Reduced direct products

by

T. Frayne, A. C. Morel, and D. S. Scott (San Francisco)

Introduction. The *reduced direct product* of a sequence of relational systems is a generalization of the standard algebraic notion of the (complete) direct product; in fact, a reduced product is a special type of homomorphic image of a corresponding direct product—though it is not always a quotient of this direct product except in the case of algebraic systems. An extreme case of the reduced product is the *prime reduced product*. The first definition and properties of prime products (in a quite different formulation from that employed here) were given by Łoś in [29]. A useful special case of reduced products was applied by Chang and Morel in [1]. Subsequently Tarski suggested that these two methods might have something in common and that the idea could be applied to giving a proof of the compactness theorem in the theory of models. In particular Tarski realized that the construction of Chang and Morel gave a proof of compactness for classes of systems, defined by certain special types of first order sentences. These suggestions led to the formulation of the general definition by Frayne, and several results were obtained by Frayne and Morel. Independently Scott gave the same definition and found overlapping results. Scott had been stimulated by many discussions with Kochen, who wanted to extend Skolem's method of [35] for obtaining models of arithmetic to more general situations. The connection between Skolem's method and reduced products is explained in [33]. The definition and results were first announced by the authors together with Tarski in [4], [5], [6], and [21]. More recently the idea has been used by Kochen [23], [24], [25], [26], [28], Keisler [12], [13], [14], [15], [17], [18], [19], [20], [21] and Rabin [32] in several different applications.

This paper is divided into two sections together with an appendix. In Section 1 the basic algebraic (or relation-theoretic) definitions are given, and several theorems are presented of a general algebraic and set-theoretic nature. In Section 2 the notions from the theory of models are recalled and the applications of reduced products to this theory are indicated. In particular a proof of the compactness theorem is given in such a way that one sees how to obtain a model for a set of sentences from given models of the finite subsets of the set of sentences: the resulting model is a certain prime reduced product of the given models. The method is not completely constructive since the axiom of choice is needed to form the prime product; however, in some special cases the method can be made more effective. In any case this proof shows for the first time a direct algebraic connection between the given models and the models of all the sentences. Furthermore it is now possible to characterize elementary classes of relational systems by means of some closure conditions involving prime products. The theorems given here are not the best possible, for using the generalized continuum hypothesis Keisler has found a way of giving simpler characterizations. The exact situation, as well as references to work of Keisler and Kochen, is indicated in the relevant place in Section 2. The appendix introduces some topological notions, and it is shown how some of the result of Section 2 could have been derived in an abstract setting.

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1. General properties of reduced products. The empty set as well as the ordinal zero is denoted by 0. In general, an ordinal is identified with the set of all smaller ordinals. Lower case Greek letters will range over ordinals and sometimes functions on ordinals. The infinite initial ordinals are denoted by ω_α, α an ordinal. We write ω for ω_0. The axiom of choice is applied freely, but cardinals are not identified with ordinals. Lower case German letters will range over cardinals both finite and infinite. The cardinal of a set A is denoted by $|A|$. We write $\aleph_\alpha = |\omega_\alpha|$. The generalized continuum hypothesis is not assumed, and we write $2^{\mathfrak{m}}$ to denote the cardinality of the power set of a set of cardinality $\mathfrak{m}$. This exponential notation should not be confused with the convention explained below, where, for example, 2^ω denotes the set of all functions on ω with values 0 or 1. At one place we use 2^ν, where $\nu < \omega$, to denote the value of the ordinary exponential function, rather than a set of functions. The context should always make the meaning clear, however.

A finite set of elements is denoted by $\{x_0, \ldots, x_{\nu-1}\}$; while the expressions $\{x_i\colon i \in I\}$ and $\{x_i\colon \ldots i \ldots\}$ denote respectively the collection of all elements x_i where $i \in I$ and the collection of all elements x_i where i satisfies the condition ... i ... We shall also sometimes write $\{i \in I\colon \ldots i \ldots\}$ to denote the collection of those $i \in I$ satisfying the condition ... i ...; similarly for $\{J \subseteq I\colon \ldots J \ldots\}$, and so on. A finite sequence of elements is denoted by $\langle x_0, \ldots, x_{\nu-1}\rangle$; while the expressions $\langle x_i\colon i \in I\rangle$ and $\langle x_i\colon \ldots i \ldots\rangle$ denote respectively the sequence (or better, the function) with the value x_i associated with i, where in the first case $i \in I$, and in the second, i satisfies the condition ... i .... The sequence $\langle x_i\colon i \in I\rangle$ is called an *I-termed sequence*; indeed, any function f with domain I is an I-termed sequence, and we have $f = \langle f(i)\colon i \in I\rangle$. The restriction of f to a subset J of I is denoted by $f \restriction J$, a J-termed sequence.

If $\langle A_i\colon i \in I\rangle$ is a sequence of sets, then $\mathrm{P}\langle A_i\colon i \in I\rangle$ denotes the *cartesian product* of the sets A_i and is taken to be the set of all I-termed sequences $\langle a_i\colon i \in I\rangle$ such that $a_i \in A_i$ for all $i \in I$. If $A = A_i$ for all $i \in I$, then we write $A^I = \mathrm{P}\langle A_i\colon i \in I\rangle$. A^I is the *cartesian power* of the set A with respect to I. For finite sequences of sets we write $A_0 \times \ldots \times A_{\nu-1} = \mathrm{P}\langle A_0, \ldots, A_{\nu-1}\rangle$. If I is empty, then $\mathrm{P}\langle A_i\colon i \in I\rangle = \{0\}$. The *intersection* and *union* of a sequence of sets are denoted respectively by $\bigcap \langle A_i\colon i \in I\rangle$ and $\bigcup \langle A_i\colon i \in I\rangle$; while if $\mathcal{F}$ is a family of sets, then we write $\bigcap \mathcal{F} = \bigcap \langle A\colon A \in \mathcal{F}\rangle$ and $\bigcup \mathcal{F} = \bigcup \langle A\colon A \in \mathcal{F}\rangle$. The intersection operation is never applied in the cases where I or $\mathcal{F}$ is empty. For finite sequences we write as usual $A_0 \cap \ldots \cap A_{\nu-1}$ and $A_0 \cup \ldots \cup A_{\nu-1}$. The *relative complement* of B in A is denoted by $A \sim B$. A sequence of sets $\langle A_i\colon i \in I\rangle$ is called an *I-termed partition* of set B if $B = \bigcup \langle A_i\colon i \in I\rangle$ and $A_i \cap A_j = 0$, for $i, j \in I$ with $i \neq j$.

