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On ramification points in the classical sense

by

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Introduction. I call any point p of an arbitrary point set X a *point of order r in the classical sense*—or here briefly a *point of order r*—if p is a unique common end-point of every two of exactly r simple arcs contained in X. A point of order $r \geqslant 3$ will be called a *ramification point* (1).

Hilton and Wylie (see [1] (2), p. 380) constructed for every mapping $f\colon X \to Y$ a space Y_f called a *mapping cylinder* of f. We may understand Y_f as a cylinder with X as its top and with its base embedded in Y, the generators being segments connecting a point $x \in X$ with its image $f(x) \in Y$.

The first purpose of this paper is to prove that for each continuum Q and for each continuous mapping f of the Cantor set onto Q the mapping cylinder K_1 of f can be realized in the Euclidean space of dimension $2\dim Q+3$ as a continuum which is a union of straight segments, disjoint one from another out of Q and such that Q is the set of all ramification points of the continuum K_1. Namely one can place a cell I^k and a straight line L in the $(k+2)$-dimensional Euclidean space so that this straight line and the k-dimensional hyperplane containing the cell are *skew*, i.e. that there is no hyperplane of dimension $k+1$ which contains both those objects. Then the straight segments joining arbitrary points of the straight line L with arbitrary points of the cell I^k have at most the end-points in common. We then obtain the continuum K_1 by placing the Cantor set C in the straight line as well as the continuum $Q = f(C)$ in the cell I^k, and by joining every point $x \in C$ with its image $y = f(x) \in Q$ by the straight segment.

A further result is a construction of another two continua, namely the continuum K_2 having the same property, but in which the order of each ramification point is $2^{\aleph_0}$, and the continuum K_3, in which the

(1) A methodical investigation of sets of ramification points in the classical sense in the continua, i.e. in compact and connected metric spaces, was initiated by Professor B. Knaster in his Topological Seminar in Wrocław (Institute of Mathematics of the Polish Academy of Sciences). I am indebted to him for the project of this paper and for the idea of the proof of Theorem 1. He suggested also the existence of the singularity realized in the final part of this paper (see the dendroid Δ).

(2) The numbers in brackets denote the references, pp. 251 and 252.

orders of all ramification points are raised by $3\dim Q+3$ at most. The application of this method to *dendroids*, i.e. acyclic and arcwise connected curves (see two other definitions, p. 239), leads in particular to a paradoxical example of a *dendroid homeomorphic with the set of all ramification points of itself*. Such a dendroid does not exist in the plane.

Preliminaries. Let K be a continuum and r a cardinal number belonging to the sequence

(1) $$0, 1, 2, \ldots, \aleph_0, 2^{\aleph_0}.$$

If any point $p \in K$ is a point of order at most r in K in the sense defined above, i.e. if p is a common end-point of at most r arcs disjoint one from another out of p and contained in K, we write $\mathrm{Ord}_p K \leqslant r$. Correspondingly, we define the notions of a point of order at least r in K (writing $\mathrm{Ord}_p K \geqslant r$) and of order r in K (writing $\mathrm{Ord}_p K = r$).

In particular a point $p \in K$ will be said (as announced) to be a *ramification point* of K when $\mathrm{Ord}_p K \geqslant 3$ and an *end-point* of K when $\mathrm{Ord}_p K = 1$, i.e. when it is an end-points of every arc containing it.

I denote by $E(K)$ the set of all end-points of K and by $R(K)$ the set of all ramification points of K.

Let $\mathcal{I}$ be the segment $0 \leqslant x \leqslant 1$, and let $\mathcal{I}^k$ be the cartesian product of k segments $\mathcal{I}$ with itself. Henceforth always

(2) $$k = 1, 2, \ldots, \aleph_0.$$

A distance between two points $x' = (x_1', x_2', \ldots)$ and $x'' = (x_1'', x_2'', \ldots)$ of the cell $\mathcal{I}^k$, i.e. where $0 \leqslant x_j' \leqslant 1$ and $0 \leqslant x_j'' \leqslant 1$ for $j = 1, 2, \ldots$, is defined for finite k by the formula

(3) $$\varrho(x', x'') = \sqrt{\sum_{j=1}^{k} |x_j'' - x_j'|^2},$$

and for infinite k by the formula

(4) $$\varrho(x', x'') = \sum_{j=1}^{\infty} \frac{1}{2^j} |x_j'' - x_j'|.$$

Let I be the segment defined by formulae

(5) $$0 \leqslant x_1 \leqslant 1, \quad x_j = 0 \quad \text{for} \quad j = 2, 3, \ldots, k+2,$$

and let C be the Cantor set of points of this segment, i.e. the set of points having coordinates (5), where

(6) $$x_1 = \sum_{l=0}^{\infty} \frac{2c_l}{3^l} \quad \text{with} \quad c_l = 0 \text{ or } 1.$$

Apart from this we denote by $\mathcal{C}$ the Cantor set of numbers, i.e. the set of numbers of form (6).

The set C can be effectively decomposed on the union of r sets C_a disjoint one from another and homeomorphic with it as follows:

(7) $$C = \bigcup_{a \in A} C_a,$$

(8) $$A \text{ is compact}, \quad \dim A = 0 \quad \text{and} \quad \overline{\overline{A}} = r,$$

(9) $$C_{a_1} \cap C_{a_2} = 0 \quad \text{for} \quad a_1 \neq a_2.$$

Indeed, take for $r = 1, 2, \ldots, \aleph_0, 2^{\aleph_0}$ the decomposition of the Cartesian product $C \times A$ on r its horizontal linear parts P_a, i.e. all the points of the a-th part of $C \times A$ having the ordinates equal to a (if $r = 2^{\aleph_0}$, we take $A = \mathcal{C}$). Of course, the sets P_a are uncountable, perfect, disjoint and $C \times A = \bigcup_{a \in A} P_a$. Further, $C \times A$ is a 0-dimensional perfect set by (8), and thus homeomorphic with C. Then, if we denote by h an arbitrary homeomorphism of $C \times A$ onto C, the decomposition $C = \bigcup_{a \in A} h(P_a)$ obviously satisfies conditions (7)-(9) setting $C_a = h(P_a)$.

Further, let Y be a k-dimensional cell given by the formulae

(10) $$x_1 = 0, \quad x_2 = 1, \quad 0 \leqslant x_j \leqslant 1 \quad \text{for} \quad j = 3, 4, \ldots, k+2.$$

Notice that the segment I is, according to (5), contained in the straight line S determined by the equations

$$x_j = 0 \quad \text{for} \quad j = 2, 3, \ldots$$

(i.e. lies on the coordinate axis Ox_1), but the cell Y, according to (10), lies in the hyperplane H the equations of which in the $(k+2)$-dimensional Euclidean space are

$$x_1 = 0, \quad x_2 = 1$$

(i.e. H is the k-dimensional hyperplane parallel to the hyperplane $Ox_3x_4\ldots x_{k+2}$). We infer from this that there is no $(k+1)$-dimensional hyperplane containing the straight line S and the hyperplane H. Therefore S and H are skew (according to the definition of this notion, p. 229). Then, to simplify, we shall also say that the segment I defined by (5) and the cell Y defined by (10) are skew.

The following notations will henceforth be used: xy will denote an arbitrary arc with end-points x and y, $\overline{xy}$ the straight segment with the same end-points, $\delta(X)$ will be the diameter of X, and $\varrho(x, A)$ the distance between the point x and the set A, i.e. $\varrho(x, A) = \min\{\varrho(x, a) \colon a \in A\}$.

Construction of the continuum $B(Q)$. First we prove the following two lemmas:

LEMMA 1. *If I is the segment* (5) *and Y the cell* (10), *then there is a number $\eta > 0$ such that for every point p of the segment joining points $x \in I$ and $y \in Y$ the following inequality holds:*

$$\varrho(p, Y) \geqslant \eta \cdot \varrho(p, y).$$

Proof. Let $x = (\bar{x}_1, 0, 0, \ldots, 0) \in I$ and $y = (0, 1, y_3, \ldots, y_{k+2}) \in Y$, where $0 \leqslant \bar{x}_1 \leqslant 1$ and $0 \leqslant y_j \leqslant 1$ for $j = 3, 4, \ldots, k+2$. The segment $\overline{xy}$ joining x and y is determined by the following parametric equations in which $0 \leqslant t \leqslant 1$ and $j = 3, 4, \ldots, k+2$:

$$x_1 = \bar{x}_1 \cdot (1-t), \quad x_2 = t, \quad x_j = y_j \cdot t. \tag{11}$$

We consider separately finite and infinite k.

