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On the representation of α-complete lattices *

by

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This paper is concerned with the problem of representation for α-complete lattices. It is well known that a lattice is isomorphic with a ring of sets if and only if it is distributive. However for an α-complete lattice L even the condition of (α, α) distributivity is not sufficient for it to be isomorphic with an α-ring of sets. A necessary and sufficient condition for such a representation is the following: whenever $x \not\leqslant y$, there exists an α-complete prime ideal P containing x such that $L-P$ contains y and is α-complete. On the other hand, necessary and sufficient conditions for a Boolean algebra to be α-representable (that is, to be isomorphic with an α-field of sets modulo an α-ideal) are known ([1], [5], [8], [4], [7]). In this paper, we deal with the problem of representing an α-complete lattice as an α-ring of sets modulo an α-ideal. Such lattices are called *α-representable.*

We shall present a characterization of α-representable lattices which is a natural generalization of a known characterization for α-representable Boolean algebras ([5], [1]). There are several differences between the results for Boolean algebras and those we obtain for lattices. For instance, while every ω-complete Boolean algebra is ω-representable ([3], [6]) in order that an ω-complete lattice L be ω-representable, it is necessary and sufficient that L satisfy the condition of $(2, \omega)$ distributivity, which is satisfied by all Boolean algebras. Also, while every α-complete, (α, α) distributive Boolean algebra is α-representable, we shall give an example of a complete, completely distributive lattice which is not α-representable for any $\alpha \geqslant 2^{\omega}$. The paper concludes with a discussion of α-representable chains.

1. Definitions. If α is a cardinal, an *α-system* is a system $\{x_i\}$, $i \in I$, whose index set I has power $\leqslant \alpha$. By an *α-complete lattice*, we mean a lattice L in which every non-empty α-system $\{x_i\}$, $i \in I$, has a least upper bound $\sum_{i\in I} {}_L x_i$, and a greatest lower bound $\prod_{i\in I} {}_L x_i$. We do not require that L have a smallest or largest element.

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By an *α-normal sublattice* of a lattice M, we mean a subset L such that if $\{x_i\}$ is any *α-system* contained in L such that $\sum_{i} {}_L x_i$ exists, then $\sum_{i} {}_M x_i$ exists and is equal to $\sum_{i} {}_L x_i$, and similarly for products. In particular, if L has a smallest element 0, then 0 is also the smallest element of M.

An *α-ring* of sets is a family of sets closed under unions and intersections of non-empty α-systems. An *α-field* of sets is an α-ring of sets closed under complements. We denote the empty set by Ø. By an *α-ideal* in an α-ring R of sets, we mean an ideal in R which is closed under unions of non-empty α-systems. If I is an α-ideal in an α-ring R of sets, then I determines a congruence relation in R: $x \equiv y \,(\mathrm{mod}\, I)$ if and only if $x+z = y+z$ for some z in I. The set R/I of congruence classes is an α-complete lattice. Let x/I denote the congruence class containing x. We have $x/I \leqslant y/I$ if and only if $x \subset y \cup z$ for some z in I. If $\{x_i\}$ is a non-empty α-system in R, then $\sum_i (x_i/I) = (\bigcup_i x_i)/I$, and $\prod_i (x_i/I) = (\bigcup_i x_i)/I$.

A lattice L is called (α, β) *distributive* if it satisfies the following conditions:

If $\{x_{ij}\}$, $i \in I$, $j \in J$, *is any* α, β*-system* (that is, I has power $\leqslant \alpha$, and J has power $\leqslant \beta$) *in* L *such that* $\sum_{i \in I} \prod_{j \in J} x_{ij}$ *exists, and* $\sum_{i \in I} x_{i,f(i)}$ *exists for every* $f \in J^I$, *then* $\sum_{f \in J^I} \prod_{i \in I} x_{i,f(i)}$ *exists and is equal to* $\sum_{i \in I} \prod_{j \in J} x_{ij}$. *In addition, we require the dual of this condition.*

An (α, ∞) *distributive lattice is one which is* (α, β) *distributive for all* β.