A *dual ideal* (or *filter*) over a set I is a family $\mathcal{D}$ of subsets of I such that $I \in \mathcal{D}$; if $X, Y \in \mathcal{D}$, then $X \cap Y \in \mathcal{D}$; and if $X \in \mathcal{D}$ and $X \subseteq Y \subseteq I$, then $Y \in \mathcal{D}$. Often we shall write for short 'ideal' rather than 'dual ideal'. The ideal $\mathcal{D}$ is *proper* if $0 \notin \mathcal{D}$; while $\mathcal{D}$ is *principal* if $\bigcap \mathcal{D} \in \mathcal{D}$. A maximal proper ideal $\mathcal{D}$ is called *prime* (or an *ultrafilter*) and is characterized as being prime by the condition that whenever $X \subseteq I$, then either $X \in \mathcal{D}$ or $I \sim X \in \mathcal{D}$. As is well known every family $\mathcal{F}$ of subsets of I having the *finite intersection property* (i.e. no finite intersection of members of $\mathcal{F}$ is empty) can be extended to a prime dual ideal $\mathcal{D}$ with $\mathcal{F} \subseteq \mathcal{D}$. A principal ideal $\mathcal{D}$ is prime if and only if $\bigcap \mathcal{D} = \{i\}$ for some $i \in I$. A dual ideal $\mathcal{D}$ is called *$\mathfrak{m}$-complete* if whenever $0 \neq \mathcal{F} \subseteq \mathcal{D}$ and $|\mathcal{F}| < \mathfrak{m}$, then $\bigcap \mathcal{F} \in \mathcal{D}$. All ideals are $\aleph_0$-complete; an $\aleph_1$-complete ideal is called *countably complete*. The ideal $\mathcal{D}$ is principal if and only if it is $\mathfrak{m}$-complete for all $\mathfrak{m}$. A prime dual ideal $\mathcal{D}$ is $\mathfrak{m}$-complete if and only if whenever $\bigcup \mathcal{F} \in \mathcal{D}$ and $|\mathcal{F}| < \mathfrak{m}$, then $\mathcal{F} \cap \mathcal{D} \neq 0$. If $\mathcal{D}$ is a dual ideal over I and $J \subseteq I$, then the family $\{X \cap J\colon X \in \mathcal{D}\}$ is a dual ideal over J called the *restriction* of $\mathcal{D}$ to J and is denoted by $\mathcal{D} \restriction J$. If $\langle J_k\colon k \in K\rangle$ is a partition of I then the sequence $\langle \mathcal{D} \restriction J_k\colon k \in K\rangle$ is called the *partition* of $\mathcal{D}$ induced by the given partition of I. If $\mathcal{D}$ is $\mathfrak{m}$-complete and $|K| < \mathfrak{m}$, then for $X \subseteq I$, $X \in \mathcal{D}$ if and only if $X \cap J_k \in \mathcal{D} \restriction J_k$ for all $k \in K$; and so $\mathcal{D}$ can be recaptured from its partition in this special case. If $\mathcal{D}$ is prime, then $\mathcal{D} \restriction J$ is prime over J if and only if $I \sim J \notin \mathcal{D}$. If $I \sim J \in \mathcal{D}$, then $\mathcal{D} \restriction J$ is not proper.

Given any dual ideal $\mathcal{D}$ over a set I, an equivalence relation $=_{\mathcal{D}}$ is induced between I-termed sequences by the condition that $f =_{\mathcal{D}} g$ if and only if $\{i \in I\colon f(i) = g(i)\} \in \mathcal{D}$. That $=_{\mathcal{D}}$ is reflexive and symmetric is obvious; while the transitivity follows from the formula

$$\{i \in I\colon f(i) = g(i)\} \cap \{i \in I\colon g(i) = h(i)\} \subseteq \{i \in I\colon f(i) = h(i)\}\,.$$

Thus any cartesian product $\mathrm{P}\langle A_i\colon i \in I\rangle$ is divided into equivalence classes by $=_{\mathcal{D}}$. For each $f \in \mathrm{P}\langle A_i\colon i \in I\rangle$, $f/{=_{\mathcal{D}}}$ will denote the equivalence class of f in the set $\mathrm{P}\langle A_i\colon i \in I\rangle$. The notation is slightly ambiguous since the product is not mentioned in the expression $f/{=_{\mathcal{D}}}$; however, the context will make the sense clear. Further, we shall sometimes write f^* for $f/{=_{\mathcal{D}}}$ when there is no question as to which ideal $\mathcal{D}$ is meant. The collection $\{f/{=_{\mathcal{D}}}\colon f \in \mathrm{P}\langle A_i\colon i \in I\rangle\}$ is called the *reduced cartesian product* of the sets A_i relative to (or reduced by) the dual ideal $\mathcal{D}$, and it is denoted by $\mathrm{P}_{\mathcal{D}}\langle A_i\colon i \in I\rangle$. If $A_i = A$ for all $i \in I$, then we write $A^I_{\mathcal{D}} = \mathrm{P}_{\mathcal{D}}\langle A_i\colon i \in I\rangle$. $A^I_{\mathcal{D}}$ is the *reduced cartesian power*. If $\mathcal{D} = \{I\}$, then clearly $=_{\mathcal{D}}$ is the identity relation; in which case $f/{=_{\mathcal{D}}} = \{f\}$, and there is an obvious one-one correspondence between $\mathrm{P}_{\mathcal{D}}\langle A_i\colon i \in I\rangle$ and $\mathrm{P}\langle A_i\colon i \in I\rangle$.

We shall not investigate further the set-theoretical properties of reduced cartesian products of sequences of sets. Instead we shall recast the definition of the notion within the framework of the theory of relational systems. Many of the subsequent theorems will have a purely set-theoretical character, but we have stated them in terms of relational systems, because systems rather than sets are our main interest in this paper.

For simplicity we shall not consider arbitrary relational systems but only those involving a single ternary relation. A (*ternary*) *relational system* is a pair $\mathfrak{A} = \langle A, R\rangle$, where A is a non-empty set and $R \subseteq A^3$. Capital German letters will range over relational systems. We write $R(x, y, z)$ for $\langle x, y, z\rangle \in R$. The cardinal of $\mathfrak{A}$ is $|A|$ and will sometimes be denoted by $|\mathfrak{A}|$. $\mathfrak{A}$ is called *finite* if $|\mathfrak{A}|$ is finite. The system $\langle\{0\}, \{0\}^3\rangle$ is denoted by $\mathfrak{U}$ and is called the *unit system*. $\mathfrak{A}$ is a *zero system* if $R = 0$.