(i) *k is finite.* A point p of the segment $\overline{xy}$ having the coordinates (11), the distance between the points p and y by (3) is equal to $[\bar{x}_1^2 \cdot (1-t)^2 + (1-t)^2 + \sum_{j=3}^{k+2} y_j^2 \cdot (1-t)^2]^{1/2}$, i.e.

$$\varrho(p, y) = (1-t) \cdot \Big(\bar{x}_1^2 + 1 + \sum_{j=3}^{k+2} y_j^2\Big). \tag{12}$$

The minimum of $\varrho(p, y)$ for $y \in Y$ is realized when $y_j = 0$; thus by the definition of $\varrho(p, Y)$

$$\varrho(p, Y) = (1-t) \cdot (\bar{x}_1^2 + 1)^{1/2}.$$

Hence by (12)

$$\varrho(p, Y) = \varrho(p, y) \cdot (\bar{x}_1^2 + 1)^{1/2} \cdot \Big(\bar{x}_1^2 + 1 + \sum_{j=3}^{k+2} y_j^2\Big)^{-1/2}.$$

Since

$$(\bar{x}_1^2 + 1)^{1/2} \cdot \Big(\bar{x}_1^2 + 1 + \sum_{j=3}^{k+2} y_j^2\Big)^{-1/2} \geqslant (k+1)^{-1/2}$$

for all points x of I and all points y of Y, we have

$$\varrho(p, Y) \geqslant \varrho(p, y) \cdot (k+1)^{-1/2}$$

and $\eta = (k+1)^{-1/2}$.

(ii) *k is infinite.* In this case the distance $\varrho(p, y)$ accordind to (4) is equal to $\frac{1}{2}\bar{x}_1 \cdot (1-t) + \frac{1}{4}(1-t) + \sum_{j=3}^{\infty} (1/2^j) y_j \cdot (1-t)$, i.e.

$$\varrho(p, y) = (1-t) \cdot \Big(\frac{1}{2}\bar{x}_1 + \frac{1}{4} + \sum_{j=3}^{\infty} \frac{1}{2^j} y_j\Big). \tag{13}$$

The minimum of $\varrho(p, y)$ for $y \in Y$ being realized when $y_j = 0$, we have

$$\varrho(p, Y) = (1-t) \cdot (\tfrac{1}{2}\bar{x}_1 + \tfrac{1}{4}).$$

Hence by (13)

$$\varrho(p, Y) = \varrho(p, y) \cdot \Big(\frac{1}{2}\bar{x}_1 + \frac{1}{4}\Big) \cdot \Big(\frac{1}{2}\bar{x}_1 + \frac{1}{4} + \sum_{j=3}^{\infty} \frac{1}{2^j} y_j\Big)^{-1}.$$

Since

$$\Big(\frac{1}{2}\bar{x}_1 + \frac{1}{4}\Big) \cdot \Big(\frac{1}{2}\bar{x}_1 + \frac{1}{4} + \sum_{j=3}^{\infty} \frac{1}{2^j} y\Big)^{-1} \geqslant \frac{1}{2}$$

for all points x of I and all points y of Y, we have

$$\varrho(p, Y) \geqslant \varrho(p, y) \cdot \tfrac{1}{2}$$

and $\eta = \frac{1}{2}$, what finishes the proof.

Since the segment I and the cell Y defined above are skew, the following lemma holds [3]:

LEMMA 2. *If I is the segment* (5) *and Y is the cell* (10), *then every two different straight segments joining points $x \in I$ and $y \in Y$ either are disjoint or have only one end-point in common.*

Proof. Let

$$\begin{aligned} x' &= (x_1', 0, 0, \ldots, 0) \in I, \\ x'' &= (x_1'', 0, 0, \ldots, 0) \in I, \end{aligned} \tag{14}$$

$$\begin{aligned} y' &= (0, 1, y_3', y_4', \ldots, y_{k+2}') \in Y, \\ y'' &= (0, 1, y_3'', y_4'', \ldots, y_{k+2}'') \in Y, \end{aligned} \tag{15}$$

where $0 \leqslant y_j' \leqslant 1$ and $0 \leqslant y_j'' \leqslant 1$ for $j = 3, 4, \ldots, k+2$. The segments $\overline{x'y'}$ and $\overline{x''y''}$ joining x' and y', as well as x'' and y'' are defined by the following parametric equations in which $0 \leqslant t \leqslant 1$ and $j = 3, 4, \ldots, k+2$:

$$x_1 = x_1' \cdot (1-t), \quad x_2 = t, \quad x_j = y_j' \cdot t, \tag{16}$$

$$x_1 = x_1'' \cdot (1-t), \quad x_2 = t, \quad x_j = y_j'' \cdot t. \tag{17}$$

Suppose that two segments

$$\overline{x'y'} \neq \overline{x''y''} \tag{18}$$

defined by equations (16) and (17) have a point of intersection p with the value t_0 of the parameter t:

$$p = \big(x_1(t_0), x_2(t_0), \ldots, x_{k+2}(t_0)\big).$$

---

[3] It is, clearly, partial case of the general theorem in analytic geometry about arbitrary skew hyperplanes of dimension k and l, i.e. not lying in a $(k+l)$-dimensional hyperplane. The proof of this theorem is exactly the same, but because it is unnecessary as a whole in the continuation of this paper, I restrict myself to the proof of case $l = 1$.

Then by (16) and (17) we have

$$x_1' \cdot (1-t_0) = x_1''(1-t_0),$$
$$y_j' \cdot t_0 = y_j'' \cdot t_0 \quad \text{for} \quad j = 3, 4, \ldots, k+2.$$

If $0 < t_0 < 1$, then it follows from these equalities by (14) and (15) that $x' = x''$ and $y' = y''$; thus $\overline{x'y'} = \overline{x''y''}$, contrary to (18). Hence $t_0 = 0$ or $t_0 = 1$, which leads to $p = x' = x''$ or $p = y' = y''$. This proves the lemma.

Let Q be a continuum. I call a continuum K a *brush-continuum* with the base Q, and I denote it by $B(Q)$, if

(19) $$Q \subset K,$$

and if K is the union of straight segments $\overline{xy}$, called its *generators*, such that

(20) there is a number $\delta > 0$ such that for every x the inequality $\varrho(x, Q) \geqslant \delta$ holds,

(21) there is a number $\eta > 0$ such that for every generator $\overline{xy}$ of K and for every point p of $\overline{xy}$ the inequality $\varrho(p, Q) \geqslant \eta \cdot \varrho(p, y)$ holds,

(22) for every x the set $\overline{xy} - y$ is a component of the set $K - Q$.

Note that condition (21) implies the following one:

(23) $\overline{xy} \cap Q = y$ for every generator $\overline{xy}$ of $B(Q)$.

Denoting for every $y \in Q$ by $T(y)$ the union of all generators $\overline{xy}$ having the point y in common, we have

(24) $$B(Q) = \bigcup_{y \in Q} T(y).$$

Such a brush-continuum $B(Q)$ exists for every continuum Q; in fact, for every mapping f of the Cantor set C onto the continuum Q it is sufficient to form the mapping cylinder of f according to the result of Hilton and Wylie ([1], p. 380) cited above (see p. 229).

I prove more, namely that for every continuum Q and for every mapping $f\colon C \to Q$ such a brush-continuum $B(Q)$ can be constructed even in the Euclidean space of dimension $2\dim Q + 3$.

THEOREM 1. *For every continuous mapping f of the Cantor set C onto the continuum Q the set*

(25) $$K = \bigcup_{x \in C} \overline{xy}$$

is a brush-continuum $B(Q)$ which lies in the Euclidean cell of dimension $2\dim Q + 3$ and is the mapping cylinder of f.

