, in the neighborhood terminology, the same as those for topological spaces. For symmetric g-topologies, (T_0) and (T_1) are evidently equivalent. A g-topology τ' on a set S is finer than a g-topology τ on S if, for each point p of S, each τ neighborhood of p contains a τ' neighborhood of p.

A uniformity [10] for a set S is a non-empty family $\mathcal{U}$ of subsets of $S\times S$ such that:

(U-I) *Each member of* $\mathcal{U}$ *contains the diagonal*, $\Delta = \{(x, x)\colon x \text{ in } S\}$.

(U-II) *If* $\mathcal{U}$ *contains* U, *then* $\mathcal{U}$ *contains* $U^{-1} = \{(x, y)\colon (y, x) \text{ in } S\}$ (symmetry).

(U-III) *If* $\mathcal{U}$ *contains* U, *then* $\mathcal{U}$ *contains some* V *such that* $V\circ V\subset U$, *where* $V\circ V = \{(x, z)\colon$ *for some* y, (x, y) *and* (y, z) *are in* $V\}$.

(U-IV) *If* $\mathcal{U}$ *contains* U *and* V, *it contains* $U\cap V$.

(U-V) $\mathcal{U}$ *is closed under the formation of supersets.*

A *generalized uniformity* (g-uniformity) is a family of subsets of $S\times S$ that satisfies (U-I). A g-uniformity $\mathcal{U}'$ for a set S is finer than a g-uniformity for S if each member of $\mathcal{U}$ contains a member of $\mathcal{U}'$. Given a g-uniformity $\mathcal{U}$ for a set S, for each U in $\mathcal{U}$ and p in S, let $N_p(U) = \{q \text{ in } S\colon (p, q) \text{ in } U\}$. The collection $\{N_p(U)\colon p \text{ in } S,\ U \text{ in } \mathcal{U}\}$ is the *g-neighborhood system* $\mathcal{N}(\mathcal{U})$ *associated with the g-uniformity* $\mathcal{U}$.

A collection $\{U\}$ of subsets of $S\times S$ is separated if $\bigcap U$ is the diagonal, Δ. We will need the fact that, in the presence of (U-V), the following weaker axiom is equivalent to (U-IV):

(U-IV′) *If* $\mathcal{U}$ *contains* U *and* V, *it contains a non-empty subset of* $U\cap V$.

In particular, a collection $\mathcal{U}$ which satisfies either (U-I)-(U-IV) or (U-I)-(U-IV′) is a base for a uniformity.

The next theorem results immediately from checking the definitions.

THEOREM 3.1. *If* (S, F) *is an SM space and* X *and* S *are not empty, then*: $\mathcal{N}(X)$ *is the collection of neighborhoods for a symmetric g-topological space of type* V; $\mathcal{U}(X)$ *is a symmetric* (⁷) *g-uniformity*; $\mathcal{N}(X)$ *is the neighborhood system associated with* $\mathcal{U}(X)$.

Theorem 3.1 permits the following definition.

DEFINITION 3.2. The g-topology for an SM space that corresponds to a fixed subset X of $\{(u, v)\colon u, v>0\}$ is called the *X g-topology*, and is denoted by $\tau(X)$.

The strong g-topology of [7] is obtained when X is the entire positive quadrant.

(⁷) A space of type V may be symmetric without the associated generalized uniformizing family being symmetric, but not conversely.

Remark. Since X g-topologies for SM spaces are symmetric, the (T_0) and (T_1) separation axioms are equivalent.

Even the strong g-topology on an SM space need not be a topology as the following example shows.

EXAMPLE 3.3. Consider the SM space consisting of the points in the plane and the following distribution functions: Let $F_{pq}(x) = H(x-1)$ if $d(p, q)\geqslant 1$ (d is the euclidean distance) or if the slope of the line joining p and q is irrational. If $d(p, q)<1$ and the line joining p and q has rational slope, let $F_{pq}(x) = 1-d(p, q)$ when $0<x\leqslant 1$ and 1 when $x>1$. The axioms (SM-I)-(SM-IV) are readily verified. Every strong g-topology neighborhood of a point p contains a strong g-topology neighborhood of the form $S_p(n)$, $n = 1, 2, \dots$, where $S_p(n)$ is the set of all points q in the plane such that $d(p, q)<1/n$ and the line joining p and q has rational slope. Thus the subcollection $\{S_p(n)\}$, $n = 1, 2, \dots$, is a family of neighborhoods for the strong g-topology. Using these families of neighborhoods, it is readily seen that N1 fails and N2 is valid. Hence, the space is of type V_D but is not topological.