If $\mathfrak{A} = \langle A, R\rangle$ and $\mathfrak{B} = \langle B, S\rangle$ are relational systems, and if h is a function with domain A and range B, then h is a *homomorphism* of $\mathfrak{A}$ onto $\mathfrak{B}$, if whenever $R(x, y, z)$, then $S\big(h(x), h(y), h(z)\big)$. In this case $\mathfrak{B}$ is called a *homomorphic image* of $\mathfrak{A}$. If h is one-one and h is a homomorphism of $\mathfrak{A}$ onto $\mathfrak{B}$, and h^{-1} is a homomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$, then h is called an *isomorphism*, and $\mathfrak{A}$ and $\mathfrak{B}$ are said to be *isomorphic*, a relation denoted by $\mathfrak{A} \cong \mathfrak{B}$. On the other hand if $A \subseteq B$ and $R = S \cap A^3$, then $\mathfrak{A}$ is called a *subsystem* of $\mathfrak{B}$. $\mathfrak{A}$ is said to be *embeddable* in $\mathfrak{B}$ if $\mathfrak{A}$ is isomorphic to a subsystem of $\mathfrak{B}$. An equivalence relation E over A is a *congruence relation* over $\mathfrak{A}$ if whenever $R(x, y, z)$ and xEx', yEy', and zEz', then $R(x', y', z')$. Letting x/E denote the equivalence class of x and A/E the set of all equivalence classes, a relation R/E is defined over A/E by the condition that $R/E\ (x/E, y/E, z/E)$ if and only if $R(x, y, z)$. The system $\mathfrak{A}/E = \langle A/E, R/E\rangle$ is called the *quotient system* of $\mathfrak{A}$ modulo E. If E is a congruence relation, then $\mathfrak{A}/E$ is a homomorphic image of $\mathfrak{A}$; but not every homomorphic image of $\mathfrak{A}$ is isomorphic to a quotient of $\mathfrak{A}$.

If $\langle\mathfrak{A}_i\colon i \in I\rangle$ is a sequence of relational systems, where $\mathfrak{A}_i = \langle A_i, R_i\rangle$ for all $i \in I$, then the system $\langle B, S\rangle$, where $B = \mathrm{P}\langle A_i\colon i \in I\rangle$ and $S \subseteq B^3$ is defined by the condition that $S(f, g, h)$ holds if and only if $R_i\big(f(i), g(i), h(i)\big)$ holds for all $i \in I$, is called the *direct product* of the systems $\mathfrak{A}_i$ and is denoted by $\mathfrak{P}\langle\mathfrak{A}_i\colon i \in I\rangle$. In case $\mathfrak{A}_i = \mathfrak{A}$ for all $i \in I$, then we write $\mathfrak{A}^I = \mathfrak{P}\langle A_i\colon i \in I\rangle$. $\mathfrak{A}^I$ is the *direct power* of the system $\mathfrak{A}$ with respect to I. For finite sequences we write $\mathfrak{A}_0 \times \ldots \times \mathfrak{A}_{\nu-1} = \mathfrak{P}\langle\mathfrak{A}_0, \ldots, \mathfrak{A}_{\nu-1}\rangle$. Note that $\mathfrak{A} \times \mathfrak{A} \cong \mathfrak{A}$, and if $\mathfrak{B}$ is a zero system, then so is $\mathfrak{A} \times \mathfrak{B}$. Indeed, an arbitrary direct product is a zero system if and only if at least one of its factors is a zero system. If $\mathcal{D}$ is a dual ideal over I, then another relation $S_{\mathcal{D}}$ can be defined over $B = \mathrm{P}\langle A_i\colon i \in I\rangle$ by the condition that $S_{\mathcal{D}}(f, g, h)$ holds if and only if $\big\{i \in I\colon R_i\big(f(i), g(i), h(i)\big)\big\} \in \mathcal{D}$. This definition of $S_{\mathcal{D}}$ is similar to that of the relation $=_{\mathcal{D}}$. Notice that $=_{\mathcal{D}}$ is a congruence relation over the system $\langle B, S_{\mathcal{D}}\rangle$, for we have

$$\{i \in I\colon f(i) = f'(i)\} \cap \{i \in I\colon g(i) = g'(i)\} \cap \{i \in I\colon h(i) = h'(i)\} \cap$$
$$\cap \big\{i \in I\colon R_i\big(f(i), g(i), h(i)\big)\big\} \subseteq \big\{i \in I\colon R_i\big(f'(i), g'(i), h'(i)\big)\big\}\,.$$

The system $\langle B, S_{\mathcal{D}}\rangle/{=_{\mathcal{D}}}$ is called the *reduced* (*direct*) *product* of the systems $\mathfrak{A}_i$ relative to (or reduced by) the dual ideal $\mathcal{D}$, and it is denoted by $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$. The reduced product is called *proper* if the ideal $\mathcal{D}$ is a proper dual ideal. If $\mathfrak{A}_i = \mathfrak{A}$ for all $i \in I$, then we write $\mathfrak{A}^I_{\mathcal{D}} = \mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$. $\mathfrak{A}^I_{\mathcal{D}}$ is the *reduced* (*direct*) *power* of the system $\mathfrak{A}$. The case of finite I is of no interest in as much as $\mathcal{D}$ would then be a principal dual ideal. Note that $B/{=_{\mathcal{D}}}$ is the set $\mathrm{P}_{\mathcal{D}}\langle A_i\colon i \in I\rangle$. In the case that $\mathcal{D}$ is a prime dual ideal, then $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$ is called a *prime* (*reduced direct*) *product*, and $\mathfrak{A}^I_{\mathcal{D}}$ is called a *prime* (*reduced direct*) *power*.

The notation of the last paragraph with the relation of the system $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$ being denoted by $S_{\mathcal{D}}/{=_{\mathcal{D}}}$ is clumsy. Hence, we shall often use the fact that if $\mathfrak{B} = \langle B, S\rangle$ is any system such that $\mathfrak{B} = \mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$, then for $f, g, h \in \mathrm{P}\langle A_i\colon i \in I\rangle$, $S(f^*, g^*, h^*)$ holds if and only if $\big\{i \in I\colon R_i\big(f(i), g(i), h(i)\big)\big\} \in \mathcal{D}$. Here f^* denotes the equivalence class $f/{=_{\mathcal{D}}}$ in $\mathrm{P}\langle A_i\colon i \in I\rangle$ as mentioned before.

