for all j, then $\prod_j a_{ij} > z$, since $z \neq \prod_j a_{ij}$. Hence $L-Q$ preserves all the products $\prod_j a_{ij}$. Thus L satisfies $(T_{\alpha\alpha})$ and by Theorem 4, $L \epsilon R_{\alpha\alpha}$.

COROLLARY. *If L is an α-complete chain with a smallest element, then L is α-representable if and only if every densely ordered interval of L has power $> \alpha$.*

Proof. Let L be an α-complete chain in $R_{\alpha\alpha}$. Let $[y, x]$ be a closed interval of L without jumps. If $[y, x]$ has power $\leqslant \alpha$, then by the α-completeness of L, $[y, x]$ has no gaps. Therefore, by Theorem 5, $[y, x]$ must have power $> \alpha$. Conversely, if every densely ordered interval of L has power $> \alpha$, then L is α-representable by Theorem 5 and Lemma 1.

THEOREM 6. *There exists a complete chain L (and therefore a complete, completely distributive lattice L) such that for every $\alpha \geqslant 2^{\omega}$, L is not α-representable.*

Proof. Let L be the set of all real numbers in the closed interval $[0, 1]$ with the natural ordering. By the corollary to Theorem 5, L is not α-representable for any $\alpha \geqslant 2^{\omega}$.

4. A Boolean algebra B with an ordered basis is an algebra which is generated by a chain. If B is generated by a chain L (or even by any sublattice L), and B is isomorphic with an α-normal subalgebra of an α-field of sets modulo an α-ideal, then $L \epsilon R_{\alpha\alpha}$. The converse does not hold, as may be shown by the example where L consists of all irrationals in $[0, 1]$, and $\alpha \geqslant 2^{\omega}$. Theorem 4 and its analogue for Boolean algebras can be used to give a criterion that B be so representable. However no criterion as simple as that of Theorem 5 seems to hold.

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On α-homomorphic images of α-rings of sets*

by

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In this paper we consider the question of characterizing those α-complete lattices which are α-homomorphic images of α-rings of sets. In [2] a necessary and sufficient condition for a lattice to be isomorphic with an α-ring of sets modulo an α-ideal was given. However, in contrast with the situation for Boolean algebras, not every homomorphic image of a ring of sets is isomorphic with a quotient of the ring by an ideal.

It is not hard to see that the class K_α of all α-homomorphic images of α-rings of sets is closed under the operations of taking direct products, α-sublattices, and α-homomorphisms. Therefore, by the extension of Birkhoff's Theorem [1] to algebras with infinitary operations, K_α is an equational class. We shall determine a set of equations which characterizes K_α. A simple sufficient condition is $(\alpha, 2^\alpha)$ distributivity in either sense. Finally the class of α-retracts of α-rings of sets is discussed.

1. Definitions. We adopt the terminology of [2]. Let α be an infinite cardinal. An α-complete lattice is not assumed to have a largest or smallest element.

An *α-sublattice* of an α-complete lattice L is a subset M such that $\sum_{L,i} x_i \epsilon M$, and $\prod_{L,i} x_i \epsilon M$ for any non-empty α-system $\{x_i\}$ in M.

A family F of sets is called *α-independent* if the intersection of an α-system $\{x_i\}$ in F is contained in the union of an α-system $\{y_j\}$ in F only when some $x_i =$ some y_j. There exist α-independent families of any power. For example, if β is any cardinal, then for each $i \epsilon \beta$, let x_i be the set of all subsets of β which contain i. The family $\{x_i\}$ is α-independent for any α.

Let K_α be the set of all α-homomorphic images of α-rings of sets. A lattice L in K_α is said to be a *free lattice* of class K_α with β generators if L has a subset W with the following properties:

1) W has power β.

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2) The α-sublattice generated by W is L.

3) Any mapping of W into a member L' of K_α can be extended to an α-homomorphism of L into L'.

Any two free lattices of class K_α with the same number of generators are isomorphic. See for example the proof of 12.1 in [4].

2. We are going to deal with formal infinitary polynomials. In order to make the argument precise, it will be convenient to introduce a formalism involving expressions of infinite length. Such languages are discussed in [3].

The variables of our formal system are the symbols v_i, $i \in \alpha$. Formulas are defined inductively as follows:

1) Any variable is a formula.