Proof. Let $\dim Q = m$, where m is a natural number or $\aleph_0$. By virtue of the imbedding theorem of Menger-Nöbeling ([3], § 40, VII, 1, p. 69)

we can homeomorphically imbed Q in the cell I^{2m+1}. Therefore put $2m+1 = k$, and imbed Q in the k-dimensional cell Y defined by (10). Further, take an arbitrary continuous mapping f of the Cantor set C on the segment I defined by (5) onto Q and join every point $x \in C$ with its image $f(x) = y \in Q$ by the straight segment $\overline{xy}$. The set (25) so obtained is connected ([3], § 41, II, 2, p. 82) because no segment $\overline{xy}$ is separated from Q. In consequence of the compactness of the set C and of the continuity of the mapping f, we see that if $x^n \in C$ and $x^0 = \lim\limits_{n\to\infty} x^n$, then $x^0 \in C$ and $y^0 = f(x^0) = \lim\limits_{n\to\infty} y^n$, where $y^n = f(x^n)$. Therefore, by virtue of the straightness of segments $\overline{xy}$, we have $\operatorname*{Lim}\limits_{n\to\infty} \overline{x^n y^n} = \overline{x^0 y^0}$, which proves by (25) that K is compact. Hence K is a continuum. Moreover, by $Q \subset Y$, we conclude from Lemma 1 that condition (21) is satisfied and, by virtue of Lemma 2, that condition (22) holds. Since the mapping f is onto, the inclusion (19) is true. Finally, every point $x \in C \subset I$ has by (5) the coordinate $x_2 = 0$ and every point $y \in Q \subset Y$ has by (10) the coordinate $x_2 = 1$. Therefore the distance $\varrho(x, Q)$ between the point x and the continuum Q satisfies by (3) and (4) the inequality $\varrho(x, Q) \geqslant 1/4$ for every x. Thus condition (20) is also satisfied. Hence K is a $B(Q)$ and, by (25), where $y = f(x)$, K is the mapping cylinder of f.

THEOREM 2. *For every continuum Q there exists a continuum K_1 which is a brush-continuum $B(Q)$ such that*

(26) *Q is the set of all its ramification points.*

Proof. Let $r \geqslant 3$ be a number belonging to sequence (1) and let $C \subset I$ be decomposed according to (7) with conditions (8) and (9), where C_a are—as before (see Preliminaries)—homeomorphic with C. Further, h being a homeomorphism of $C \times A$ onto C, let φ_a be an arbitrary continuous mapping of $P_a = C \times (a)$ onto Q such that

(27) $\varphi_{a_1}(x, a_1) = \varphi_{a_2}(x, a_2)$ for every $x \in C$ and every pair $a_1, a_2 \in A$.

Denote by φ the mapping of $C \times A$ onto Q determined by the condition

(28) $$\varphi | P_a = \varphi_a.$$

Putting

(29) $$f = \varphi h^{-1},$$

we can easily see by virtue of (27) and (28) that f maps continuously the Cantor set C onto Q. Thus by Theorem 1 the continuum K defined by (25) is a brush-continuum $B(Q)$. Moreover, by (7), (8) and (9), we have

(30) $$\overline{\overline{f^{-1}(y)}} \geqslant r \quad \text{for every} \quad y \in Q.$$

From the hypothesis that $r \geqslant 3$ it follows by (30) that every point $y \in Q$ is the common end-point of at least three segments $\overline{xy}$; hence $Q \subset R(K)$. Conversely, because by virtue of Lemma 2 the segments $\overline{xy}$ are disjoint except—perhaps—the end-points $y \in Q$, there are in K no other ramification points except the end-points y of the segments $\overline{xy}$; hence $R(K) \subset Q$. Both inclusions give (26). It suffices then to denote K by K_1.

COROLLARY 1. *For every continuum Q there exists a continuum K_2 which is a brush-continuum $B(Q)$ such that (26) is satisfied and that*

$$\text{(31)} \qquad \operatorname{Ord}_y K_2 = 2^{\aleph_0} \quad \textit{for every} \quad y \in Q.$$

Indeed, it suffices to take $r = 2^{\aleph_0}$ in the proof of Theorem 2 and to denote K_1 by K_2.

Let us remark that condition (22) implies the following

COROLLARY 2. *The intersection of the continuum K, K_1 or K_2 with the hyperplane $x_2 = x_2^0$, where $0 < x_2^0 \leqslant 1$, is 0-dimensional.*

Estimations. In the previous construction of the continuum K_2 the orders of all points y of the continuum Q become maximally raised by $2^{\aleph_0}$ (see (31)). This suggests the inverse problem: how to construct a brush-continuum $B(Q)$ (possibly having property (26)) such that the difference $\operatorname{Ord}_y B(Q) - \operatorname{Ord}_y Q$ be as small as possible? With this in view we first prove the following supplement of Theorem 1:

THEOREM 3. *For every continuous mapping f of the Cantor set C onto a continuum Q such that*

$$\text{(32)} \qquad \overline{\overline{f^{-1}(y)}} \leqslant s \quad \textit{for every} \quad y \in Q,$$

we have

$$\text{(33)} \qquad \operatorname{Ord}_y K \leqslant \operatorname{Ord}_y Q + s.$$

Proof. In fact, it follows from inequality (32) that every point $y \in Q$ is a common end-point of at most s straight segments $\overline{xy}$, which by virtue of the disjointness of these segments one from another out of point y (see Lemma 2) and by (25) gives (33).

COROLLARY 3. *For every continuum Q there exists a brush-continuum $B(Q)$ such that for every point $y \in Q$*

$$\text{(34)} \qquad \operatorname{Ord}_y B(Q) \leqslant \operatorname{Ord}_y Q + \dim Q + 1.$$

Indeed, let $\dim Q = m$. Let us take in the construction of the continuum K (Theorems 1 and 3) the value $r = 1$. It is known ([3], § 40, II, 1, p. 54) that there exists a continuous mapping f of the Cantor set C onto the continuum Q such that

$$\text{(35)} \qquad \overline{\overline{f^{-1}(y)}} \leqslant m+1 \quad \text{for every} \quad y \in Q.$$

Hence, by substituting in (32) and (33) $m+1$ for s and $B(Q)$ for K, we have (34).

COROLLARY 4. *For every continuum Q there exists a brush-continuum $B(Q)$ such that (26) is satisfied and that for every point $y \in Q$*

$$\text{(36)} \qquad \operatorname{Ord}_y B(Q) \leqslant \operatorname{Ord}_y Q + 3\dim Q + 3.$$

It suffices to remark that if $\overline{\overline{\varphi_a^{-1}(y)}} \leqslant s$ for every $y \in Q$, where $\alpha \in A$, then the mapping f defined by (29) satisfies by (28) the inequality

$$\text{(37)} \qquad \overline{\overline{f^{-1}(y)}} \leqslant r \cdot s.$$

By taking in the proof of Theorem 2 the value $r = 3$ and by choosing the mappings φ_a in such a manner that for each of them inequality (35) holds ([3], § 40, II, 1, p. 54), where $m = \dim Q$, and denoting K_1 by $B(Q)$ we obtain by (37) and by Theorem 2 inequality (36).

We now prove that estimation (34) cannot be lowered. We infer this conclusion from the following Lemma 3 and Theorem 4.

LEMMA 3. *Decomposition (24) is upper semicontinuous.*

Proof. Let for y^n and y, points of Q, and for $p^n \in T(y^n)$

$$\text{(38)} \qquad \lim_{n\to\infty} y^n = y \quad \text{and} \quad \lim_{n\to\infty} p^n = p.$$

Thus the segments $\overline{p^n y^n} \subset T(y^n)$ tend by their straightness to the segment $\overline{py}$. By virtue of the compactness of $B(Q)$ we have

$$\text{(39)} \qquad \overline{py} \subset B(Q).$$

Note that it follows from (21) by (38) that

$$\text{(40)} \qquad \overline{py} \cap Q = y.$$

Let us consider two cases:

Case 1: $p \in Q$. In this case we have the equality $p = y$, because otherwise the distance $\varrho(p^n, Q)$ would tend to zero and $\varrho(p^n, y^n)$ would tend to $\varrho(p, y)$; thus we would have for every fixed number $\eta > 0$ and for a sufficiently great n the inequality

$$\varrho(p^n, Q) < \eta \cdot \varrho(p^n, y^n)$$

contrary to (21). Thus, p being y, we have $p \in T(y)$.

Case 2: $p \in B(Q) - Q$. In this case by (24) p belongs to some $T(y') - y'$, and thus to some $\overline{xy'} - y'$, which is by (22) a component of $B(Q) - Q$. Since the set $\overline{py} - y$ is by (39) and (40) a connected subset of $B(Q) - Q$, we have

$$\overline{py} - y \subset \overline{xy'} - y';$$

thus by (23) $y = y'$, which implies that $p \in T(y)$ in the case 2 also.

The condition $p \in T(y)$ implies by (38) that the set-valued mapping which assigns to every point $y \in Q$ the set $T(y)$ is upper semicontinuous. It is equivalent ([3], § 39, V, 1, p. 42) to the upper semicontinuity of decomposition (24).