The choice of terminology and notation has not been easy for the authors, and no doubt the choices made will not please everyone. We may mention some of the variations that have been used. Since it is common in topology to refer to a dual ideal as a filter, Kochen [27] uses the term *filtered products*. Prime dual ideals are *ultrafilters* and so Kochen [28], Keisler [14], and Lyndon [30] use the term *ultraproducts*, which in certain combinations is more pleasant to some ears than the alliterative term *prime products*. Further, Keisler [11] uses $\mathfrak{A}^I/\mathcal{D}$ for $\mathfrak{A}^I_{\mathcal{D}}$. We have preferred to keep the symbol / for the indication of quotient systems. Lyndon [30] uses a further abreviation for direct products (and uses ideals instead of dual ideals), but we prefer to have different symbols for products and powers. Also we prefer the style that uses capital German letters for systems and the capital Roman letters for sets of elements of systems, a convention reflected in our symbols for products.

At last we shall state some consequences of the foregoing definitions.

THEOREM 1.1. (i) *If* $\mathcal{D} = \{I\}$, *then*

$$\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle \cong \mathfrak{P}\langle\mathfrak{A}_i\colon i \in I\rangle;$$

(ii) *If* $\mathcal{D}$ *is not proper, then*

$$\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle \cong \mathfrak{U}.$$

Proof. Obvious.

THEOREM 1.2. *Let* $\mathfrak{m}$ *be an infinite cardinal and* $\mathcal{D}$ *be an* $\mathfrak{m}$*-complete dual ideal over* I. *Suppose that* $\langle J_k\colon k \in K\rangle$ *is a partition of* I *with* $|K| < \mathfrak{m}$. *Let* $\mathcal{D}_k = \mathcal{D}\upharpoonright J_k$ *for* $k \in K$. *Then*

$$\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle \cong \mathfrak{P}\langle\mathfrak{P}_{\mathcal{D}_k}\langle\mathfrak{A}_i\colon i \in J_k\rangle\colon k \in K\rangle.$$

Proof. As pointed out above, a set $X \in \mathcal{D}$ if and only if $X \cap J_k \in \mathcal{D}_k$ for all $k \in K$. Hence, it follows at once that $f =_{\mathcal{D}} g$ if and only if $f\upharpoonright J_k =_{\mathcal{D}_k} g\upharpoonright J_k$ for all $k \in K$. In other words, the function h defined on the elements of $\mathrm{P}_{\mathcal{D}}\langle A_i\colon i \in I\rangle$ by the equation

$$h(f/=_{\mathcal{D}}) = \langle f\upharpoonright J_k/=_{\mathcal{D}_k}\colon k \in K\rangle$$

is one-one. It is clearly a mapping onto the cartesian product of the $\mathrm{P}_{\mathcal{D}_k}\langle A_i\colon i \in J_k\rangle$ by virtue of the fact that $\langle J_k\colon k \in K\rangle$ is a partition of I. The conclusion that h also preserves the relation of the reduced product, and hence is an isomorphism, can be easily verified in the same way in which it was shown that h is one-one.

The preceding result may be viewed as an associative law, a more general form of which will be given in 1.10.

COROLLARY 1.3. *Let* $\mathcal{D}$ *be any ideal over* I *and* $J \in \mathcal{D}$. *Then*

$$\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle \cong \mathfrak{P}_{\mathcal{D}\upharpoonright J}\langle\mathfrak{A}_i\colon i \in J\rangle.$$

Proof. Apply 1.2 to the case of the partition $\langle J, I\sim J\rangle$ with $\mathfrak{m} = \aleph_0$. Note that $\mathcal{D}\upharpoonright I\sim J$ is not proper and so the second factor of the direct product by 1.1 (ii) is a trivial factor isomorphic to $\mathfrak{U}$.

If $\mathcal{D}$ is prime and the hypotheses of 1.2 apply, then there must be a $k \in K$ with $J_k \in \mathcal{D}$. Thus in this case we see that the direct product in 1.2 is isomorphic to one of its factors.

COROLLARY 1.4. *If* $\mathcal{D}$ *is a principal dual ideal over* I, *then*

$$\mathfrak{P}_{\mathcal{D}}\langle A_i\colon i \in I\rangle \cong \mathfrak{P}\langle A_i\colon i \in \bigcap \mathcal{D}\rangle.$$

Proof. Apply 1.3 and then 1.1 (i).

COROLLARY 1.5. *If* $\mathcal{D}$ *is a principal prime dual ideal over* I *and* $j \in \bigcap \mathcal{D}$, *then*

$$\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle \cong \mathfrak{A}_j.$$

Proof. Apply 1.4 and the fact that the direct product with a single factor is isomorphic to that factor.

THEOREM 1.6. *If* $\mathcal{D}$ *is a proper dual ideal over* I *and* $\mathfrak{A}_i$ *is a zero system for each* $i \in I$, *then* $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$ *is a zero system.*

Proof. Let $\mathfrak{A}_i = \langle A_i, R_i\rangle$, where $R_i = 0$, for all $i \in I$. If the reduced product were not a zero system, then there would be sequences $f, g, h \in \mathrm{P}\langle A_i\colon i \in I\rangle$ such that

$$J = \{i \in I\colon R_i(f(i), g(i), h(i))\} \in \mathcal{D}.$$

But $J = 0$, which contradicts the assumption that $\mathcal{D}$ is proper.

COROLLARY 1.7. *If* $\mathcal{D}$ *is a dual ideal over* I *and* $J = \{i \in I\colon \mathfrak{A}_i$ *is a zero system*$\}$, *then* $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$ *is a zero system if and only if* $I\sim J \notin \mathcal{D}$.

Proof. If $I\sim J \notin \mathcal{D}$, then $\mathcal{D}\upharpoonright J$ is proper and by 1.6, $\mathfrak{P}_{\mathcal{D}\upharpoonright J}\langle\mathfrak{A}_i\colon i \in J\rangle$ is a zero system. By 1.2 the full reduced product is isomorphic to $\mathfrak{P}_{\mathcal{D}\upharpoonright J}\langle\mathfrak{A}_i\colon i \in J\rangle \times \mathfrak{P}_{\mathcal{D}\upharpoonright I\sim J}\langle\mathfrak{A}_i\colon i \in I\sim J\rangle$, and hence is a zero system. If $I\sim J \in \mathcal{D}$, then by 1.3 the reduced product is isomorphic to a reduced product of non-zero factors, which is easily seen to be a non-zero system.

THEOREM 1.8. *If* $\mathcal{D}$ *is a dual ideal over* I, *no* $\mathfrak{A}_i$ *is a zero system for* $i \in I$, *and if* $J = \{i \in I\colon |\mathfrak{A}_i| > 1\}$, *then*

$$\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle \cong \mathfrak{P}_{\mathcal{D}\upharpoonright J}\langle\mathfrak{A}_i\colon i \in I\rangle.$$

Proof. As before, the full reduced product is isomorphic to

$$\mathfrak{P}_{\mathcal{D}\upharpoonright J}\langle\mathfrak{A}_i\colon i \in J\rangle \times \mathfrak{P}_{\mathcal{D}\upharpoonright I\sim J}\langle\mathfrak{A}_i\colon i \in I\sim J\rangle.$$

Under the assumptions the second factor is isomorphic to $\mathfrak{U}$.