2) If φ_j is a formula for each $j \in \beta$, where β is a non-empty ordinal $\leqslant \alpha$, then $\bigvee(\varphi_0\varphi_1\ldots\varphi_j\ldots)$ and $\bigwedge(\varphi_0\varphi_1\ldots\varphi_j\ldots)$ are formulas. We abbreviate these as $\bigvee_{j\in\beta}\varphi_j$ and $\bigwedge_{j\in\beta}\varphi_j$.

If L is an α-complete lattice, then by an *L-assignment*, we mean a function on α to L. If f is an L-assignment, we can associate with f a unique function $\tilde{f}$ on the set of all formulas such that $\tilde{f}(v_i) = f(i)$ for each $i \in \alpha$, and $\tilde{f}(\bigvee\varphi_j) = \sum \tilde{f}(\varphi_j)$, and $\tilde{f}(\bigwedge\varphi_j) = \prod \tilde{f}(\varphi_j)$.

An equation $\varphi = \psi$ is said to be *satisfied identically* in L if $\tilde{f}(\varphi) = \tilde{f}(\psi)$ for every L-assignment f.

3. LEMMA 1. *If A is a non-empty subset of an α-complete lattice L, then every member of the α-sublattice generated by A is of the form $\tilde{f}(\varphi)$, where φ is a formula, and f is an L-assignment with range contained in A.*

Proof. Let B be the set of all elements of the form $\tilde{f}(\varphi)$, where f is an L-assignment with range in A, and φ is a formula. Clearly $B \supset A$. Suppose that $\tilde{f}_j(\varphi_j)$ is a member of B for each $j \in \beta$, where β is a non-empty ordinal $\leqslant \alpha$. Divide α into disjoint subsets S_j, $j \in \beta$, each of power α, and let θ_j be a one-to-one mapping of α onto S_j. Let ψ_j be the result of replacing in φ_j each variable v_i by the variable v_k, where $k = \theta_j(i)$. Finally, let g be the L-assignment such that for each $j \in \beta$ and each $i \in S_j$, $g(i) = f_j(\theta_j^{-1}(i))$. Then $\tilde{g}(\psi_j) = \tilde{f}_j(\varphi_j)$ for each $j \in \beta$. Therefore

$$\sum_j \tilde{f}_j(\varphi_j) = \sum_j \tilde{g}(\psi_j) = \tilde{g}(\bigvee_j \psi_j) \in B .$$

Thus B is closed under sums of non-empty α-systems. A similar argument applies to products.

LEMMA 2. *Let h be an α-homomorphism of an α-complete lattice L_1 into an α-complete lattice L_2. Let f_k be L_k-assignments, $k = 1, 2$, such that $h(f_1(i)) = f_2(i)$ for each $i \in \alpha$. Then $h(\tilde{f}_1(\varphi)) = \tilde{f}_2(\varphi)$ for every formula φ.*

Proof. The proof is immediate by induction on the rank of φ.

4. DEFINITION 1. To each formula φ we associate a family $A(\varphi)$ of subsets of α according to the following rules:

1) If $\varphi = v_i$, then $A(\varphi) = \{\{i\}\}$, the family whose only member is $\{i\}$.

2) If $\varphi = \bigvee_j \varphi_j$, then $A(\varphi) = \bigcup_j A(\varphi_j)$.

3) If $\varphi = \bigwedge_j \varphi_j$, then $A(\varphi)$ consists of all sets of the form $\bigcup_j \lambda(j)$, where λ varies over the Cartesian product $\underset{j}{P} A(\varphi_j)$.

Notice that $A(\varphi)$ is of power $\leqslant 2^\alpha$ for any φ.

DEFINITION 2. If F is a function, and S is a subset of its domain, then $F[S]$ denotes the *set of images of members of S*.

LEMMA 3. *Let F be a function on α to α. Let $\bar{\varphi}$ be the result of replacing in φ each variable v_i by $v_{F(i)}$. Then $A(\bar{\varphi})$ consists of all sets of the form $F[S]$, where S varies over $A(\varphi)$.*

Proof. This is easily proved by induction on the rank of φ, using the fact that $F[\bigcup_j S_j] = \bigcup_j F[S_j]$.