Remark. The g-topology in example 3.3 is (T_2), symmetric, of denumerable character, and of type V_D, but it is not of type V_α. No example of a g-topological space with all these properties seems to have been given previously.

THEOREM 3.4. *The g-uniformity* $\mathcal{U}(X)$ *is separated (implying* $\tau(X)$ *is* (T_1)) *iff for each pair* p, q *of distinct points in* S *there is a* (u, v) *in* X *such that* $v\leqslant G_{pq}(u)$.

Proof. We have $v\leqslant G_{pq}(u)$ for some (u, v) in X iff (p, q) is not in $U(u, v)$ for some (u, v) in X, and this is so iff (p, q) is not in $\bigcap U$.

COROLLARY 3.5. *The strong g-topology is* (T_1) *and the associated g-uniformity is separated.*

In contrast, the next example shows that the (T_2) (Hausdorff) separation axiom can fail for the strong g-topology, even if the SM space involved is a Menger space.

EXAMPLE 3.6. Let the set S consist of the positive integers and two other points, a and b. Let $F_{an}(x)$ and $F_{bn}(x)$ have the values: nx if $0\leqslant x\leqslant 1/2n$; $1/2+(4/\pi)\tan^{-1}(nx-1/2)$ if $1/2n<x\leqslant 1$; and 1 if $x>1$. Let $F_{mn}(x)$, $m\neq n$, and $F_{ab}(x)$ be $H(x-1)$. Of course, $F_{pp} = H$ and $F_{pq} = F_{qp}$ for all p and q. Direct reference to the axioms shows that (S, F, T_w) is a Menger space. In particular, (SM-IVm) is true because if p, q, r is an arbitrary triple of points, and x, y is a fixed pair of numbers, either $F_{pq}(x)$ and $F_{qr}(y)$ are less than 1, in which case the right hand side of (SM-IVm) is 0, or one of $F_{pq}(x)$ and $F_{qr}(y)$ is 1. Suppose that $F_{pq}(x)$ is 1. Then $x>1$ so that $x+y>1$, whence $F_{pr}(x+y)$ equals 1.

The (u, v) spheres are a neighborhood basis for a topology ([2], I, § 1, n° 2). The topology is not (T_2) because a and b do not have disjoint neighborhoods.

In example 3.6 we showed that the strong g-topology of a Menger space having T_w as t-norm need not be Hausdorff. Theorem 3.7 provides a sifficient condition, valid even for monotonic functions T, that the strong g-topology be Hausdorff.

THEOREM 3.7. *If (S, F) satisfies (SM-IVm) with a monotonic function T (not necessarily a t-norm), then the strong g-topology is Hausdorff if, for each pair (p, q), of distinct points,*

$$\sup_{t<1} T(t, t) > \inf_{x>0} F_{pq}(x) .$$

Proof. Suppose that (S, F) is not Hausdorff. Then there are two points, p and q, and a sequence of points $\{m\}$, such that $\lim_m F_{qm}(x)$ is 1 for each fixed positive x. By hypothesis (2), we can choose numbers $x, y > 0$ and $t < 1$ such that $F_{pq}(x+y) < T(t, t)$. On the other hand, for sufficiently large m, $F_{pm}(x)$ and $F_{qm}(y)$ both exceed t. Hence, by (SM-IVm) and (T-II), $F_{pq}(x+y) \geqslant T\big(F_{pm}(x), F_{mq}(x)\big) \geqslant T(t, t)$. This is a contradiction.

COROLLARY 3.8. *If (S, F, T) is a Menger space such that $T > T_w$ and the F_{pq} are right-continuous (and hence continuous) at 0, then the strong g-topology is Hausdorff.*

DEFINITION 3.9. If X and X' are subsets of the plane, we write $X \geqslant X'$ and say that X is *finer than* X' (and X' *coarser than* X) iff for each point (a', b') of X' there exists a point (a, b) of X such that $a \leqslant a'$ and $b \leqslant b'$. If both $\geqslant$ and $\leqslant$ hold, we write $=$ (equivalent to). If $\geqslant$ is true and $\leqslant$ is false, we write $>$.