Notice that if there were zero systems among the factors, the best we could say is that either the second factor is isomorphic to $\mathfrak{U}$ or to one of the zero factors $\mathfrak{A}_i$ with $i \in I\sim J$.

Corollary 1.9. *If $\mathcal{D}$ is a dual ideal over I and*

$$J = \{i \in I\colon |\mathfrak{A}_i| > 1 \text{ and } \mathfrak{A}_i \text{ is not a zero system}\},$$

then either $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$ is a zero system, or

$$\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle \cong \mathfrak{P}_{\mathcal{D}\restriction J}\langle\mathfrak{A}_i\colon i \in J\rangle .$$

Proof. Apply 1.7, 1.3, and finally 1.8.

The import of 1.9 is that either a reduced product is trivial, or it is isomorphic to a reduced product of its non-trivial factors. Clearly, similar results could be formulated for getting rid of factors of the form $\langle A, A^3\rangle$, or at least reducing the occurence of such factors to a single instance. As another example of the elimination of factors, we may mention that by 1.3 it is possible to assume without loss of generality that the dual ideal $\mathcal{D}$ contains sets all of the same cardinality; for apply 1.3 where $J \in \mathcal{D}$ is chosen so that $|J|$ is a minimum. Finally, as a last example of the rearrangement of factors, we have the general *associative law.*

Theorem 1.10. *Let $\langle J_k\colon k \in K\rangle$ be a partition of I, and let $\mathcal{D}_k$ be a dual ideal over J_k for each $k \in K$. Let $\mathcal{D}'$ be a dual ideal over K and define a dual ideal $\mathcal{D}$ over I by the equation*

$$\mathcal{D} = \{X \subseteq I\colon \{k \in K\colon X \cap J_k \in \mathcal{D}_k\} \in \mathcal{D}'\} .$$

Then

$$\mathfrak{P}_{\mathcal{D}'}\langle\mathfrak{P}_{\mathcal{D}_k}\langle\mathfrak{A}_i\colon j \in J_k\rangle\colon k \in K\rangle \cong \mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle .$$

Proof. It is only necessary to check that the function h such that

$$h(f/{=_{\mathcal{D}}}) = \langle\langle f(i)\colon i \in J_k\rangle/{=_{\mathcal{D}_k}}\colon k \in K\rangle/{=_{\mathcal{D}'}}$$

is the required isomorphism. We leave this to the reader, since it is an easy but lengthy computation on sequences.

Notice that 1.2 is really a special case of 1.10 where $\mathcal{D}' = \{K\}$. Of course, the hypothesis that $|K| < \mathfrak{m}$ cannot be eliminated from 1.2. In 1.10, if K were finite, then by 1.4 we may as well assume that $\mathcal{D}' = \{K\}$. Let $K = \{0, \ldots, \nu-1\}$ then $\mathcal{D}$ takes on a simpler form:

$$\mathcal{D} = \{X_0 \cup \ldots \cup X_{\nu-1}\colon X_k \in \mathcal{D}_k \text{ for } k < \nu\} .$$

In the case of arbitrary K, if $\mathcal{D}'$ and the $\mathcal{D}_k$ are all prime, then so is $\mathcal{D}$, as may be easily verified. In the finite case with $|K| > 1$, $\mathcal{D}' = \{K\}$, and the $\mathcal{D}_k$ proper for $k \in K$, $\mathcal{D}$ is never prime. However, the ideals $\mathcal{D}$ obtained from prime $\mathcal{D}_k$ in this way are of interest, and we shall return to them in 1.30-31. Next we want to discuss the questions of the cardinality of reduced products, an area in which there are still some unsolved

problems. First, we shall give some simple facts about homomorphisms, isomorphisms, and embeddings.

Theorem 1.11. *If $\mathcal{D}$ and $\mathcal{D}'$ are dual ideals over I and $\mathcal{D} \subseteq \mathcal{D}'$, then $\mathfrak{P}_{\mathcal{D}'}\langle\mathfrak{A}_i\colon i \in I\rangle$ is a homomorphic image of $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$.*

Proof. It need only be checked that the function h such that $h(f/{=_{\mathcal{D}}}) = f/{=_{\mathcal{D}'}}$ is well-defined and is the required homomorphism.

Theorem 1.12. *If $\mathcal{D}$ is a dual ideal over I and $\mathfrak{A}_i \cong \mathfrak{B}_i$ for all $i \in I$, then*

$$\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle \cong \mathfrak{P}_{\mathcal{D}}\langle\mathfrak{B}_i\colon i \in I\rangle .$$

Theorem 1.13. *If $\mathcal{D}$ is a dual ideal over I and $\mathfrak{B}_i$ is a homomorphic image of $\mathfrak{A}_i$ for all $i \in I$, then $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{B}_i\colon i \in I\rangle$ is a homomorphic image of $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$.*

Theorem 1.14. *If $\mathcal{D}$ is a dual ideal over I and $\mathfrak{A}_i$ is embeddable in $\mathfrak{B}_i$ for all $i \in I$, then $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$ is embeddable in $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{B}_i\colon i \in I\rangle$.*

Proofs. Obvious.

A result analogous to 1.12-1.14 with the hypothesis that $\mathfrak{B}_i$ is a direct power of $\mathfrak{A}_i$ does not hold, for we shall see a case below where

$$(\mathfrak{A}^J)^I_{\mathcal{D}} \ncong (\mathfrak{A}^I_{\mathcal{D}})^J .$$

Somewhat more interesting than the above is the fact that every relational system can be reconstructed from its finite subsystems, in the precise sense of the next theorem.