If L is a distributive lattice, and $x \in L$, we denote by $\hat{x}$ the set of prime filters containing x. As is well known, the family of all sets $\hat{x}$ is a ring $\hat{L}$ of sets isomorphic with L.

2. Let $R_{\alpha\beta}$ be the set of all lattices isomorphic with a β-normal sublattice of an α-ring of sets modulo an α-ideal.

THEOREM 1. *Let* α *and* β *be infinite cardinals with* $\beta \leqslant \alpha$. *A lattice* L *is in* $R_{\alpha\beta}$ *if and only if* L *is isomorphic with a* β*-normal sublattice of an* α*-field of sets modulo an* α*-ideal.*

Proof. Let L be a β-normal sublattice of R/I, where R is an α-ring of sets and I is an α-ideal. Let F be the α-field of sets generated by R, and J be the ideal in F generated by I. Then J is an α-ideal. If x, y are in R, then $x/I \leqslant y/I$ if and only if $x/J \leqslant y/J$. The mapping φ defined by $\varphi(x/I) = x/J$ is therefore an isomorphism of R/I into F/J. Furthermore φ is an α-homomorphism. Thus R/I is isomorphic with an α-normal sublattice of F/J, and therefore L is isomorphic with a β-normal sublattice of F/J.

LEMMA 1. *Let* L *be an* α*-complete lattice with a smallest element. If* L *is in* $R_{\alpha\alpha}$, *then* L *is* α*-representable.*

Proof. Suppose that L is an α-normal sublattice of R/I, where R is an α-ring of sets and I is an α-ideal. Let S be the set of elements x of R such that $x/I \in L$. Then S is an α-ring of sets containing I. The mapping φ defined by $\varphi\big((x/I)_S\big) = (x/I)_R$ is clearly an isomorphism of S/I onto L.

LEMMA 2. *Any* $(2, \alpha)$ *distributive lattice* L *is isomorphic with an* α*-normal sublattice of a complete Boolean algebra.*

Proof. We identify L with its representation $\hat{L}$ as a ring of subsets of the set X of all prime ideals of L. Let F be the field of sets generated by L. Every element of F is a finite intersection $\bigcap_{k \leqslant n} (\bar{a}_k \cup b_k)$, where a_k and b_k are members of $L \cup \{Ø\} \cup \{X\}$, and $\bar{a}_k$ is the complement of a_k. Let $x = \sum_{i} {}_L x_i$, where $\{x_i\}$ is an α-system in L. We wish to show $x = \sum_{i} {}_F x_i$. Let $y \in F$ and $y \supset x_i$ for all i. If $y = \bigcap_k (\bar{a}_k \cup b_k)$, then $a_k x_i \subset b_k$ for each i and k. Using the $(2, \alpha)$ distributivity of L, it follows that $x a_k \subset b_k$ for all k, and hence $x \subset y$. We omit the dual proof for products. If B is the normal completion of F, L is thus an α-normal sublattice of B.

In the above proof, F is independent of α. Therefore we have a very simple proof of the following result of Funayama [2].

THEOREM 2. *A lattice* L *is normally embeddable in a Boolean algebra if and only if* L *is* $(2, \infty)$ *distributive.*

THEOREM 3. *A lattice* L *is in* $R_{\omega\omega}$ *if and only if* L *is* $(2, \omega)$ *distributive.*

Proof. The necessity of the condition is clear. Suppose that L is $(2, \omega)$ distributive. By Lemma 2, L is isomorphic with an ω-normal sublattice of an ω-complete Boolean algebra. By the Loomis-Sikorski Theorem ([3], [6]), every such Boolean algebra is ω-representable.

COROLLARY. *An* ω*-complete lattice* L *with a smallest element is* ω*-representable if and only if* L *is* $(2, \omega)$ *distributive.*

Proof. This follows immediately from Theorem 3 and Lemma 1.

DEFINITION. A filter P is said *to preserve the product* $\prod_i x_i$ if $\prod_i x_i \in P$ if and only if $x_i \in P$ for all i. An ideal Q is said *to preserve the sum* $\sum_i x_i$ if $\sum_i x_i \in Q$ if and only if $x_i \in Q$ for all i.