DEFINITION 3. A lattice L is said to be (α, β) *distributive in the $\prod\sum$ sense* if it satisfies the following condition: If $\{x_{ij}\}$, $i \in I$, $j \in J$, is any α,β-system (that is I has power $\leqslant \alpha$, and J has power $\leqslant \beta$) in L such that $\prod_{i\in I}\sum_{j\in J} x_{ij}$ exists, and $\prod_{i\in I} x_{i,f(i)}$ exists for every $f \in J^I$, then $\sum_{f\in J^I}\prod_{i\in I} x_{i,f(i)}$ exists and is equal to $\prod_{i\in I}\sum_{j\in J} x_{ij}$. We also have a dual definition of (α, β) distributivity in the $\sum\prod$ sense.

LEMMA 4. *Let L be an α-complete lattice which is $(\alpha, 2^\alpha)$ distributive in the $\prod\sum$ sense. Then for any L-assignment f and any formula φ, we have*

$$\tilde{f}(\varphi) = \sum_{S\in A(\varphi)} \prod_{i\in S} f(i) .$$

Proof. This is obvious when φ is a variable. Suppose that the statement holds for φ_j, $j \in \beta$, where β is a non-empty ordinal $\leqslant \alpha$. Let $\varphi = \bigvee_{j\in\beta}\varphi_j$. Then

$$\tilde{f}(\varphi) = \sum_{j\in\beta}\tilde{f}(\varphi_j) = \sum_{j\in\beta}\sum_{S\in A(\varphi_j)}\prod_{i\in S} f(i) = \sum_{S\in A(\varphi)}\prod_{i\in S} f(i) .$$

If $\varphi = \bigwedge_{j\in\beta}\varphi_j$, then

$$\tilde{f}(\varphi) = \prod_{j\in\beta}\tilde{f}(\varphi_j) = \prod_{j\in\beta}\sum_{S\in A(\varphi_j)}\prod_{i\in S} f(i) = \sum_{\lambda\in \underset{j\in\beta}{P} A(\varphi_j)}\prod_{j\in\beta}\prod_{i\in\lambda(j)} f(i)$$
$$= \sum_{\lambda\in \underset{j\in\beta}{P} A(\varphi_j)}\prod_{i\in\bigcup_{j\in\beta}\lambda(j)} f(i) = \sum_{S\in A(\varphi)}\prod_{i\in S} f(i)$$

Lemma 5. *If R is an α-ring of sets, and f is an R-assignment, then* $\tilde{f}(\varphi) = \bigcup_{S \in A(\varphi)} \bigcap_{i \in S} f(i)$, *for any formula φ.*

Proof. Since R is an α-sublattice of the lattice of all subsets of a set, the result follows from Lemma 4 and Lemma 2 with h as the identity map.

Definition 4. If φ and ψ are formulas, we write $\varphi \sim \psi$ whenever every member of $A(\varphi)$ contains a member of $A(\psi)$, and every member of $A(\psi)$ contains a member of $A(\varphi)$.

Theorem 1. *Let L be an α-complete lattice. Then L is an α-homomorphic image of an α-ring of sets if and only if the equations $\varphi = \psi$ are satisfied identically in L whenever $\varphi \sim \psi$.*

Theorem 2. *The free lattice of class K_α with β generators is the α-ring generated by an α-independent family of sets which has power β.*

Proofs. We first prove the necessity of the condition in Theorem 1. Suppose that h is an α-homomorphism of an α-ring R of sets onto L. Let φ, ψ be formulas such that every member of $A(\varphi)$ contains a member of $A(\psi)$, and let f be any L-assignment. We need only prove $\tilde{f}(\varphi) \leqslant \tilde{f}(\psi)$. Let g be an R-assignment such that $h(g(i)) = f(i)$ for each $i \in \alpha$. If $S \in A(\varphi)$, let $T(S)$ be a member of $A(\psi)$ such that $T(S) \subset S$. For each such S,

$$\bigcap_{i \in S} g(i) \subset \bigcap_{i \in T(S)} g(i) \subset \bigcup_{T \in A(\psi)} \bigcap_{i \in T} g(i) = \tilde{g}(\psi),$$

by Lemma 4. Therefore by Lemma 4, $\tilde{g}(\varphi) \subset \tilde{g}(\psi)$. Hence by Lemma 2,

$$\tilde{f}(\varphi) = h(\tilde{g}(\varphi)) \leqslant h(\tilde{g}(\psi)) = \tilde{f}(\psi).$$

Now let L be an α-complete lattice satisfying the condition of Theorem 1. Let W be a non-empty α-independent family of sets, and let θ be a function on W into L. We prove Theorem 2 by showing that θ can be extended to an α-homomorphism of the α-ring R of sets generated by W into L. By choosing W and θ so that θ is onto L, this will also prove the sufficiency of the condition of Theorem 1.