Theorem 1.15. *Let $\langle\mathfrak{A}_i\colon i \in I\rangle$ be a sequence containing all finite subsystems of a system $\mathfrak{A}$. Then there is a proper dual ideal $\mathcal{D}$ over I such that if $\mathcal{D}' \supseteq \mathcal{D}$ is a proper dual ideal over I, then $\mathfrak{A}$ is embeddable in $\mathfrak{P}_{\mathcal{D}'}\langle\mathfrak{A}_i\colon i \in I\rangle$.*

Proof. Without loss of generality, we may assume that each $\mathfrak{A}_i$ is a finite subsystem of $\mathfrak{A}$. Let $\mathfrak{A}_i = \langle A_i, R_i\rangle$ and $\mathfrak{A} = \langle A, R\rangle$. For each $j \in I$ let

$$J_j = \{i \in I\colon \mathfrak{A}_j \subseteq \mathfrak{A}_i\} .$$

If $K \subseteq I$ is a finite set, then it is clear that $\bigcap\langle J_j\colon j \in K\rangle \neq 0$. Hence, the least dual ideal $\mathcal{D}$ over I with $J_j \in \mathcal{D}$ for $j \in I$ is a proper ideal. Let $\mathcal{D}'$ be a proper dual ideal containing $\mathcal{D}$. Let $\langle d_i\colon i \in I\rangle$ be a fixed sequence such that $d_i \in A_i$ for $i \in I$. For each $a \in A$, let $h(a)$ be that sequence $\langle a_i\colon i \in I\rangle$ such that $a_i = a$ if $a \in A_i$, and $a_i = d_i$ otherwise. If $a, b \in A$ and $h(a) =_{\mathcal{D}'} h(b)$, then

$$J' = \{i \in I\colon h(a)(i) = h(b)(i)\} \in \mathcal{D}' .$$

Let $j \in I$ be chosen so that $a, b \in A_j$. Now $J_j \in \mathcal{D}'$ and $J_j \cap J' \in \mathcal{D}'$. But $\mathcal{D}'$ is proper, and hence there must be a $k \in J_j \cap J'$. By the choice of k, we know that $a, b \in A_k$. By the definition of h, it follows that $h(a)(k) = a$ and $h(b)(k) = b$. On the other hand, $k \in J'$, so we conclude that $a = b$. In as much as the range of h is included in $\mathrm{P}\langle A_i\colon i \in I\rangle$, this last argument shows that the function h^* such that $h^*(a) = h(a)/{=_{\mathcal{D}'}}$ is a one-one function mapping A into $\mathrm{P}_{\mathcal{D}'}\langle A_i\colon i \in I\rangle$. Finally, let $a, b, c \in A$. Choose $j \in I$ so that $a, b, c \in A_j$. From the definition of h and the fact that the $\mathfrak{A}_i$ are all subsystems of $\mathfrak{A}$, it is easy to see that $R(a, b, c)$ holds if and only if

$$J_j \subseteq \{i \in I\colon R_i(h(a)(i), h(b)(i), h(c)(i))\},$$

which in turn is equivalent to the statement

$$J_j \cap \{i \in I\colon R_i(h(a)(i), h(b)(i), h(c)(i))\} \neq 0 .$$

These equivalences imply at once that h^* is an isomorphism from $\mathfrak{A}$ onto a subsystem of $\mathfrak{P}_{\mathcal{D}'}\langle\mathfrak{A}_i\colon i \in I\rangle$.

The idea of the proof of 1.15 is not new, for similar constructions in algebra have been known for some time. For example, Saunders Mac Lane pointed out to Tarski that the direct limit of algebraic systems may always be obtained as a subsystem of a homomorphic image of a direct product of the systems. The main point of such formulations as 1.15 is that we are isolating for closer study the exact kind of homomorphic images needed.

Corollary 1.16. *Let* **K** *be a class of relational systems, and let* $\mathfrak{A}$ *be a system such that each finite subsystem of* $\mathfrak{A}$ *is embeddable in some system in* **K**. *Then* $\mathfrak{A}$ *is embeddable in a prime reduced product of systems in* **K**.

Proof. By 1.14 and 1.15.

A different type of embedding theorem was noticed by Chang and Keisler and is included next with their permission. This theorem shows a relation between a direct product and all of its "subproducts" over finite subsets of the index set.

Theorem 1.17. *Let* $\langle J_k\colon k \in K\rangle$ *be a sequence of subsets of* I *containing all the finite subsets of* I. *Then there is a proper dual ideal* $\mathcal{D}$ *over* K *such that if* $\mathcal{D}' \supseteq \mathcal{D}$ *is a proper dual ideal over* K *and* $\langle\mathfrak{A}_i\colon i \in I\rangle$ *is any sequence of relational systems, then* $\mathfrak{P}\langle\mathfrak{A}_i\colon i \in I\rangle$ *can be embedded in* $\mathfrak{P}_{\mathcal{D}'}\langle\mathfrak{P}\langle\mathfrak{A}_j\colon j \in J_k\rangle\colon k \in K\rangle$.

Proof. Without loss of generality, we may assume that each J_k is a finite subset of I. For each $i \in I$ let

$$L_i = \{k \in K\colon i \in J_k\}.$$

It is obvious that the least dual ideal $\mathcal{D}$ over K with $L_i \in \mathcal{D}$ for $i \in I$ is a proper ideal. Let $\mathcal{D}' \supseteq \mathcal{D}$ be a proper dual ideal over K and let $\langle\mathfrak{A}_i\colon i \in I\rangle$

be a sequence of systems with $\mathfrak{A}_i = \langle A_i, R_i\rangle$ for $i \in I$. Define a function h on $\mathrm{P}\langle A_i\colon i \in I\rangle$ such that for $f \in \mathrm{P}\langle A_i\colon i \in I\rangle$,

$$h(f) = \langle f \upharpoonright J_k\colon k \in K\rangle/{=_{\mathcal{D}'}} .$$

If $f, g \in \mathrm{P}\langle A_i\colon i \in I\rangle$ and $f \neq g$, then $f(i) \neq g(i)$ for some $i \in I$. Hence, $L_i \subseteq \{k \in K\colon f \upharpoonright J_k \neq g \upharpoonright J_k\}$, and so $h(f) \neq h(g)$. A similar argument will show that the one-one function h is actually an embedding.

Theorems 1.15 and 1.17 only show that for some dual ideals an embedding exists; in the next theorem we see that in the case of powers, we can say that an embedding exists for all proper dual ideals.

Theorem 1.18. *If* $\mathcal{D}$ *is a proper dual ideal over* I, *then* $\mathfrak{A}$ *is embeddable in* $\mathfrak{A}^I_{\mathcal{D}}$.

Proof. Let $\mathfrak{A} = \langle A, R\rangle$. The function h defined by the obvious equation $h(a) = \langle a\colon i \in I\rangle/{=_{\mathcal{D}}}$ for $a \in A$ (where the sequence mentioned is a constant sequence) is the required embedding.

We shall call the function indicated in the proof of 1.18 the *canonical embedding* of $\mathfrak{A}$ into $\mathfrak{A}^I_{\mathcal{D}}$. More information on this embedding can be obtained from the results of Section 2 (see esp. 2.10).

Corollary 1.19. *If* $\mathfrak{m} \leqslant |\mathfrak{A}_i|$ *for* $i \in I$ *and* $\mathcal{D}$ *is a proper dual ideal over* I, *then* $\mathfrak{m} \leqslant |\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle|$.