THEOREM 4. *If* $\beta \leqslant \alpha$, *a lattice* L *is in* $R_{\alpha\beta}$ *if and only if* L *satisfies the following condition:*

$(T_{\alpha\beta})$: *If* $x, y \in L$, *and* $x \nleqslant y$, *and* $\{a_{ij}\}$, $\{b_{ij}\}$, $i \in I$, $j \in J$, *are* α, β *systems in* L *such that* $\prod_{j \in J} a_{ij}$ *and* $\sum_{j \in J} b_{ij}$ *exist for each* $i \in I$, *then there exists a prime filter* P *such that* $x \in P$, $y \notin P$, *and such that for each* $i \in I$, P *preserves the product* $\prod_j a_{ij}$ *and* L--P *preserves the sum* $\sum_j b_{ij}$.

Proof. Let $L \in R_{\alpha\beta}$. By Theorem 1, we may assume that L is a β-normal sublattice of F/E, where F is an α-field of sets and E is an α-ideal. Let x, y, $\{a_{ij}\}$, $\{b_{ij}\}$ satisfy the hypotheses of $(\mathrm{T}_{\alpha\beta})$. By [4], there exists a prime filter P in F/E containing $x-y$ and preserving the products $\prod_j a_{ij}$ and $\prod_j \bar{b}_{ij}$ for each $i \in I$. Let Q be the prime ideal $F/E-P$. Then $x \in P$, $y \in Q$, and Q preserves the sums $\sum_j b_{ij}$. If we set $P_1 = P \cap L$ and $Q_1 = Q \cap L$, then P_1 is a filter in L, Q_1 is an ideal in L, and $P_1 \cap Q_1 = \emptyset$. It is well known that P_1 can be enlarged to a prime filter P_2 disjoint from Q_1. Clearly P_2 is the desired prime filter.

Conversely, suppose that L satisfies condition $(\mathrm{T}_{\alpha\beta})$. We first show L is distributive. Let x, y, z be elements of L. Obviously $x(y+z) \geqslant xy+xz$. Suppose $x(y+z) \nleqslant xy+xz$. Let P be a prime filter containing $x(y+z)$ but not $xy+xz$. P contains x and $y+z$, and therefore P contains either y or z. But then P contains xy or xz, contradicting $xy+xz \notin P$.

Let S be the α-ring of sets generated by $\hat{L}$. Let E be the α-ideal in S generated by all sets of the form $\bigcap_j \hat{x}_j - \prod_{\hat{L}}{}_j \hat{x}_j$, or of the form $\sum_{\hat{L}}{}_j \hat{x}_j - \bigcup_j \hat{x}_j$, where $\{x_j\}$, $j \in J$, is any β-system in L such that $\sum_j x_j$ or $\prod_j x_j$ exists. We define a mapping of L into S/E by $\varphi(x) = \hat{x}/E$. φ is obviously monotone. Suppose $\varphi(x) \leqslant \varphi(y)$, but $x \nleqslant y$. Then $\hat{x} \subset \hat{y} \cup z$ for some $z \in E$. There exist α, β systems $\{a_{ij}\}$, $\{b_{ij}\}$, $i \in I$, $j \in J$, such that for each $i \in I$, $\prod_j a_{ij}$ and $\sum_j b_{ij}$ exist and

$$z \subset \bigcup_i \Big(\bigcap_j \hat{a}_{ij} - \prod_j \hat{a}_{ij}\Big) \cup \bigcup_i \Big(\sum_j \hat{b}_{ij} - \bigcup_j \hat{b}_{ij}\Big). \tag{1}$$

Observe that a prime filter P of L belongs to the right side of (1) if and only if P fails to preserve $\prod_j a_{ij}$ for some $i \in I$, or $L-P$ fails to preserve $\sum_j b_{ij}$ for some $i \in I$. Since $x \nleqslant y$, the hypothesis implies that there exists a prime filter P such that $P \in \hat{x}$, $P \notin \hat{y}$, and $P \notin z$. Since this contradicts $\hat{x} \subset \hat{y} \cup z$, we see that φ is an isomorphism. Let $\{x_j\}$ be a β-system in L such that $x = \sum_L x_j$ exists. Then $\hat{x} \equiv \bigcup \hat{x}_j \ (\mathrm{mod}\, E)$, and therefore $\varphi(x) = \hat{x}/E = (\bigcup \hat{x}_j)/E = \sum_{S/E}(\hat{x}_j/E) = \sum_{S/E}\varphi(x_j)$. Similarly φ preserves all existing products of β-systems in L. Therefore L is isomorphic with a β-normal sublattice of S/E. The proof is complete.