THEOREM 4. *For every continuum Q, for its every brush-continuum $B(Q)$ and for every point $y \in Q$ it follows from the condition*

(41) $$\mathrm{Ord}_y B(Q) \leqslant \mathrm{Ord}_y Q + m + 1$$

that

(42) $$\dim Q \leqslant m .$$

Proof. Let Q be a continuum and let $B(Q) = \bigcup \overline{xy}$ be an arbitrary brush-continuum with the base Q. Let us take in every generator $\overline{xy}$ of $B(Q)$ a point z such that $\varrho(y, z) = \delta$, where the number δ satisfies condition (20), and let Z be the set of all such points z. By the upper semicontinuity of decomposition (24) (see Lemma 3) the set Z is compact; thus by virtue of (22) we have

(43) $$\dim Z = 0 .$$

Let f be a mapping which assigns to each point $z \in \overline{xy}$ the end-point $y \in Q$ of the generator $\overline{xy}$. By virtue of Lemma 3 the mapping f is continuous and by (19) we clearly have the equality

(44) $$f(Z) = Q .$$

It follows from (41) that every point $y \in Q$ is a common end-point of at most $m+1$ generators of the brush-continuum $B(Q)$. Thus we have $\overline{\overline{f^{-1}(y)}} \leqslant m+1$, and (42) follows from the Hurewicz theorem ([3], § 40, I, 2, p. 52) by (43) and (44).

Now, in order to see that estimation (34) cannot be lowered, it suffices to remark that otherwise it would imply that for every continuum Q there exists a brush-continuum $B(Q)$ such that for every point $y \in Q$

$$\mathrm{Ord}_y B(Q) < \mathrm{Ord}_y Q + \dim Q + 1 ,$$

therefore that there exists a continuum Q, a brush-continuum $B(Q)$ and a point $y \in Q$ such that

$$\mathrm{Ord}_y B(Q) \leqslant \mathrm{Ord}_y Q + m + 1$$

and $m < \dim Q$. But that would be contrary to Theorem 4.

The possibility of a reduction of estimation (36) for all continua Q (with the preservation of condition (26)) remains an open problem.

However, with some additional hypotheses about the continuum Q this estimation can immediately be improved. With this purpose notice that the using of the value $r = 3$ for the construction of $B(Q)$ in Corollary 3 had in view the ensuring of property (26) for every Q. In fact, if there exists in Q a point y lying in no arc (when Q is, for instance, a hereditarily indecomposable continuum), then in order that $y \in R(B(Q))$ it is necessary and sufficient that y be the common end-point of at least three arcs, necessarily disjoint in this case from Q except at that end-point. But if Q is a union of arcs, i.e. if there exists in Q for every $y \in Q$ an arc containing y, we can take $r = 2$ instead of $r = 3$, because in this case, besides at least two arcs (i.e. the segments $\overline{xy}$) having the end-point y only in common with Q, there exists in Q at least one more such arc (and when $y \in E(Q)$, exactly one). So condition (26) is also satisfied. In this case estimation (36) can thus be improved: namely we have the following

COROLLARY 5. *For every arcwise connected continuum Q there exists a brush-continuum $B(Q)$ such that* (26) *is satisfied and that for every point $y \in Q$*

(45) $$\mathrm{Ord}_y B(Q) \leqslant \mathrm{Ord}_y Q + 2\dim Q + 2 .$$

Applications to dendroids. A space X is said to be *unicoherent* provided that it is connected and for every decomposition $X = A \cup B$ on closed and connected sets the intersection $A \cap B$ is connected ([3], § 41, X, p. 104). I call—with B. Knaster—a *dendroid* each arcwise connected and hereditarily unicoherent continuum. This definition is equivalent ([7], Theorem 1.1, p. 179) to the following:

(46) A *dendroid* is an arcwise connected continuum every two points of which can be joined by exactly one irreducible continuum between them.

In particular, it follows that

(47) *No dendroid contains any indecomposable continuum.*

Indeed, if we suppose that a and b are points of an indecomposable continuum N which is irreducible between a and b ([3], § 43, VII, 7, p. 150) and contained in a dendroid Δ, then these points are joined in Δ by two irreducible continua, namely by an arc (Δ being arcwise connected) and by the continuum N.

It follows from (47) that

(48) *Every dendroid is a curve*, i.e. a continuum of dimension 1.

Indeed, any compact space of higher dimension contains indecomposable continua ([3], § 43, V, p. 144).

Dendroids are a generalization of dendrites: every dendrite is a dendroid, and every locally connected dendroid is a dendrite ([6], X, 2, Theorems 1 and 2, p. 306).

From the definition of the dendroid it follows also that

(49) *Every subcontinuum of a dendroid is a dendroid.*

Especially, from (49) it follows that $E(\Delta)$, i.e. the set of all end-points of the dendroid Δ, does not contain any non-degenerate continuum ([4], 2.1, p. 302). Thus

(50) *If $E(\Delta)$ is F_σ, then* $\dim E(\Delta) = 0$.

The hypothesis of (50) that $E(\Delta)$ is F_σ is essential, because Lelek has given in [4], p. 314 an example of a dendroid D such that $E(D)$ is not F_σ and that $\dim E(D) = 1$.

The following examples of curves lying in the plane Ox_1x_2 are dendroids:

E 1. The *harmonic fan* M_h consisting of a straight segment joining the point $(0, 1)$ with the origin and of straight segments joining the same point with points $(1/n, 0)$ for $n = 1, 2, \ldots$

The set $E(M_h)$ is countable and compact (the harmonic sequence and the origin).

E 2. The *Cantor fan* M_C consisting of straight segments joining the point $(1/2, 1/2)$ with all points $(x_1, 0)$, where x_1 has the form (6).

$E(M_C)$ is the Cantor set.

E 3. The *Cantor hooked fan* M_C^v defined in the following manner:

Let T^1 be the set of the left end-points of components of $I - C$, and T^2 the set of their right end-points. The set $T^1 \cup T^2$ is countable and $\overline{T^1 \cup T^2} = C$.

Let us assign to every point $p = (x_1, 0)$ of $T^1 \cup T^2$ the point $p' = (x_1', 1/4)$ defined by the formula

$$x_1' = \frac{1}{2}x_1 + \frac{1}{4} - (-1)^i \cdot \frac{1}{6 \cdot 3^n},$$

where $1/3^n$ is the length of the interval of which p is an end-point, and $i = 1$ or 2 according to whether $p \in T^1$ or $p \in T^2$. We put

$$M_C^v = M_C \cup \bigcup \overline{pp'},$$

where $p \in T^1 \cup T^2$.

The set $E(M_C^v)$ consists of a countable set of the points p' and of a G_δ-set $C - (T^1 \cup T^2)$.

E 4. The brush-continuum $B(J)$ constructed as follows:

Let J be the segment $0 \leqslant x_1 \leqslant 1$, $x_2 = 1$, and f the *Cantor stair-function* ("fonction scalariforme" de Cantor) onto the segment J, i.e. the mapping which maps continuously ([2], § 24a, VIa, p. 236) every point $x = (x_1, 0) \in C \subset I$, where x_1 has form (6) and I is the segment (5), onto the point $y = (x_1', 1) \in J$, where $x_1' = \sum_{l=0}^{\infty} c_l/2^l$. Thus, $f(C) = J$. The continuous mapping f is by definition non-decreasing; thus, if $x^1 = (x_{1,1}, 0)$ and $x^2 = (x_{1,2}, 0)$ are points of the set C and $x_{1,1} < x_{1,2}$, then for its images $y^1 = f(x^1) = (x_{1,1}', 1)$ and $y^2 = f(x^2) = (x_{1,2}', 1)$ we have $x_{1,1}' \leqslant x_{1,2}'$. Hence different segments $\overline{xy}$ either are disjoint or have only end-points y in common. The brush-continuum $B(J)$ is obtained by putting $B(J) = \bigcup_{x \in C} \overline{xy}$.

$E(B(J))$ is the Cantor set.

THEOREM 5. *Each brush-continuum $B(\Delta)$ of a dendroid Δ is also a dendroid.*

Proof. The continuum $B(\Delta)$ is by definition of the form

$$B(\Delta) = \bigcup \overline{xy},$$

where the generators $\overline{xy}$ satisfy conditions (20)-(22). Let $p' \in \overline{x'y'}$ and $p'' \in \overline{x''y''}$. If $\overline{x'y'} = \overline{x''y''}$, then the partial segment $p'p''$ is by (22) a single irreducible continuum between p' and p'' in $B(\Delta)$. If $x'y' \neq x''y''$, we join y' with y'' by an arc $y'y''$ in Δ (this arc can shrink to one point if $y' = y''$). Then $\overline{p'y'} \cup y'y'' \cup \overline{y''p''}$ is obviously an arc $p'p''$ in $B(\Delta)$ and it may easily be seen by (22) that there is in $B(\Delta)$ no other irreducible continuum between these points. Thus the conditions of definition (46) are satisfied.