Proof. It is sufficient to argue for zero systems. In that case a zero system $\mathfrak{A}$ with $|\mathfrak{A}| = \mathfrak{m}$ is embeddable in each $\mathfrak{A}_i$. Apply 1.14 and 1.18.

Theorem 1.20. *If* $|\mathfrak{A}| < \mathfrak{m}$ *and* $\mathcal{D}$ *is an* $\mathfrak{m}$*-complete prime dual ideal over* I, *then* $\mathfrak{A}^I_{\mathcal{D}} \cong \mathfrak{A}$.

Proof. Let $\mathfrak{A} = \langle A, R\rangle$. We need only show that the canonical embedding mentioned in 1.18 has as its range $A^I_{\mathcal{D}}$. For this purpose, let h be that embedding and let $f \in A^I$. Obviously, $\bigcup\{\{i \in I\colon f(i) = a\}\colon a \in A\} = I \in \mathcal{D}$. By hypothesis the number of terms of this union is less than $\mathfrak{m}$, and so there must be an element $a \in A$ such that $\{i \in I\colon f(i) = a\} \in \mathcal{D}$. Hence, $h(a) = f/{=_{\mathcal{D}}}$, which establishes the desired conclusion.

Corollary 1.21. *If* $|\mathfrak{A}_i| \leqslant \mathfrak{n} < \mathfrak{m}$ *for all* $i \in I$ *and* $\mathcal{D}$ *is an* $\mathfrak{m}$*-complete prime dual ideal over* I, *then* $|\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle| \leqslant \mathfrak{n}$.

Proof. For zero systems apply 1.14 and 1.20.

In view of the fact that every dual ideal is $\aleph_0$-complete, it follows from 1.21 that a prime reduced product of finite systems of bounded cardinality is always finite. Theorem 1.15 shows there can be no bound given for arbitrary prime reduced products of finite systems. The question of which cardinals can be obtained as the cardinals of reduced products is not completely settled.

The cases where $\mathfrak{A}^I_{\mathcal{D}} \cong \mathfrak{A}$ seem to be very rare when $\mathfrak{A}$ is infinite, as will be shown in the next sequence of theorems. To shorten the statements of the hypotheses, we shall call a dual ideal $\mathcal{D}$ over I a *uniform* ideal if for each $J \in \mathcal{D}$ we have $|J| = |I|$. As noted in the remarks following 1.9, the only interesting cases (as far as the question of cardinality is concerned) are those where the ideal is uniform.

LEMMA 1.22. *Let $\mathfrak{m}$ be an infinite cardinal and suppose that $|A| \geqslant |I| = \mathfrak{m}$. Then there exists a family $\mathcal{F} \subseteq A^I$ such that*

(i) $|\mathcal{F}| > \mathfrak{m}$;

(ii) *if $f, g \in \mathcal{F}$ and $f \neq g$, then*

$$|\{i \in I\colon f(i) = g(i)\}| < \mathfrak{m}\,.$$

Proof. Similar to the lemma in [2], p. 549; cf. also [34].

THEOREM 1.23. *Let $\mathfrak{m}$ be an infinite cardinal and suppose that $|\mathfrak{A}| \geqslant |I| = \mathfrak{m}$. If $\mathcal{D}$ is a uniform dual ideal over I, then $|\mathfrak{A}^I_{\mathcal{D}}| > \mathfrak{m}$.*

Proof. Use the elements $f/{=_{\mathcal{D}}}$ of $\mathfrak{A}^I_{\mathcal{D}}$ where f belongs to the family $\mathcal{F}$ of 1.22. A similar argument was used in [2].

Unfortunately, the proof of 1.22 does not seem to yield the conclusion that $|\mathcal{F}| = 2^{\mathfrak{m}}$, in as much as the diagonal argument only shows that each family $\mathcal{F}$ with $|\mathcal{F}| = \mathfrak{m}$ which satisfies 1.22 (ii) is not maximal with respect to condition (ii). A different type of argument due to Tarski in [38] gives in several cases this additional information without the necessity of using the generalized continuum hypothesis.

LEMMA 1.24. *Let $\mathfrak{m}$ be an infinite cardinal, and let $\mathfrak{n}$ be the least cardinal such that $\mathfrak{m} < 2^{\mathfrak{n}}$. Suppose that $|A| \geqslant \mathfrak{m}$ and $|I| = \mathfrak{n}$. Then there exists a family $\mathcal{F} \subseteq A^I$ such that*

(i) $|\mathcal{F}| = 2^{\mathfrak{n}}$;

(ii) *if $f, g \in \mathcal{F}$ and $f \neq g$, then*

$$|\{i \in I\colon f(i) = g(i)\}| < \mathfrak{n}\,.$$

THEOREM 1.25. *Let $\mathfrak{m}$ be an infinite cardinal, and let $\mathfrak{n}$ be the least cardinal such that $\mathfrak{m} < 2^{\mathfrak{n}}$. Suppose that $|\mathfrak{A}| \geqslant \mathfrak{m}$ and $|I| = \mathfrak{n}$. If $\mathcal{D}$ is a uniform dual ideal over I, then $|\mathfrak{A}^I_{\mathcal{D}}| \geqslant 2^{\mathfrak{n}}$.*

Proof. By 1.24.

Both 1.23 and 1.25 required the hypothesis that $|\mathfrak{A}| \geqslant \mathfrak{m}$, whereas from 1.26 below it is clear that if $\mathfrak{A}$ is merely infinite, then $\mathfrak{A}^I_{\mathcal{D}}$ can be made arbitrarily large even using prime dual ideals $\mathcal{D}$. Using 1.17, Chang and Keisler noticed that the place where $\mathfrak{A}^I_{\mathcal{D}}$ is large can be estimated rather closely. A weaker result was announced by Keisler in [18] and another weaker result was included by Kochen in [28]; a similar proof was found independently by D. Monk.

THEOREM 1.26. *Let $\mathfrak{m}$ be an infinite cardinal and let $|I| = \mathfrak{m}$. Then there is a proper dual ideal $\mathcal{D}$ over I such that if $\mathcal{D}' \supseteq \mathcal{D}$ is a proper dual ideal over I and $\mathfrak{A}$ is any infinite system, then $|\mathfrak{A}^I_{\mathcal{D}'}| \geqslant 2^{\mathfrak{m}}$.*

Proof. Since $\mathfrak{m}$ is infinite, the cardinality of the set of finite subsets of I is $\mathfrak{m}$ also. Let $\langle J_i\colon i \in I\rangle$ be a sequence including all the finite subsets of I and only finite subsets of I. Applying 1.17 (with $I = K$), there is a $\mathcal{D}$ such that if $\mathcal{D}' \supseteq \mathcal{D}$ is proper, then $\mathfrak{A}^I$ is embeddable in $\mathfrak{P}_{\mathcal{D}'}\langle \mathfrak{A}^{J_i}\colon i \in I\rangle$. But since $\mathfrak{A}$ is infinite and each J_i is finite, $|\mathfrak{A}^{J_i}| = |\mathfrak{A}|$. It easily follows that $2^{\mathfrak{m}} \leqslant |\mathfrak{A}^I| \leqslant |\mathfrak{P}_{\mathcal{D}'}\langle \mathfrak{A}^{J_i}\colon i \in I\rangle| = |\mathfrak{A}^I_{\mathcal{D}'}|$.