THEOREM 3.10. *For every SM space: If $X \geqslant X'$, then $\mathcal{U}(X) \geqslant \mathcal{U}(X')$ and $\tau(X) \geqslant \tau(X')$. If $X = X'$, then $\mathcal{U}(X) = \mathcal{U}(X')$ and $\tau(X) = \tau(X')$. A sufficient (but not necessary) condition for $X > X'$ to imply $\mathcal{U}(X) > \mathcal{U}(X')$ and hence $\tau(X) > \tau(X')$, is that there exist p and q in S and (u, v) in X such that, simultaneously for all (u', v') in X', $G_{pq}(u) \geqslant v$ and $G_{pq}(u') \leqslant v'$.*

Proof. The first two statements are immediate consequences of the definitions. To prove sufficiency in the last statement note that if the G_{pq} and $(u, v) = x$ of the hypothesis exist, $U(x)$ does not contain (p, q) but (p, q) is in $\bigcap \mathcal{U}(X') \supset U(x)$.

Example 3.11 shows the condition is not necessary.

EXAMPLE 3.11. Let $S = \{1, 2, 3\}$, $X = \{(\frac{1}{4}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{2}), (\frac{3}{4}, \frac{3}{4})\}$, $X' = \{(\frac{1}{4}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{2})\}$. $F_{12}(x) = \frac{1}{3}, 0 < x \leqslant 1$. $F_{13}(x) = \frac{1}{5}$, $0 < x \leqslant \frac{1}{3}$; $F_{13}(x) = F_{23}$. $F_{ij}(x) = 1$, $1 < x$.

COROLLARY. *The strong g-topology and g-uniformity are finer than any other $\tau(X)$ and $\mathcal{U}(X)$, respectively.*

If X is directed, i.e., given x_1, x_2 in X there is an x in X such that $x \geqslant x_1$ and $x \geqslant x_2$, then $U(x) \subset U(x_1) \cap U(x_2)$ and $\mathcal{U}(X)$ satisfies (U-IV′). If X is not directed, Example 3.12 shows that (U-IV′) may still hold.

EXAMPLE 3.12. $S = \{1\}$. $X = \{(\frac{1}{4}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{2})\}$.

Theorem 3.10 shows that, for a given SM space, the relation "X_1 induces the same g-topology as X_2" is an equivalence relation for the subsets of the positive quadrant of the plane. If $[X]$ is the equivalence class containing a given X, we choose the canonical representative X^* to be the largest member of the class $[X]$. It is given by: $X^* = \{(u^*, v^*)\colon u \geqslant u, v^* \geqslant v$, for some (u, v) in $X\}$.

The directed subsets divide into four equivalence classes. If (u_0, v_0) refers to a fixed pair of numbers, their canonical representatives are:

(D-I) $\{(u, v)\colon u \geqslant u_0 > 0;\ v \geqslant v_0 > 0\}$,
(D-II) $\{(u, v)\colon u > u_0 \geqslant 0;\ v \geqslant v_0 > 0\}$,
(D-III) $\{(u, v)\colon u \geqslant u_0 > 0;\ v > v_0 \geqslant 0\}$,
(D-IV) $\{(u, v)\colon u > u_0 \geqslant 0;\ v > v_0 \geqslant 0\}$.

For those among these four types of sets which exist for a fixed (u_0, v_0), the corresponding topologies are ordered as follows: I $\geqslant$ (II, III) $\geqslant$ IV. Note that the strong g-topology is of type IV with $u_0 = v_0 = 0$. It is evident that $\mathcal{U}(X)$ has a countable base whenever X is equivalent to any one of the four types of sets.

THEOREM 3.13. *Suppose that for a given SM space, (SM-IVm) holds for a monotonic function T. Then $\mathcal{U}(X)$ satisfies (U-III) in each of the following cases:*

1. *For* $x > 1-v_0$, $T(x, x) > 1-v_0$; $X = \{(u, v)\colon v \geqslant v_0 > 0\}$ ((D-II) with $u_0 = 0$).