Theorems 2 and 5 imply the following

COROLLARY 6. *For every dendroid Δ there exists a brush-continuum $B(\Delta)$ which is a dendroid such that Δ is the set of all its ramification points.*

Moreover, we conclude from Corollary 5 by virtue of (46) and (48) that Corollary 6 can be specified as follows:

COROLLARY 7. *For every dendroid Δ there exists a brush-continuum $B(\Delta)$ which is a dendroid such that Δ is the set of all its ramification points and that for every point $y \in \Delta$*

$$\mathrm{Ord}_y B(\Delta) \leqslant \mathrm{Ord}_y \Delta + 4.$$

By using a slightly different construction, the estimation of the order of ramification points may be lowered for some types of dendroids, namely those having the compact set of end-points. For this purpose we prove, in the first place, the following lemma concerning the prolongation of dendroids beyond the set of all their end-points:

LEMMA 4. *If the set $E(\Delta)$ of a dendroid Δ is F_σ, then there exists a dendroid $\Delta' \supset \Delta$ such that*

$$\text{(51)} \qquad \mathrm{Ord}_p \Delta' = \begin{cases} \mathrm{Ord}_p \Delta, & \text{when } p \in \Delta - E(\Delta), \\ 2, & \text{when } p \in E(\Delta), \\ r \leqslant 2, & \text{when } p \in \Delta' - \Delta, \end{cases}$$

and that

(52) *the set* $(\Delta'-\Delta)\cup E(\Delta)$ *is the union of straight segments which are its components.*

Proof. As a curve (see (48)), every dendroid can, by virtue of the Menger-Nöbeling theorem ([3], § 40, VII, 1, p. 69), be homeomorphically imbedded in the 3-dimensional cell Y defined by (10) for $k=3$. Let $E(\Delta)=\bigcup_{n=1}^{\infty}F_n$, where F_n are compact, and let $p=(0,0,x_3',x_4',x_5')\in F_n$. Further, let $I(n,p)$ be a segment defined by equations (in which $n=1,2,\ldots$)

(53) $$x_1=x_2=0\,,\quad x_3=x_3'\,,\quad x_4=x_4'\,,\quad x_5=x_5'\,,\quad 0\leqslant x_6\leqslant 1/n \quad \text{and} \quad x_j=0 \quad \text{for} \quad j>6\,.$$

Then

(54) $$\Delta'=\Delta\cup\bigcup_{n=1}^{\infty}\bigcup_{p\in F_n}I(n,p)\,.$$

Since the segments $I(n,p)$, except their end-points $y\in E(\Delta)$, lie beyond Y, they have no other common points with $\Delta\subset Y$. It is easy to see that property (51) is therefore verified. The hereditary unicoherence of Δ implies that of Δ'. The length of segments $I(n,p)$ tends by their definition to zero; thus the limit points of $\Delta'-\Delta$ lie in Δ, namely in the set $\overline{E(\Delta)}\subset\Delta$. Hence Δ' is compact. Δ' is connected as the union of the continuum Δ and of segments $I(n,p)$ having common end-points with it. Moreover, it is easy to see that Δ' is arcwise connected; hence it is a dendroid. Condition (52) is satisfied by virtue of definition (54) and by (50).

The hypothesis of Lemma 4 that $E(\Delta)$ is F_σ is essential. In fact, the dendroid M_C^v (see example E 3, p. 256), whose end-point set consists of a countable F_σ and of a G_δ of power $2^{\aleph_0}$ (the set $C-(T^1\cup T^2)$), cannot be prolonged beyond all its end-points with preserving property (51). Indeed, suppose that a dendroid M' exists satisfying conditions (51) and (52) with $\Delta=M_C^v$ and $\Delta'=M'$. Consider the G_δ-set $C-(T^1\cup T^2)\subset E(M_C^v)$ and denote for every point $q\in C-(T^1\cup T^2)$ by q' the other end-point of the prolonging segment $\overline{qq'}$, i.e. the segment which has one common point with M_C^v only and is, accordingly to (52), a component of the set $(M'-M_C^v)\cup E(M_C^v)$. Let $m=1,2,\ldots$, and let E_m be the subset of $C-(T^1\cup T^2)$ consisting of all the points q such that

(55) $$\varrho(q',M_C^v)\geqslant\frac{1}{m}\,.$$

Every E_m is compact. In order to see it, consider a convergent sequence $\{q_i\}$ of points of a given E_m and put $q_0=\lim_{i\to\infty}q_i$. If q_0 were a point $p\in T^1\cup T^2$, then the prolonging segments $\overline{q_iq_i'}$ would tend to a straight segment $\overline{pq_0'}$ such that $\overline{pq_0'}\cap\overline{pp'}=p$, because by the definition of M_C^v every point $a\neq p$ of $\overline{pp'}$ lies in the interior of a triangle whose inferior vertices are end-points of a complementary interval of C and thus cannot be a limit point of those segments $\overline{q_iq_i'}$ which lie outside this triangle. The point p would then be a ramification point of M', contrary to (51). It follows that $q_0\in C-(T^1\cup T^2)$ and, the end-point q_0' of the limit segment $\overline{q_0q_0'}$ obviously satisfying (55) also, we conclude that $q_0\in E_m$. As a compact subset of the boundary set $C-(T^1\cup T^2)$, the set E_m is non-dense in C. We would have hence $C=T^1\cup T^2\cup\bigcup_{m=1}^{\infty}E_m$, which is impossible by virtue of Baire's category theorem.

From Lemma 4 follows

Corollary 8. *If the set* $E(\Delta)$ *of a dendroid* Δ *is compact, then there exists a dendroid* $\Delta^*\supset\Delta$ *satisfying conditions* (51) *and* (52) (*with* Δ^* *instead of* Δ') *and such that*

(56) *there is a* $\delta>0$ *such that for every component* $\overline{xy}$ *of* $(\Delta^*-\Delta)\cup E(\Delta)$ *the inequality* $\varrho(x,\Delta)\geqslant\delta$ *holds.*

Indeed, it is sufficient to put in the proof of Lemma 4 $F_n=E(\Delta)$ for every $n=1,2,\ldots$ Then, clearly, we have $I(n+1,p)\subset I(n,p)$ for every natural n and for every $p\in E(\Delta)$. Hence by putting

(57) $$I(p)=I(1,p)$$

and by substituting in (54) Δ^* for Δ' we obtain

(58) $$\Delta^*=\Delta\cup\bigcup_{p\in E(\Delta)}I(p)\,.$$

The proof of conditions (51) and (52) for Δ^* instead of Δ' is the same as in Lemma 4. Condition (56) with $\delta=1$ holds by the definition of the segments $I(p)$.

Theorem 6. *If the set* $E(\Delta)$ *of a dendroid* Δ *is compact, then there exists a brush-continuum* $B(\Delta)$ *such that* Δ *is the set of all its ramification points and that*

(59) $$\mathrm{Ord}_yB(\Delta)\leqslant\mathrm{Ord}_y\Delta+2 \quad \textit{for every} \quad y\in\Delta-E(\Delta)\,,$$

(60) $$\mathrm{Ord}_yB(\Delta)\leqslant\mathrm{Ord}_y\Delta+3 \quad \textit{for every} \quad y\in E(\Delta)\,.$$

Proof. Let Δ^* be the dendroid considered in Corollary 8 and defined by (58). Let us homeomorphically imbed Δ^* in the 3-dimensional cell Y defined by (10) for $k=3$, and let f be a continuous mapping of the Cantor set $C\subset I$ onto the dendroid $\Delta\subset\Delta^*$

(61) $$f(C)=\Delta$$

such that (see [3], § 40, II, 1, p. 54)

$$\overline{\overline{f^{-1}(y)}} \leqslant 2 \quad \text{for every} \quad y \in \Delta . \tag{62}$$

Let us join every point $x \in C$ with its image $y = f(x) \in \Delta$ by the straight segment $\overline{xy}$ and put

$$B(\Delta) = \Delta^* \cup \bigcup_{x \in C} \overline{xy} . \tag{63}$$

The set $B(\Delta)$ defined in this manner is connected as the union of the dendroid Δ^* and the straight segments having common points with it. $B(\Delta)$ is compact by the continuity of f. Thus it is a continuum. By (58) and (63) we see, firstly, that $\Delta \subset B(\Delta)$, whence condition (19) of the definition of a brush-continuum is satisfied, and, secondly, that $B(\Delta)$ is the union of the segments $I(p)$, where $p \in E(\Delta)$, and the segments $\overline{xy}$, where $x \in C$. To prove condition (21) we recall that the dendroid Δ is imbedded in the cell $Y \subset I^5$ defined by (10) for $k = 3$ and that the segments $I(p)$ are defined according to (57) by equations (53) for $n = 1$. It follows that $I(p) - p \subset I^6 - I^5$ for every $p \in E(\Delta)$. Hence $\varrho(q, \Delta) = \varrho(q, p)$ for every $q \in I(p)$. Condition (21) is then satisfied for $I(p)$ putting $\eta = 1$; for the segments $\overline{xy}$ the same condition holds by Lemma 1. Also (22) holds by (52) for the segments $I(p)$ and by Lemma 2 for $\overline{xy}$. Finally, condition (20) is satisfied for $I(p)$ by (56) and for $\overline{xy}$ by (5) and (10). Thus $B(\Delta)$ is a brush-continuum, and by virtue of Theorem 5, a dendroid.