By Lemma 1, every member of R is of the form $\tilde{f}(\varphi)$, where f is an R-assignment with range contained in W. For each such f, let f' be the L-assignment such that $f'(i) = \theta(f(i))$ for all $i \in \alpha$. Suppose that an element of R has two representations $\tilde{f}(\varphi) = \tilde{g}(\psi)$. It will be shown that for each $S \in A(\varphi)$, there exists a member T of $A(\psi)$ such that $g[T] \subset f[S]$. If not, there exists an $S \in A(\varphi)$ such that every $T \in A(\psi)$ contains an element $i(T)$ such that $g(i(T)) \neq f(j)$ for all $j \in S$. By Lemma 5,

$$\bigcap_{j \in S} f(j) \subset \tilde{f}(\varphi) = \tilde{g}(\psi) = \bigcup_{T \in A(\psi)} \bigcap_{i \in T} g(i) \subset \bigcup_{T \in A(\psi)} g(i(T)).$$

This contradicts the α-independence of W. Similarly, for each $T \in A(\psi)$, there exists an $S \in A(\varphi)$ such that $f[S] \subset g[T]$.

Arrange the members of the union of the ranges of f and g in a non-repeating sequence $\{a_j\}$, $j \in \beta$, where β is a cardinal $\leqslant \alpha$. Let F be the function on α to β such that $F(i) = j$ whenever $f(i) = a_j$, and let G be the function on α to β such that $G(i) = j$ whenever $g(i) = a_j$. Let $\bar{\varphi}$ be the result of replacing in φ each variable v_i by $v_{F(i)}$, and let $\bar{\psi}$ be the result of replacing in ψ each variable v_i by $v_{G(i)}$. It is easily seen that if S and T are subsets of α, then $g[T] \subset f[S]$ implies $G[T] \subset F[S]$. Therefore by the previous paragraph and Lemma 3, we have $\bar{\varphi} \sim \bar{\psi}$. Let k be an R-assignment such that $k(i) = a_i$ for $i \in \beta$, and $k(i)$ is arbitrary for $i \in \alpha - \beta$. Clearly, $\tilde{k}(\bar{\varphi}) = \tilde{f}(\varphi)$, $\tilde{k}'(\bar{\varphi}) = \tilde{f}'(\varphi)$, $\tilde{k}(\bar{\psi}) = \tilde{g}(\psi)$, and $\tilde{k}'(\bar{\psi}) = \tilde{g}'(\psi)$. Since $\bar{\varphi} \sim \bar{\psi}$, the hypothesis implies $\tilde{k}'(\bar{\varphi}) = \tilde{k}'(\bar{\psi})$, and therefore $\tilde{f}'(\varphi) = \tilde{g}'(\psi)$.

We have shown that $\tilde{f}(\varphi) = \tilde{g}(\psi)$ implies $\tilde{f}'(\varphi) = \tilde{g}'(\psi)$. We may therefore define a mapping h of R into L by $h(\tilde{f}(\varphi)) = \tilde{f}'(\varphi)$. Suppose $y = \bigcup_j \tilde{f}_j(\varphi_j)$, where $\{\tilde{f}_j(\varphi_j)\}$ is a non-empty α-system in R. As in the proof of Lemma 1, there exist formulas ψ_j and an R-assignment g such that $\tilde{g}(\psi_j) = \tilde{f}_j(\varphi_j)$, $\tilde{g}'(\psi_j) = \tilde{f}'_j(\varphi_j)$, and $y = \tilde{g}(\bigvee_j \psi_j)$. Then

$$h(y) = \tilde{g}'(\bigvee_j \psi_j) = \sum_j \tilde{g}'(\psi_j) = \sum_j \tilde{f}'_j(\varphi_j) = \sum_j h(\tilde{f}_j(\varphi_j)).$$

A similar argument for products shows that h is an α-homomorphism.