2. $\sup_{t>v_0} T(1-t, 1-t) = 1-v_0$; $X = \{(u, v)\colon v > v_0 \geqslant 0\}$ ((D-IV) with $u_0 = 0$).

Proof. Given $U(2u, v)$, we establish the theorem under each hypothesis by showing that $U(u, v_0) \circ U(u, v_0) \subset U(2u, v)$.

Using the first hypothesis: By (SM-IVm), $F_{pr}(2u) \geqslant T\big(F_{pq}(u), F_{qr}(u)\big)$. If (p, q) and (q, r) are in $U(u, v_0)$, then $F_{pq}(u)$ and $F_{qr}(u)$ each exceed $1-v_0$, and hence by the hypothesis, $T\big(F_{pq}(u), F_{qr}(u)\big) > 1-v_0$. Therefore (p, r) is in $U(2u, v_0)$, hence in $U(2u, v)$.

Using the second hypothesis, we can choose v' such that $v' > v_0$, yet $T(1-v', 1-v') > 1-v$. Then, if (p, q) and (q, r) are in $U(u, v')$, $F_{pr}(2u) \geqslant T\big(F_{pq}(u), F_{qr}(u)\big) \geqslant T(1-v', 1-v') > 1-v$ so (p, r) is in $U(2u, v)$.

THEOREM 3.14. *The X g-topology for an SM space satisfying the hypotheses of theorem 3.13 is pseudo-metrizable. The X g-topology is metrizable if, in addition, $\mathcal{U}$ is separated.*

Proof. By theorems 3.1 ((U-I), (U-II)), 3.13 ((U-III)), the discussion preceding example 3.12 ((U-IV′)), and the discussion preceding theorem 3.13 (countable base), $\mathcal{U}(X)$ is a basis for a uniform structure with a countable base. Therefore ([4], pp. 174-180) the space is pseudo-metrizable, and is metrizable iff $\mathcal{U}$ is separated.

The main result of [8] follows as a corollary.

COROLLARY 3.15. *A Menger space with the strong g-topology is metrizable if* $\sup_{x<1} T(x,x)=1$.

Proof. The hypotheses of theorem 3.14 and corollary 3.5 are satisfied.

For particular SM spaces, e.g. simple spaces, more can often be proven.

THEOREM 3.16. *In a simple space with any X g-topology, $N_p(u,v)$ is an ordinary spherical neighborhood of p in the generating M-space. Conversely, any spherical neighborhood of p in the generating M-space has the form $N_p(u,v)$ for some choice of (u,v) if there is a point $(u',v')\leqslant X$ such that $G(u')=1-v'<1$ and $t>u'$ implies $G(t)>G(u')$.*

Proof. The first assertion holds because q is in $N_p(u,v)$ iff $F_{pq}(u)=G\big(u/d(p,q)\big)>1-v$ iff $d(p,q)<k_0$ for some k_0 depending on G, u and v ([7], theorem 7.1).

To prove the second assertion, consider the ordinary spherical neighborhood $S_p(k)=\{q\colon d(p,q)<k\}$. Due to the left continuity of the distribution function G, there always is (u',v') satisfying the second half of our hypothesis. However, in the general case, (u',v') may not be $\leqslant X$. Hence the first half of the hypothesis is necessary. Now choose $u=ku'$ and $v=v'$. Then q is in $S_p(k)$ iff $d(p,q)<k$ iff $u/d(p,q)>u/k$ iff $G\big(u/d(p,q)\big)>G(u/k)=1-v$ iff $F_{pq}(u)>1-v$.

COROLLARY 3.17. *Simple spaces with the strong g-topology are metrizable with the metric of the generating space.*

Proof. The strong g-topology satisfies the conditions of the theorem for any G. Hence the collection of $u-v$ spheres is identical with the collection of metric spheres.

4. The *g*-écart *g*-topology. In this section we study a g-topology quite different from the X g-topologies. We develop this g-topology by exploiting certain structural similarities between the non-negative real numbers and the set of distribution functions. We assume that the SM spaces which we consider consist of more than one point.

Let S be a set and P be a partially ordered ($<$) set with least element 0. A *generalized écart* (g-écart) is a mapping G from $S\times S$ into P (compare [1] p. 138). If a g-écart G satisfies $G(p,p)=0$ and the set S consists of more than one point, the *g-écart g-topology* for S is the g-topology determined from G, and its partially ordered range set P, as follows.