Further, every point $y \in \Delta$ is a common end-point of at least three arcs disjoint one from another out of y. In fact, for every point $y \in \Delta$ there exists by (61) at least one segment $\overline{xy}$ disjoint, neglecting y, from Δ^* and hence also from Δ. Moreover, if $y \in \Delta - E(\Delta)$, then we have $\mathrm{Ord}_y \Delta \geqslant 2$ by the arcwise connectedness of the dendroid Δ; then there exist in Δ at least two arcs with the common end-point y, disjoint beyond this point. If $y \in E(\Delta)$, the point y is in Δ^* a common end-point of exactly two arcs also disjoint beyond y: one of them lies in Δ, and the other—the segment $I(p)$—lies in $\Delta^* - \Delta \cup y$. Thus $\Delta \subset R(B(\Delta))$. Further, we have $R(B(\Delta)) \subset \Delta$ by the disjointness of the segments $I(p)$ from the dendroid Δ beyond their end-points p and by the disjointness of the segments $\overline{xy}$ from the dendroid Δ^* beyond their end-points y. The equality $R(B(\Delta)) = \Delta$ is proved.

At last there exist by (62) for every point $y \in \Delta$ at most two arcs $\overline{xy}$ with the common end-point y. If $y \in \Delta - E(\Delta)$, then all other arcs with end-point y are contained in Δ by the construction, which proves (59); and if $y \in E(\Delta)$, then there exists by (51) and (52) only one more arc, namely the segment $I(y)$ disjoint, neglecting y, from the dendroid Δ and from the arcs $\overline{xy}$. Estimation (60) follows.

It is interesting to note that the aforesaid improvement of estimation (conditions (59) and (60)) can be obtained by virtue of Lemma 4 also in the more general case when $E(\Delta)$ is F_σ if we omit condition (20) in the definition of the brush-continuum (this generalized notion of brush-continuum I call a *nearly brush-continuum*).

Indeed, let Δ' be the dendroid that exists by virtue of Lemma 4 and is defined by equality (54). By treating the Δ' as the Δ^* in the proof of Theorem 6 we obtain the nearly brush-continuum $B'(\Delta) = \Delta' \cup \bigcup_{x \in C} \overline{xy}$. The continuation of the proof is analogical to that of Theorem 6.

The dendroid Δ. Lelek has constructed in the Hilbert fundamental cube an example of a dendroid homeomorphic with the set of all its ramification points (not published). I give here, with his kind consent, a modification of his example. Namely I construct a dendroid Δ having besides the previous property the property $\mathrm{Ord}_y \Delta \leqslant 4$ for every point $y \in \Delta$.

Construction. Begin to numerate the coordinate axes of Hilbert cube $I^{\aleph_0}$ from number 0 instead of 1. We shall thus have the axes Ox_0, Ox_1 and so on. The axis Ox_0 will be reserved for the end of the construction.

Consider the segment I defined by (5) and let $I_n \subset I$, where $n = 0, 1, 2, \ldots$, be the segment defined by the conditions

$$2^{-(n+1)} \leqslant x_1 \leqslant 2^{-n}, \quad x_j = 0 \quad \text{for} \quad j = 2, 3, \ldots \tag{64}$$

Denoting by p_0 the end-point of I with abscissa $x_1 = 0$ we have by (5) and (64)

$$I = p_0 \cup \bigcup_{n=0}^{\infty} I_n = \overline{\bigcup_{n=0}^{\infty} I_n} . \tag{65}$$

Let $m = 1, 2, \ldots, 2n+1$ and $n = 0, 1, 2, \ldots$, and let I_n^m be the m-dimensional cell defined by the conditions

$$\begin{aligned} 2^{-(n+1)} \leqslant x_j \leqslant 2^{-n} \quad &\text{for} \quad j = 1, 2, \ldots, m, \\ x_j = 0 \quad &\text{for} \quad j = m+1, m+2, \ldots \end{aligned} \tag{66}$$

Thus

$$\lim_{n\to\infty} \delta(I_n^m) = 0 , \tag{67}$$

whence

$$p_0 = \mathop{\mathrm{Lim}}_{n\to\infty} I_n^m \quad \text{for each sequence of values of } m . \tag{68}$$

For instance the union $\bigcup_{n=0}^{\infty} I_n^{2n+1}$ is then obviously a non-dense subcontinuum of $I^{\aleph_0}$ (a known weakly infinite dimensional continuum). In this union the fundamental one third Δ_0 of the example Δ will be constructed.

Denoting by J_n the segment defined by the three conditions

$$(69)\qquad \begin{aligned} x_j &= 2^{-(n+1)} && \text{for} \quad j = 1, 2, \ldots, 2n, \\ 0 \leqslant x_j &\leqslant 2^{-(n+1)} && \text{for} \quad j = 2n+1, \\ x_j &= 0 && \text{for} \quad j = 2n+2, 2n+3, \ldots, \end{aligned}$$

we see by (66) and (69) that $J_n \subset I_n^{2n+1}$, and that J_n and I_n^{2n-1} are skew.

Further, let C_n be the Cantor set on the segment J_n.

At last, let h be the homothetic transformation with centre p_0 and with ratio 1/2. Thus

$$(70)\qquad h(p_0) = p_0,$$

and by the definition of the cell I_n^m we have for every m and n

$$(71)\qquad h(I_n^m) = I_{n+1}^m.$$

We now define by induction the sequence $\{B_n\}$ of dendroids (nearly brush-continua) which will be the important parts of the dendroid A_0. Put $B_0 = I_0$. Let f_1 be the Cantor stair-function which maps continuously the Cantor set $C_1 \subset J_1$ onto the set $I_1^1 = I_1$ (see [2], § 24a, VIa, p. 236, also the example E4 in this paper). It is known that

$$(72)\qquad \overline{\overline{f_1^{-1}(y)}} \leqslant 2 \quad \text{for every } y \in I_1,$$

$$(73)\qquad \overline{\overline{f_1^{-1}(y)}} = 1 \quad \text{if} \quad y \in E(I_1).$$

Joining every point $x \in C_1$ with its image $y = f_1(x) \in I_1$ by a straight segment, put

$$(74)\qquad B_1 = \bigcup_{x \in C_1} \overline{xy}.$$

It can be shown as in the proof of Theorem 1 that the set B_1 defined in this way is a brush-continuum with the base I_1. Therefore B_1 is a dendroid by Theorem 5.