Theorem 3. *If L is an α-complete lattice which is $(\alpha, 2^\alpha)$ distributive in either sense, then $L \in K_\alpha$.*

Proof. Since the dual of any lattice in K_α is also in K_α, it suffices to prove the theorem when L is $(\alpha, 2^\alpha)$ distributive in the $\prod\sum$ sense. Let $\varphi \sim \psi$, and let f be any L-assignment. Each member S of $A(\varphi)$ contains a member $T(S)$ of $A(\psi)$. Therefore by Lemma 4, for each such S we have

$$\prod_{i \in S} f(i) \leqslant \prod_{i \in T(S)} f(i) \leqslant \sum_{T \in A(\psi)} \prod_{i \in T} f(i) = \tilde{f}(\psi).$$

Therefore by Lemma 4, $\tilde{f}(\varphi) \leqslant \tilde{f}(\psi)$, and similarly $\tilde{f}(\psi) \leqslant \tilde{f}(\varphi)$.

A simple direct proof of Theorem 3 is the following. Let R be the family of all non-empty hereditary subsets H of L such that H has a least upper bound $\sum(H)$, and H is generated by a subset of power $\leqslant 2^\alpha$. R consists of all sets of the form $\bigcup_{j \in J} I(a_j)$, where J is a non-empty set of power $\leqslant 2^\alpha$, $I(a_j)$ is the principal ideal with upper element a_j, and $\sum_{j \in J} a_j$ exists. Using this representation, it is easy to show that R is an α-ring of sets, and the mapping h defined by $h(H) = \sum(H)$ is an α-homomorphism of R onto L.

Corollary. *Every α-complete chain is in K_α.*

Remarks. Theorem 1 may be proved using Birkhoff's Theorem as follows. By the remarks in the introduction, K_α is the smallest equational class containing all α-rings of sets. Let D be the class of all α-complete lattices which are $(\alpha, 2^\alpha)$ distributive in the $\prod\sum$ sense. Every α-ring is an α-sublattice of a member of D (see the proof of Lemma 5), and we have given a direct proof of Theorem 3. Therefore K_α is also the smallest equational class containing D. It follows than an α-complete lattice is in K_α if and only if it satisfies every equation in our language with α variables which is satisfied identically by all members of D. If $\varphi\sim\psi$, then by Lemma 4, we see that $\varphi = \psi$ is satisfied identically by every member of D. Conversely, if $\varphi = \psi$ is satisfied in every member of D, then it is satisfied identically in every α-ring of sets. If $\varphi\sim\psi$ does not hold, it is easy to find an assignment f whose range is contained in an α-independent family of sets such that $f(\varphi) \neq f(\psi)$. This completes the proof of Theorem 1.

In the case of α-representable Boolean algebras, a very simple class of characterizing equations was found. The equations given in Theorem 1 are certainly not independent. There remains the question whether it is possible to reduce their number significantly.

In the case of α-complete Boolean algebras, (α, α) distributivity is sufficient for α-representability. We suspect that (α, α) distributivity is not sufficient for an α-complete lattice to be in K_α, but we have no counterexample.

5. An α-complete lattice L is called an *α-retract* of an α-ring R of sets if L is isomorphic with a sublattice M of R, and there exists an α-homomorphism h of L onto M such that $h(x) = x$ for all $x \in M$. We do not assume that M is an α-sublattice of R. Let L_α be the set of α-retracts of α-rings of sets.

In order to state conditions for membership in L_α, we first dualize Definition 1.

DEFINITION 5. If φ is a formula, let $B(\varphi)$ be a family of subsets of α such that:

1) If $\varphi = v_i$, then $B(\varphi) = \{\{i\}\}$.

2) If $\varphi = \bigvee_j \varphi_j$, then $B(\varphi) = \bigcup_j B(\varphi_j)$.

3) If $\varphi = \bigwedge_j \varphi_j$, then $B(\varphi)$ consists of all sets $\bigcup_j \lambda(j)$, where λ varies over $\mathrm{P}B(\varphi_j)$.

The duals of Lemmas 4 and 5, obtained by replacing $A(\varphi)$ by $B(\varphi)$, interchanging unions and intersections, and interchanging sums and products, are obviously valid.