For each $f>0$ in P and each p in S, the *f-sphere for p* is $N_p(f)=\{q \text{ in } S\colon G(p,q)<f\}$. Then for each p in S the collection of f-spheres, $\mathcal{N}_p(P)=\{N_p(f)\colon f>0 \text{ and in } P\}$, is a family of neighborhoods for p.

A *generalized metric* (g-metric) for a set S is a g-écart G for which the following axioms hold ([1], p. 138).

(GM-I) $P\neq\{0\}$.

(GM-II) *For each pair f, $g>0$, there is an $h>0$ such that $h\leqslant f$ and $h\leqslant g$.*

(GM-III) *For p,q in S, $G(p,q)=0$ if and only if $p=q$.*

(GM-IV) *For each $f>0$ there is a $g>0$ such that: $G(p,q)<g$ implies $G(q,p)<f$ for all p,q in S.*

(GM-V) *For each $f>0$ there is a $g>0$ such that: $G(p,q)<g$ and $G(q,r)<g$ implies $G(p,r)<f$ for all p,q,r in S.*

To keep the analogy with metric space methods as close as possible, we generally will work with the tails, G_{pq}, rather than the F_{pq}. We partially order ($<$) the tails as follows. If f and g are two tails, $f<g$ means $f(x)\leqslant g(x)$ for all x, with inequality for at least one x. The least element under this ordering is the tail of H, which we will denote by L. These considerations permit us to make the following definitions.

DEFINITION 4.1. The *g-écart associated with an* SM *space* (S,F) is the mapping G defined by $G(p,q)=G_{pq}$.

THEOREM 4.1. *For any* SM *space, the associated g-écart satisfies* (GM-I)-(GM-IV).

Proof. (GM-I) is obvious; (GM-II) follows from the fact that the sup of two distribution functions is again a distribution function—so that the inf of two tails is again a tail; (GM-III) is equivalent to (SM-I); (SM-III) implies (GM-IV).

The axiom (GM-V) introduces an interrelationship between the distribution functions of various pairs of points. It is similar in this respect to (SM-IVm). It is therefore to be expected that conditions involving (SM-IVm) will be useful in establishing (GM-V).

THEOREM 4.3. *If* (S,F) *is an* SM *space and* (SM-IVm) *holds under a function T satisfying* (T-I), (T-II) *and* $\sup_{x<1} T(x,x)=1$, *then G is a generalized metric.*

Proof. It suffices to establish (GM-V). Given a tail $f>L$, we wish to find a tail $g>L$ such that $G_{pq}<g$ and $G_{qr}<g$ implies $G_{pr}<f$ for all p,q,r, in S. By (SM-IVm) and (T-II), $F_{pr}(2x)\geqslant T\big(F_{pq}(x),F_{qr}(x)\big)\geqslant T\big(1-g(x),1-g(x)\big)$. Hence, it suffices to find a g such that $T\big(1-g(x),1-g(x)\big)\geqslant 1-f(2x)$, with inequality for some x.

Since $f > L$, there exists $a > 0$ such that $1-f(2a) < 1$. Let $g(x) = 0$, $x > a$. By hypothesis there is a number $b < 1$ such that $T(b, b) > 1-f(2a)$. Let $g(x) = 1-b$ for $0 < x \leqslant a$. Then, if $x > a$, $T\big(1-g(x), 1-g(x)\big) = T(1, 1) \geqslant 1-f(2x)$; and if $x \leqslant a$, then $T\big(1-g(x), 1-g(x)\big) = T(b, b) > 1-f(2a) \geqslant 1-f(2x)$. Thus $T\big(1-g(x), 1-g(x)\big) \geqslant 1-f(2x)$ for all x, with the inequality for $0 < x \leqslant a$.

THEOREM 4.4. *If (S, F) is an SM space, and there is a tail $g > L$ such that $g \leqslant G_{pq}$ for all distinct p and q in S, then G is a generalized metric and the g-écart g-topology is discrete.*

Proof. For each $f > L$, choose the g of the hypothesis to play the role of the g in the statement of (GM-V). Then if $G_{pq} < g$ and $G_{qr} < g$, our hypothesis yields $p = q = r$. Thus $G_{pr} = L < f$ and (GM-V) is verified. Since the g-sphere with center p consists of $\{p\}$, the g-topology is discrete.