Now let $n > 1$ and assume the dendroids

$$(75)\qquad B_{n-2} \subset I_{n-2}^{2n-3} \quad \text{and} \quad B_{n-1} \subset I_{n-1}^{2n-1}$$

to be defined. We map the dendroid B_{n-1} by the homothetic transformation h onto the dendroid $h(B_{n-1})$. There exists (by (48) and [3], § 40, II, 1, p. 54) a continuous mapping f_n of C_n onto the dendroid $h(B_{n-1})$ such that

$$(76)\qquad \overline{\overline{f_n^{-1}(y)}} \leqslant 2 \quad \text{for every} \quad y \in h(B_{n-1}).$$

Joining every point $x \in C_n$ with its image $y = f_n(x) \in h(B_{n-1})$ by a straight segment we obtain the set

$$(77)\qquad B\big(h(B_{n-1})\big) = \bigcup_{x \in C_n} \overline{xy}$$

which is a brush-continuum with the base $h(B_{n-1})$; this can be shown as in the proof of Theorem 1. The continuum B_{n-1} being a dendroid by the inductive hypothesis, continuum (77) is also dendroid by Theorem 5. Clearly

$$(78)\qquad B\big(h(B_{n-1})\big) \subset I_n^{2n+1}.$$

Consider the set $h\big(h(B_{n-2})\big)$. By the first inclusion of (75) and by (71) we have $h(B_{n-2}) \subset h(I_{n-2}^{2n-3}) = I_{n-1}^{2n-3} \subset I_{n-1}^{2n+1}$; hence

$$h\big(h(B_{n-2})\big) \subset h(I_{n-1}^{2n-3}) = I_n^{2n-3} \subset I_n^{2n+1}.$$

Now take in every $\overline{xy}$ of union (77) a point x' such that

$$(79)\qquad \varrho(x', y) = \varrho(x, y) \cdot \frac{\varrho\Big(y, h\big(h(B_{n-2})\big)\Big)}{\delta\Big(B\big(h(B_{n-2})\big)\Big)}$$

(which exists since the last fraction is smaller than 1), and let g_n be a mapping of the dendroid $B\big(h(B_{n-1})\big)$ into itself such that every partial mapping $g_n|\overline{xy}$ is the homothetic transformation of $\overline{xy}$ onto its partial segment $\overline{x'y}$ with the centre y. Thus

$$(80)\qquad g_n\Big(B\big(h(B_{n-1})\big)\Big) \subset B\big(h(B_{n-1})\big),$$

and in consequence of (79) the mapping g_n is continuous.

By (49) and (80) the continuum $g_n\Big(B\big(h(B_{n-1})\big)\Big)$ is a dendroid and we see by the definition of the mapping g_n that

(81) the mapping g_n is an identity on $h(B_{n-1})$,

$$(82)\qquad g_n(\overline{xy}) = y \quad \text{for every} \quad y \in h\big(h(B_{n-2})\big).$$

Put

$$(83)\qquad B = g_n\Big(B\big(h(B_{n-1})\big)\Big);$$

thus

(84) *B_n is a dendroid.*

Inclusions (80) and (78) instantly give

$$(85)\qquad B_n \subset I_n^{2n+1};$$

we then conclude by (85) and (68) that

$$(86)\qquad p_0 = \operatorname*{Lim}_{n\to\infty} B_n,$$

and by the skewness of the segment J_n and the cell I_n^{2n-1}, that

(87) *the intersection $B_n \cap B_{n+1}$ is a single point*,

namely the common end-point of the segments I_n and I_{n+1}.

Put

(88) $$y_n = I_n \cap I_{n+1}\,.$$

Hence we have by (71)

(89) $$h(y_n) = y_{n+1}\,.$$

Now let

(90) $$\Lambda_0 = p_0 \cup \bigcup_{n=0}^{\infty} B_n\,.$$

By (86) and (90) Λ_0 is compact. The sets B_n being connected and arcwise connected by (84), each partial union of $\bigcup_{n=0}^{\infty} B_n$ is connected and arcwise connected by (87); thus Λ_0 is connected by (86) and also arcwise connected, because the segment I joins p_0 with every B_n.

Finally, Λ_0 contains only one irreducible continuum between any two points. In fact, $p_i \in B_i$ can be joined with $p_k \in B_k$ by the union of arcs $A = p_i y_i \cup \overline{y_i y_{i+1}} \cup \ldots \cup \overline{y_{k-1} y_k} \cup y_k p_k$; therefore if there existed another irreducible continuum N between p_i and p_k, then, since the continuum N must contain the points $y_i, y_{i+1}, \ldots, y_{k-1}, y_k$ of A, each arc $\overline{y_j y_{j+1}}$ must by virtue of (84) be the same in N and in A. Thus N were identical with A.

All conditions of definition (46) are satisfied, i.e. it is proved that Λ_0 is a dendroid.

Let φ be the rotation through $2\pi/3$ around the axis Ox_0 and let

(91) $$\Lambda = \Lambda_0 \cup \varphi(\Lambda_0) \cup \varphi\big(\varphi(\Lambda_0)\big)\,.$$

Thus Λ is the union of three dendroids isometric with Λ_0 and having only one point in common, namely p_0. Hence Λ is a dendroid.

Proof of the property $R(\Lambda) = h(\Lambda)$. It follows from the definition of dendroids B_n and of mappings h and g_n by (79) that

(92) $$R(B_n) = R\Big(B\big(h(B_{n-1})\big)\Big)\,.$$

It follows, further, from (77) that if $y \in h(B_{n-1})$, then y is the common end-point of at least three segments: namely of at least one segment $\overline{xy}$ because the mapping f_n is onto, and of two segments contained in I, disjoint from each another and from the segment $\overline{xy}$, neglecting the point y. Thus $h(B_{n-1}) \subset R\Big(B\big(h(B_{n-1})\big)\Big)$. By (87) it can be shown, as in the proof of Lemma 2, that the segments $\overline{xy}$ in (77) are disjoint except for their common end-points $y \in h(B_{n-1})$. Thus $R\Big(B\big(h(B_{n-1})\big)\Big) \subset h(B_{n-1})$. Both these inclusions give $R\Big(B\big(h(B_{n-1})\big)\Big) = h(B_{n-1})$, whence by (87)

(93) $$R(B_n) = h(B_{n-1})\,.$$

We infer from (86) and (90), recording that $B_0 = I_0$, the equality

$$R(\Lambda_0 - \overline{\bigcup_{n=1}^{\infty} B_n}) = 0\,;$$

it follows by virtue of (90), (87) and (93), no three of the B_n having a point in common, that

(94) $$R(\Lambda_0) = \bigcup_{n=0}^{\infty} R(B_n) = \bigcup_{n=1}^{\infty} R(B_n) = h(\bigcup_{n=1}^{\infty} B_{n-1}) = h(\bigcup_{n=0}^{\infty} B_n)\,,$$

whence by the definition of φ

(95) $$R\big(\varphi(\Lambda_0)\big) = \varphi\big(R(\Lambda_0)\big) = \varphi\big(h(\bigcup_{n=0}^{\infty} B_n)\big) = h\big(\varphi(\bigcup_{n=0}^{\infty} B_n)\big)\,,$$

and there follows an analogical formula for $R\Big(\varphi\big(\varphi(\Lambda_0)\big)\Big)$.

But the point p_0 is by the definition of φ and by (91) the common end-point of exactly three segments I, $\varphi(I)$ and $\varphi\big(\varphi(I)\big)$, disjoint one from another out of p_0. Thus

(96) $$\mathrm{Ord}_{p_0} \Lambda = 3\,;$$

therefore $p_0 \in R(\Lambda)$. Thus by (91)

(97) $$R(\Lambda) = p_0 \cup R(\Lambda_0) \cup R\big(\varphi(\Lambda_0)\big) \cup R\Big(\varphi\big(\varphi(\Lambda_0)\big)\Big)\,,$$

whence, h being a homeomorphism by definition, we obtain by (94), (95) and (70)

$$\begin{aligned} R(\Lambda) &= p_0 \cup h(\bigcup_{n=0}^{\infty} B_n) \cup h\big(\varphi(\bigcup_{n=0}^{\infty} B_n)\big) \cup h\Big(\varphi\big(\varphi(\bigcup_{n=0}^{\infty} B_n)\big)\Big) \\ &= \big(p_0 \cup h(\bigcup_{n=0}^{\infty} B_n)\big) \cup h\big(\varphi(p_0 \cup \bigcup_{n=0}^{\infty} B_n)\big) \cup h\Big(\varphi\big(\varphi(p_0 \cup \bigcup_{n=0}^{\infty} B_n)\big)\Big) \\ &= h(p_0 \cup \bigcup_{n=0}^{\infty} B_n) \cup h\big(\varphi(p_0 \cup \bigcup_{n=0}^{\infty} B_n)\big) \cup h\Big(\varphi\big(\varphi(p_0 \cup \bigcup_{n=0}^{\infty} B_n)\big)\Big)\,. \end{aligned}$$

Consequently, we have by (90) and (91)

$$\begin{aligned} R(\Lambda) &= h(\Lambda_0) \cup h\big(\varphi(\Lambda_0)\big) \cup h\Big(\varphi\big(\varphi(\Lambda_0)\big)\Big) \\ &= h\Big(\Lambda_0 \cup \varphi(\Lambda_0) \cup \varphi\big(\varphi(\Lambda_0)\big)\Big) = h(\Lambda)\,. \end{aligned}$$

Proof of the property $\mathrm{Ord}_y \Lambda \leqslant 4$. We prove by induction this property in Λ_0, separately for the points $y = y_n$ defined by (88) and for the points $y \in B_n - y_{n-1} - y_n$.