Remark. Theorem 3 may be derived directly from Theorem 4 without making use of the Loomis-Sikorski Theorem. Also, the necessity of the condition $(\mathrm{T}_{\alpha\beta})$ in Theorem 4 can be shown directly without making use of the corresponding result for Boolean algebras.

COROLLARY 1. *An α-complete lattice L with a smallest element is α-representable if and only if it satisfies condition* $(\mathrm{T}_{\alpha\alpha})$.

Proof. This follows from Theorem 4 and Lemma 1.

COROLLARY 2. *Let α and β be infinite cardinals with $\beta \leqslant \alpha$. If L is any β-complete distributive lattice, there exists a congruence relation $K_{\alpha\beta}$ in L such that $L/K_{\alpha\beta} \in R_{\alpha\beta}$, and $L \in R_{\alpha\beta}$ if and only if $K_{\alpha\beta}$ reduces to the identity relation.*

Proof. The mapping φ defined in the proof of Theorem 4 is a β-homomorphism of L onto a β-normal sublattice of S/E. Let $K_{\alpha\beta}$ be the set of all pairs (x, y) in L such that $\varphi(x) = \varphi(y)$, that is, $\hat{x}/E = \hat{y}/E$. Then $L/K_{\alpha\beta}$ is isomorphic with the range of φ. If $L \in R_{\alpha\beta}$, then L satisfies $(\mathrm{T}_{\alpha\beta})$, which implies that $(x, y) \in K_{\alpha\beta}$ if and only if $x = y$.

3. THEOREM 5. *Let L be a chain. Then $L \in R_{\alpha\alpha}$ if and only if every closed interval of L without gaps or jumps has power $> \alpha$.*

Proof. Necessity: Let $[y, x]$ be a closed interval without gaps or jumps. Suppose that $[y, x]$ has power $\leqslant \alpha$. If z is any element such that $y < z \leqslant x$, we let $\{b_{zj}\}$ be an α-system consisting of all elements w such that $y \leqslant w < z$. For each z such that $y \leqslant z < x$, we let $\{a_{zj}\}$ be an α-system consisting of all w such that $z < w \leqslant x$. Since L has no jumps, we have $z = \sum_j b_{zj}$ and $z = \prod_j a_{zj}$ for each such z. By Theorem 4, there exists an ideal Q in L such that $y \in Q$, $x \notin Q$, Q preserves each sum $\sum_j b_{zj}$ and $L-Q$ preserves each product $\prod_j a_{zj}$. Now $Q \cap [y, x]$ is an ideal in $[y, x]$, and since $[y, x]$ has no gaps, every ideal in $[y, x]$ has a least upper bound. Therefore Q has a least upper bound $u \in [y, x]$. If $u \in Q$, then $u < x$ and $a_{uj} \notin Q$ for all j. Since $\prod_j a_{uj} = u \in Q$, this contradicts the fact that $L-Q$ preserves the product $\prod_j a_{uj}$. If $u \notin Q$, then Q consists of all elements of L which are $< u$. Therefore $y < u$ and $b_{uj} \in Q$ for all j, while $\sum_j b_{uj} = u \notin Q$. This contradicts the fact that Q preserves the sum $\sum_j b_{uj}$.

Sufficiency: Let x, y, $\{a_{ij}\}$, and $\{b_{ij}\}$ satisfy the hypothesis of condition $(\mathrm{T}_{\alpha\alpha})$ of Theorem 4. If the closed interval $[y, x]$ has a jump $[u, v]$, let Q be the principal ideal with upper element u. Then Q preserves all sums, while $L-Q$ preserves all products. In a chain all proper ideals are prime ideals. Suppose that $[y, x]$ has a gap. Then there exists an ideal Q in $[y, x]$ which has no least upper bound. The ideal in L generated by Q has the same property, and its complement with respect to L has no greatest lower bound. It is easy to see that an ideal without a least upper bound preserves all sums, and dually.