Consider the family $\mathcal{U} = \{U(f)\colon f > L\}$, where $U(f) = \{g\colon g < f\}$. This family determines a g-uniformity, for it is easy to see, as in the proof of theorem 4.2, that all the uniformity axioms are satisfied except, perhaps, (U-III). This g-uniformity is separated, and it has a countable base. One such base is given by the collection $U(f_n)$, where f_n is defined by $f_n(x) = 0$, $x > 1/n$; $f_n(x) = 1/n$, $0 < x \leqslant 1/n$. Further, the g-topology obtained from this g-uniformity is the g-écart g-topology. These considerations, and the fact that (GM-V) implies (U-III) for this g-uniformity, yield the following theorem.

THEOREM 4.5. *The g-écart g-topology is metrizable if* (GM-V) *holds. In particular, the g-écart g-topology is metrizable if the hypotheses of either theorem* 4.3 *or theorem* 4.4 *are satisfied.*

THEOREM 4.6. *The g-écart g-topology is finer than the strong g-topology.*

Proof. Given a strong neighborhood $N_p(u, v)$ of a point p, it suffices to find a g-écart neighborhood inside. Let $g(x) = v$, $0 < x \leqslant u$; $g(x) = 0$, $x > u$. Then $G_{pq} < g$ implies $G_{pq}(u) < v$ so q is in $N_p(u, v)$ and $N_p(g)$ is the desired g-écart neighborhood.

We next see, as a consequence of several simple observations, that the g-écart and strong g-topologies are in general distinct.

The following lemma, stated without proof, is readily verified.

LEMMA 4.7. *A point p in S is not isolated iff $\{G_{pq}\colon q \neq p\}$ is cofinal* [4].

COROLLARY 4.8. *The g-écart g-topology is discrete iff, for each p, $\{G_{pq}\colon q \neq p\}$ is not cofinal. The écart g-topology for simple spaces is discrete.*

Proof. $\inf_{x>0} G(x) < 1$.

COROLLARY 4.9. *The g-écart g-topology and the strong g-topology need not be identical.*

Proof. By Corollaries 3.17 and 4.8, the two topologies are distinct for any simple space generated by a non-discrete metric.

It has been observed ([7], p. 315) that any given metric space (S, d) can be regarded as an SM space (S, F) of a special kind if F_{pq} is defined for $p \neq q$ by $F_{pq}(x) = H\big(x-d(p, q)\big)$. With this definition of F_{pq}, the strong g-topology is identical with the metric topology. However, the g-écart g-topology is discrete. In the next theorem, we show that it is possible to define the F_{pq} in such a way that the g-écart g-topology is identical with the metric topology.

We need a simple lemma whose proof will be omitted.

LEMMA 4.10. *A collection of tails is cofinal iff it contains a sequence $\{g_n\}$ such that $\{\big(M(g_n), m(g_n)\big)\}$ converges to $(0, 0)$, where $m(g) = \sup_{x>0} g(x)$ and $M(g) = \inf\{x\colon g(x) = 1\}$.*

THEOREM 4.11. *For each metric space (M, d) there is an* SM *space (M, f) on the same set, such that the metric topology, the g-écart g-topology, and the strong g-topology are identical.*

Proof. For each pair of distinct points p and q in M, let $G_{pq}(x) = \min\big(1, d(p, q)\big)$ if $0 < x \leqslant d(p, q)$ and $G_{pq}(x) = 0$ if $x > d(p, q)$. Since d is a metric, (SM-IV) follows from the metric triangle inequality. We show that the metric neighborhood system of a point p is a base for the g-écart g-neighborhood system. Evidently, p is isolated in (M, F) iff it is isolated in (M, d). On the other hand, p is not isolated in (M, F) iff the collection $\{G_{pq}\colon p \neq q\}$ is cofinal (lemma 4.7) iff the G_{pq} spheres are a base for the g-écart g-neighborhood system at p. But the above collection of G_{pq} spheres is in 1-1 correspondence with the set of metric spheres, $\{r\colon d(p, r) < d(p, q)\}$, i.e., for each pair of distinct points p and q, $\{r\colon G_{pr} < G_{pq}\} = \{r\colon d(p, r) < d(p, q)\}$. This collection of metric spheres is a neighborhood basis for the metric topology. Thus the metric topology and the g-écart topology are identical.