(i) $y = y_n$. If $n = 0$, then y_0 is by (73) the common end-point of exactly three arcs of Λ_0, namely of the segments I_0, I_1 and $\overline{xy_0}$, where x is one of the end-points of the segment J_1 defined by (69). Thus $\mathrm{Ord}_{y_0} \Lambda_0 = 3$.

Assume now on y_{n-1} that

(98) $$\mathrm{Ord}_{y_{n-1}} \Lambda_0 \leqslant 4 .$$

It follows by the definition of the mapping g_n with its properties (81) and (82) that there are in Λ_0 no other arcs with end-point y except the images under h of the arcs having as their common end-point the point y_{n-1}. Thus we have by (89) and (98)

(99) $$\mathrm{Ord}_{y_n} \Lambda_0 \leqslant 4 .$$

This is the end of the induction.

(ii) $y \in B_n - y_{n-1} - y_n$. If $n = 0$, we have $\mathrm{Ord}_y B_0 \leqslant 2$ for every $y \in B_0 - y_0$ because $B_0 = I_0$. If $n = 1$, we recall that $R(B_1) = I_1$ by (71) and (93), whence $\mathrm{Ord}_y B_1 \leqslant 2$ for every $y \in B_1 - I_1$. Further, every $y \in I_1 - y_0 - y_1$ is by (72) the common end-point of at most two segments $\overline{xy}$ in (74) and of exactly two segments, $\overline{yy_0}$ and $\overline{yy_1}$, contained in I_1; hence $\mathrm{Ord}_y B_1 \leqslant 4$ for $y \in I_1 - y_0 - y_1$, and thus also for $y \in B_1 - y_0 - y_1$ by virtue of the recalled equality $R(B_1) = I_1$. So $\mathrm{Ord}_y B_n \leqslant 4$ for $n = 0$ and 1.

Hence, assuming

(100) $$\mathrm{Ord}_y B_{n-2} \leqslant 4 \quad \text{for} \quad y \in B_{n-2} - y_{n-3} - y_{n-2} ,$$

it suffices to show that

(101) $$\mathrm{Ord}_y B_n \leqslant 4 \quad \text{for} \quad y \in B_n - y_{n-1} - y_n .$$

We have identically

(102) $$B_n = \big(B_n - R(B_n)\big) \cup \Big(R(B_n) - h\big(h(B_{n-2})\big)\Big) \cup h\big(h(B_{n-2})\big) .$$

It follows by the definition of $R(B_n)$ that

(103) $$\mathrm{Ord}_y B_n \leqslant 2 \quad \text{for every} \quad y \in B_n - R(B_n) .$$

Remark that by (93)

$$\begin{aligned} R(B_n) - h\big(h(B_{n-2})\big) &= h(B_{n-1}) - h\big(h(B_{n-2})\big) \\ &= h\big(B_{n-1} - h(B_{n-2})\big) = h\big(B_{n-1} - R(B_{n-1})\big) . \end{aligned}$$

Therefore we have $\mathrm{Ord}_y\Big(R(B_n) - h\big(h(B_{n-2})\big)\Big) \leqslant 2$. By (76) y is the common end-point of at most two segments disjoint from $R(B_n) - h\big(h(B_{n-2})\big)$, namely the segments $\overline{x'y}$, where $x' = g_n(x)$ and $x \in f^{-1}(y)$. It follows that

(104) $$\mathrm{Ord}_y B_n \leqslant 4 \quad \text{for} \quad y \in R(B_n) - h\big(h(B_{n-2})\big) .$$

Remark that by (100), h being a homeomorphism, we have

$$\mathrm{Ord}_y h\big(h(B_{n-2})\big) \leqslant 4 \quad \text{for} \quad y \in h\big(h(B_{n-2})\big) - h\big(h(y_{n-3})\big) - h\big(h(y_{n-2})\big) .$$

Since by (82) every segment $\overline{xy}$, where $x \in f_n^{-1}(y)$, becomes mapped by g_n onto the point y, there are in B_n no other arcs with end-point y except the arcs contained in $h\big(h(B_{n-2})\big)$. Thus we have by (89) and (99)

$$\mathrm{Ord}_y B_n \leqslant 4 \quad \text{for every} \quad y \in h\big(h(B_{n-2})\big) .$$

So (101) is proved by (102)-(104) for an arbitrary $n = 0, 1, 2, \ldots$ It follows by (84), (87) and (90) that

$$\mathrm{Ord}_y \Lambda_0 = \mathrm{Ord}_y B_n \quad \text{for every} \quad y \neq y_{n'} \text{ and } n' = 0, 1, 2, \ldots$$

Hence it follows by (99) and (101) that $\mathrm{Ord}_y \Lambda_0 \leqslant 4$ for every point $y \in \Lambda_0 - p_0$, and finally, by (91), that $\mathrm{Ord}_y \Lambda \leqslant 4$ for every $y \in \Lambda - p_0$. It remains to add (96).

Remark. As a curve, the dendroid Λ has a homeomorphic image in a 3-dimensional cell by the Menger-Nöbeling imbedding theorem (cited here p. 234 and 242). No dendroid Δ, however, having the property

(105) $$R(\Delta) = h(\Delta)$$

has a homeomorphic image in the plane. In fact, suppose that such a dendroid $\Delta \subset E^2$ exists, and let h be a homeomorphism such that (105) holds. By (105) the set $R\big(R(\Delta)\big)$ is also a dendroid. Consider an arc $L \subset R\big(R(\Delta)\big)$ and let $p \in L$. By the definition of ramification points, $p \in R\big(R(\Delta)\big)$ implies the existence of an arc

(106) $$pq \subset R(\Delta)$$

such that $pq \cap L = p$. But, Δ being a dendroid,

(107) $$p_1 \neq p_2 \quad \text{implies} \quad p_1q_1 \cap p_2q_2 = 0 .$$

Let Γ be the collection of $2^{\aleph_0}$ disjoint arcs pq, where the point p runs over the arc L. Then exist an arc $pq \subset \Gamma$ and an interior point a of this arc such that a is not accessible from the set $U = E^2 - \bigcup_{p \in L} pq$ (see [5], p. 276, Corollary 3). By (106), $a \in pq$ implies that a is an end-point of an arc $ab \subset \Delta$ such that $pq \cap ab = a$, whence $ab - a \subset U$ by (107). But this means accessibility of a from U, and hence is impossible.

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On the representation of α-complete lattices *

by

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This paper is concerned with the problem of representation for α-complete lattices. It is well known that a lattice is isomorphic with a ring of sets if and only if it is distributive. However for an α-complete lattice L even the condition of (α, α) distributivity is not sufficient for it to be isomorphic with an α-ring of sets. A necessary and sufficient condition for such a representation is the following: whenever $x \nleqslant y$, there exists an α-complete prime ideal P containing x such that $L-P$ contains y and is α-complete. On the other hand, necessary and sufficient conditions for a Boolean algebra to be α-representable (that is, to be isomorphic with an α-field of sets modulo an α-ideal) are known ([1], [5], [8], [4], [7]). In this paper, we deal with the problem of representing an α-complete lattice as an α-ring of sets modulo an α-ideal. Such lattices are called *α-representable.*

We shall present a characterization of α-representable lattices which is a natural generalization of a known characterization for α-representable Boolean algebras ([5], [1]). There are several differences between the results for Boolean algebras and those we obtain for lattices. For instance, while every ω-complete Boolean algebra is ω-representable ([3], [6]) in order that an ω-complete lattice L be ω-representable, it is necessary and sufficient that L satisfy the condition of $(2, \omega)$ distributivity, which is satisfied by all Boolean algebras. Also, while every α-complete, (α, α) distributive Boolean algebra is α-representable, we shall give an example of a complete, completely distributive lattice which is not α-representable for any $\alpha \geqslant 2^{\omega}$. The paper concludes with a discussion of α-representable chains.

1. Definitions. If α is a cardinal, an *α-system* is a system $\{x_i\}$, $i \in I$, whose index set I has power $\leqslant \alpha$. By an *α-complete lattice*, we mean a lattice L in which every non-empty α-system $\{x_i\}$, $i \in I$, has a least upper bound $\sum_{i \in I} {}_L x_i$, and a greatest lower bound $\prod_{i \in I} {}_L x_i$. We do not require that L have a smallest or largest element.

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