There remains the case where $[y, x]$ has more than α elements. Let z be an element such that $y \leqslant z < x$, and such that for all i, $z \neq \prod_j a_{ij}$. The principal ideal Q with upper element z preserves all sums. If $a_{ij} > z$

for all j, then $\prod_j a_{ij} > z$, since $z \neq \prod_j a_{ij}$. Hence $L-Q$ preserves all the products $\prod_j a_{ij}$. Thus L satisfies $(T_{\alpha\alpha})$ and by Theorem 4, $L \in R_{\alpha\alpha}$.

COROLLARY. *If L is an α-complete chain with a smallest element, then L is α-representable if and only if every densely ordered interval of L has power $> \alpha$.*

Proof. Let L be an α-complete chain in $R_{\alpha\alpha}$. Let $[y, x]$ be a closed interval of L without jumps. If $[y, x]$ has power $\leqslant \alpha$, then by the α-completeness of L, $[y, x]$ has no gaps. Therefore, by Theorem 5, $[y, x]$ must have power $> \alpha$. Conversely, if every densely ordered interval of L has power $> \alpha$, then L is α-representable by Theorem 5 and Lemma 1.

THEOREM 6. *There exists a complete chain L (and therefore a complete, completely distributive lattice L) such that for every $\alpha \geqslant 2^{\omega}$, L is not α-representable.*

Proof. Let L be the set of all real numbers in the closed interval $[0,1]$ with the natural ordering. By the corollary to Theorem 5, L is not α-representable for any $\alpha \geqslant 2^{\omega}$.

4. A Boolean algebra B with an ordered basis is an algebra which is generated by a chain. If B is generated by a chain L (or even by any sublattice L), and B is isomorphic with an α-normal subalgebra of an α-field of sets modulo an α-ideal, then $L \in R_{\alpha\alpha}$. The converse does not hold, as may be shown by the example where L consists of all irrationals in $[0,1]$, and $\alpha \geqslant 2^{\omega}$. Theorem 4 and its analogue for Boolean algebras can be used to give a criterion that B be so representable. However no criterion as simple as that of Theorem 5 seems to hold.

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On α-homomorphic images of α-rings of sets*

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In this paper we consider the question of characterizing those α-complete lattices which are α-homomorphic images of α-rings of sets. In [2] a necessary and sufficient condition for a lattice to be isomorphic with an α-ring of sets modulo an α-ideal was given. However, in contrast with the situation for Boolean algebras, not every homomorphic image of a ring of sets is isomorphic with a quotient of the ring by an ideal.

It is not hard to see that the class K_α of all α-homomorphic images of α-rings of sets is closed under the operations of taking direct products, α-sublattices, and α-homomorphisms. Therefore, by the extension of Birkhoff's Theorem [1] to algebras with infinitary operations, K_α is an equational class. We shall determine a set of equations which characterizes K_α. A simple sufficient condition is $(\alpha, 2^{\alpha})$ distributivity in either sense. Finally the class of α-retracts of α-rings of sets is discussed.

1. Definitions. We adopt the terminology of [2]. Let α be an infinite cardinal. An α-complete lattice is not assumed to have a largest or smallest element.

An *α-sublattice* of an α-complete lattice L is a subset M such that $\sum_{i L} x_i \in M$, and $\prod_{i L} x_i \in M$ for any non-empty α-system $\{x_i\}$ in M.

A family F of sets is called *α-independent* if the intersection of an α-system $\{x_i\}$ in F is contained in the union of an α-system $\{y_j\}$ in F only when some $x_i =$ some y_j. There exist α-independent families of any power. For example, if β is any cardinal, then for each $i \in \beta$, let x_i be the set of all subsets of β which contain i. The family $\{x_i\}$ is α-independent for any α.

Let K_α be the set of all α-homomorphic images of α-rings of sets. A lattice L in K_α is said to be a *free lattice* of class K_α with β generators if L has a subset W with the following properties:

1) W has power β.

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