THEOREM 4. *Let L be an α-complete distributive lattice. A necessary and sufficient condition for L to be in L_α is the following:*

If φ and ψ are formulas, and f_1, f_2 are L-assignments such that for each $S \in A(\varphi)$ and each $T \in B(\psi)$, there exist finite sets $S' \subset S$, and $T' \subset T$ such that $\prod_{i\in S'} f_1(i) \leqslant \sum_{i\in T'} f_2(j)$, then $\tilde{f}_1(\varphi) \leqslant \tilde{f}_2(\psi)$.

Proof. Necessity: Suppose that there exists an α-homomorphism h of an α-ring R of sets onto L, and a subring M of R such that h restricted to M is an isomorphism of M onto L. Let φ, ψ, f_1, and f_2 satisfy the hypothesis of the condition of Theorem 4. Let g_1, g_2 be R-assignments such that for $k = 1, 2$, $g_k(i)$ is the element x of M such that $h(x) = f_k(i)$. Then for each $S \in A(\varphi)$, $T \in B(\psi)$, we have

$$\prod_{i\in S'} f_1(i) = h\Big(\bigcap_{i\in S'} g_1(i)\Big) \leqslant h\Big(\bigcup_{i\in T'} g_2(i)\Big) = \sum_{i\in T'} f_2(i)\,.$$

Since h is an isomorphism when restricted to M, we have

$$\bigcap_{i\in S} g_1(i) \subset \bigcap_{i\in S'} g_1(i) \subset \bigcup_{i\in T'} g_2(i) \subset \bigcup_{i\in T} g_2(i)\,.$$

By Lemma 5 and its dual,

$$\tilde{g}_1(\varphi) = \bigcup_{S\in A(\varphi)} \bigcap_{i\in S} g_1(i) \subset \bigcap_{T\in B(\psi)} \bigcup_{i\in T} g_2(i) = \tilde{g}_2(\psi)\,.$$

Therefore by Lemma 2,

$$\tilde{f}_1(\varphi) = h\big(\tilde{g}_1(\varphi)\big) \leqslant h\big(\tilde{g}_2(\psi)\big) = \tilde{f}_2(\psi)\,.$$

Sufficiency: If $x \in L$, let $\hat{x}$ be the set of prime filters of L which contain x. The sets $\hat{x}$ form a ring M of sets isomorphic with L. Also if an intersection $\bigcap_{i\in I} \hat{x}_i$ is contained in a union $\bigcup_{i\in J} \hat{x}_j$, then there exist finite sets $I' \subset I$ and $J' \subset J$ such that $\bigcap_{i\in I'} \hat{x}_i \subset \bigcup_{j\in J'} \hat{x}_j$.

Let R be the α-ring of sets generated by M. Each member of R is of the form $\tilde{f}(\varphi)$, where f is an R-assignment with range contained in M. Suppose $\tilde{f}_1(\varphi) \subset \tilde{f}_2(\psi)$, where f_1, f_2 are such R-assignments. By Lemma 5 and its dual, $\bigcap_{i\in S} f_1(i) \subset \bigcup_{i\in T} f_2(i)$ for each $S \in A(\varphi)$, and $T \in B(\psi)$. By the previous paragraph, there exist finite sets $S' \subset S$, $T' \subset T$ such that $\bigcap_{i\in S'} f_1(i) \subset \bigcup_{i\in T'} f_2(i)$.

If f is any R-assignment with range contained in M, let f' be the L-assignment such that $\widehat{f'(i)} = f(i)$. By the isomorphism of M and L, we have

$$\prod_{i\in S'} f_1'(i) \leqslant \sum_{i\in T'} f_2'(i)\,.$$

Therefore by our hypothesis, $\widetilde{f}_1'(\varphi) \leqslant \widetilde{f}_2'(\psi)$. Thus $\widetilde{f}_1(\varphi) = \widetilde{f}_2(\psi)$ implies $\widetilde{f}_1'(\varphi) = \widetilde{f}_2'(\psi)$. We may therefore define a mapping h of R onto L by $h(\widetilde{f}(\varphi)) = \widetilde{f}'(\varphi)$. An argument similar to that of Theorem 1, shows that h is an α-homomorphism. Since $h(\hat{x}) = x$ for $\hat{x} \in M$, L is an α-retract of R.