The strong g-topology is readily seen to agree with the metric topology. This concludes the proof.

5. Other g-topologies generated by distribution functions.

Other generalized topologies, whose neighborhood systems are generated by the distribution functions, can be constructed by somehow restricting the generating distribution functions to a subset of the total. This leads to a generalization of the g-écart g-topology that is analogous to the generalization of the strong g-topology studied in section 3 (X g-topologies).

We might generalize still further by letting the subset of the distributions that generate the neighborhoods of the points in the space vary with the point p. One g-topology which illustrates this further generalization is sketched below.

DEFINITION 5.1. Given an SM space (S, F), for each pair of points p and r in S, the r-sphere with center p, $N_p(r)$, is defined to be the sphere $N_p(G_{pr}) = \{q\colon G_{pq} < G_{pr}\}$. The R g-topology for (S, F) is the structure whose family of neighborhoods at each point p is the collection $\{N_p(r)\colon r \text{ in } S\}$.

An immediate consequence of our definition is the following theorem:

THEOREM 5.2. *The R g-topology is coarser than the g-écart g-topology.*

It can be shown readily that the R g-topology is never coarser than all the X g-topologies. Further, examples can be constructed in which the R g-topology is strictly finer than, equal to, strictly less than, or incomparable with, any preassigned X g-topology. To illustrate, we give an example in which the R g-topology is strictly finer than the strong g-topology.

EXAMPLE 5.3. Let S be the non-negative integers. If m and n are distinct, let $F_{mn}(x) = \frac{1}{2} + (4/\pi)\tan^{-1}\big((m+n)x - \frac{1}{2}\big)$ when $x > \frac{1}{2}$. Let $F_{mn}(x) = (m+n)x$ if $0 \leqslant x \leqslant (m+n)/2$ and $(m, n) \neq (0, 1)$. Let $F_{01}(x) = \frac{1}{2}$ if $0 < x \leqslant \frac{1}{2}$. It can be verified that 0 is an isolated point in the R g-topology bur not in the strong g-topology. The neighborhood systems of the two topologies are the same at the other points.

If we delete the element 1 in example 5.3, the point 0 is no longer isolated. This is an instance of the general situation: the R g-topology of a subspace may be strictly coarser, and is never strictly finer, than the relative R g-topology. This results from the fact that the collection of distribution functions is not the same in each case. This contrasts with the g-écart g-topology, where the collection of generating distribution functions is fixed and the relative g-écart g-topology induced on a subspace is equivalent to the g-écart g-topology of the subspace.

6. Open questions. When is a generalized topology statistically metrizable, i.e. derivable from an SM space in one of the ways we have studied in sections 3 and 4?

What conditions are both necessary and sufficient for the various SM g-topologies to be of type V_D?

What conditions are necessary and sufficient for the various SM g-topologies to be topologies?

References

[1] A. Appert and Ky-Fan, *Espaces topologiques intermédiaires*, Actual. Sci. Ind. 1121, Paris.

[2] N. Bourbaki, *Topologie générale*, Actual. Sci. Ind. 858-1142 (second edition).

[3] M. Fréchet, *Les espaces abstraits*, Paris 1928.

[4] J. L. Kelley, *General topology*, Princeton 1955.

[5] K. Menger, *Statistical metrics*, Proc. Nat. Acad. of Sci., U.S.A. 28 (1942), pp. 535-537.

[6] B. Schweizer and A. Sklar, *Espaces métriques aleatoires*, Comptes Rendus Acad. Sci., Paris, 247 (1958), pp. 2092-2094.

[7] — — *Statistical metric spaces*, Pac. J. Math. 10 (1960), pp. 313-334.

[8] — — and E. Thorp, *Metrization of statistical metric spaces*, Pac. J. Math. 10 (1960), pp. 673-675.

[9] E. Thorp, *Best possible triangle inequalities for statistical metric spaces*, Proc. Am. Math. Soc. 11 (1960), pp. 134-140.

[10] A. Weil, *Sur les espaces à structure uniforme*, Paris 1937.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
and NEW MEXICO STATE UNIVERSITY

Reçu par la Rédaction le 9. 5. 1961