THEOREM 5. *Let L be an α-complete lattice which is $(\alpha, 2^\alpha)$ distributive in both senses. Then $L \in L_\alpha$.*

Proof. Let φ, ψ, f_1, and f_2 satisfy the hypothesis of the condition of Theorem 4. Then $\prod_{i \in S} f_1(i) \leqslant \sum_{i \in T} f_2(i)$ for all $S \in A(\varphi)$, and $T \in B(\psi)$. Therefore by Lemma 4 and its dual, $\widetilde{f}_1(\varphi) \leqslant \widetilde{f}_2(\psi)$.

6. Let R_α be the set of all lattices which are isomorphic with an α-ring of sets divided by an α-ideal. Let K_α' (or L_α') be the set of lattices in K_α (or L_α) which have a smallest element. The proof of Theorem 4 in [2] shows that $R_\alpha \subset L_\alpha'$ for all α, and obviously $L_\alpha' \subset K_\alpha'$ for all α. By Theorem 5, every α-complete chain is in L_α. However the chain of all reals in the closed interval $[0, 1]$ is not in R_α for any $\alpha \geqslant 2^\omega$, by Theorem 6 of [2]. Therefore $R_\alpha \neq L_\alpha'$ for all $\alpha \geqslant 2^\omega$. Since every member of K_ω is $(2, \omega)$ distributive in both senses, the corollary of Theorem 3 in [2] shows that $R_\omega = L_\omega' = K_\omega'$. It is not known whether $L_\alpha = K_\alpha$ for some $\alpha > \omega$.

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Remarques sur les relations d'équivalence

par

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Dédié amicalement à M. Béla Szőkefalvi-Nagy à l'occasion de son 50-eme anniversaire

1. On peut formuler la question traitée dans le travail [2] de M. S. Gołąb — en la généralisant de 1 à n dimensions — comme il suit: Soient

$$\boldsymbol{x} = (x^1, x^2, \ldots, x^n)$$

les coordonnées d'un point P de l'espace dans un système de coordonnées arbitraire, mais fixé. Étant données les coordonnées $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ et $\boldsymbol{x}_3$ des points P_1, P_2 et P_3, comment trouver les coordonnées $\boldsymbol{x}_4$ de l'extrémité P_4 du vecteur $\overline{P_3P_4}$ de manière qu'il soit équivalent à $\overline{P_1P_2}$? Alors

$$\boldsymbol{x}_4 = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) \tag{1}$$

et M. Gołąb a postulé comme conditions d'équivalence les suivantes:

I. réflexivité: $\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_1) = \boldsymbol{x}_2$,

II. symétrie: $\boldsymbol{f}\big(\boldsymbol{x}_3, \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3), \boldsymbol{x}_1\big) = \boldsymbol{x}_2$,

III. transitivité: $\boldsymbol{f}\big(\boldsymbol{x}_3, \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3), \boldsymbol{x}_4\big) = \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4)$,

et, ensuite, aussi la condition

IV. réversibilité: $\boldsymbol{f}\big(\boldsymbol{x}_2, \boldsymbol{x}_1, \boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)\big) = \boldsymbol{x}_3$.

Dans l'espace à n dimensions on voit aussi en posant $\boldsymbol{x}_4 = \boldsymbol{x}_1$ que II est une conséquence de III et de I et en écrivant

$$\boldsymbol{f}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = \boldsymbol{g}(\boldsymbol{x}_2, \boldsymbol{x}_1, \boldsymbol{x}_3) \tag{2}$$

I et III se transforment en

$$\boldsymbol{g}(\boldsymbol{x}_2, \boldsymbol{x}_1, \boldsymbol{x}_1) = \boldsymbol{x}_2 \tag{3}$$

et

$$\boldsymbol{g}\big(\boldsymbol{g}(\boldsymbol{x}_2, \boldsymbol{x}_1, \boldsymbol{x}_3), \boldsymbol{x}_3, \boldsymbol{x}_4\big) = \boldsymbol{g}(\boldsymbol{x}_2, \boldsymbol{x}_1, \boldsymbol{x}_4), \tag{4}$$

qui sont les équations fonctionnelles des objets géométriques à n composantes dans des espaces à n dimensions.