The counterexample mentioned after 1.14 can now be easily constructed. Let $\mathfrak{A}$ be the two element zero system; let $|J| = \aleph_0$, $|I| = 2^{\aleph_0}$, and let $\mathcal{D}'$ be a prime dual ideal chosen as in 1.26. Then $|(\mathfrak{A}^J)^I_{\mathcal{D}'}| > 2^{\aleph_0}$, by 1.26; but by 1.20, $|(\mathfrak{A}^I_{\mathcal{D}'})^J| = 2^{\aleph_0}$; hence the two systems are not isomorphic. Chang and Keisler have shown that for $\mathcal{D}$ prime over I and $\mathcal{D}'$ prime over J, the systems $(\mathfrak{A}^J_{\mathcal{D}'})^I_{\mathcal{D}}$ and $(\mathfrak{A}^I_{\mathcal{D}})^J_{\mathcal{D}'}$ need not be isomorphic even with $I = J$. The proof, however, uses the relational structure of $\mathfrak{A}$ and is not just a trivial calculation with cardinal numbers as in the example above.

The reason for stating 1.15, 1.17 and 1.26 in the form "*there is a $\mathcal{D}$ such that for every $\mathcal{D}' \supseteq \mathcal{D}$, etc.*" rather than in the more direct form "*there is a prime dual ideal $\mathcal{D}'$, etc.*", is that in the proofs the proper ideal $\mathcal{D}$ can be defined in an effective way that seemed to be rather explicit and may possibly yield further information.

The generalizations of the above results from powers to products is straightforward. The situation with finite factors is not covered by these generalizations, however, and requires a separate argument.

LEMMA 1.27. *There exists a family $\mathcal{F} \subseteq \omega^\omega$ such that*

(i) $|\mathcal{F}| = 2^{\aleph_0}$;

(ii) *if $f \in \mathcal{F}$, then $f(\nu) < 2^\nu$ for all $\nu < \omega$;*

(iii) *if $f, g \in \mathcal{F}$ and $f \neq g$, then $\{\nu < \omega\colon f(\nu) = g(\nu)\}$ is finite.*

Proof. In the statement of the Lemma, ω^ω refers to the family of all functions from ω into ω; while 2^ν stands for the ordinary finite integer 2^ν. Let 2^ω be the family of functions on ω with values 0 or 1. To each function $\varphi \in 2^\omega$ associate a function $f_\varphi \in \omega^\omega$ defined by the equation

$$f_\varphi(\nu) = \sum_{\varkappa < \nu} \varphi(\varkappa) \cdot 2^\varkappa$$

for all $\nu < \omega$. Let $F = \{f_\varphi\colon \varphi \in 2^\omega\}$. The three required properties are immediate.

Notice that this lemma strengthens slightly the conclusion of 1.24. The formulation resulted from an easy analysis of the proof of 1.24 in [38].

THEOREM 1.28. *Let $\mathcal{D}$ be a prime dual ideal over I which is not countably complete, and suppose for all $\mathfrak{n} < \aleph_0$, $\{i \in I: |\mathfrak{A}_i| = \mathfrak{n}\} \notin \mathcal{D}$. Then* $|\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i: i \in I\rangle| \geqslant 2^{\aleph_0}$.

Proof. Let $I' = \{i \in I: |\mathfrak{A}_i| < \aleph_0\}$, and let $I'' = I \sim I'$. By assumption $\mathcal{D}$ is prime; hence either $I' \in \mathcal{D}$ or $I'' \in \mathcal{D}$. In view of 1.3, we may assume without loss of generality that either $I' = I$ or $I'' = I$. In the first case we see that there must exist infinitely many $\mathfrak{n} < \aleph_0$ such that $\{i \in I: |\mathfrak{A}_i| = \mathfrak{n}\} \neq 0$ by virtue of our hypotheses on $\mathcal{D}$. It easily follows by taking unions of several of these subsets of I that there is a partition $\langle J_\nu: \nu < \omega\rangle$ of I into nonempty sets such that (i) $J_\nu \notin \mathcal{D}$, for $\nu < \omega$, and (ii) whenever $i \in J_\nu$, $\nu < \omega$, we have $|\mathfrak{A}_i| \geqslant |2^\nu|$. In the second case, where $I'' = I$, all the factors are infinite. Since we have assumed that $\mathcal{D}$ is not countably complete, it is a simple matter to show the existence of a partition $\langle J_\nu: \nu < \omega\rangle$ of I into non-empty sets such that $J_\nu \notin \mathcal{D}$. The second property of the partition follows because $|\mathfrak{A}_i| \geqslant \aleph_0$ for all $i \in I$. Thus in either case we may assume that we have a partition with the two properties (i) and (ii). The next step is to choose for $i \in J_\nu$, where $\nu < \omega$, a sequence $\langle a_{i,\varkappa}: \varkappa < 2^\nu\rangle$ of distinct elements of the system $\mathfrak{A}_i$. Let $\mathcal{F}$ be the family of functions mentioned in 1.27. For each $f \in \mathcal{F}$, let h_f be defined on I by the equation

$$h_f(i) = a_{i,f(\nu)},$$

where $\nu < \omega$ is the unique integer such that $i \in J_\nu$. An easy calculation shows that for $f, g \in \mathcal{F}$,

$$\{i \in I: h_f(i) = h_g(i)\} = \bigcup \{J_\nu: \nu < \omega \text{ and } f(\nu) = g(\nu)\}.$$

If $f \neq g$, then by the lemma the indicated union is finite, and hence is not in $\mathcal{D}$. In other words, $h_f \neq_{\mathcal{D}} h_g$. This proves that $|\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i: i \in I\rangle| \geqslant |\mathcal{F}|$, and the conclusion follows.

Suppose that $|\mathfrak{A}| = \aleph_0$. None of the foregoing theorems answers the question of whether there exists an I with $|I| = \mathfrak{m} \geqslant \aleph_0$ and a prime dual ideal $\mathcal{D}$ over I such that

$$\aleph_0 < |\mathfrak{A}^I_{\mathcal{D}}| < 2^{\mathfrak{m}}.$$

This is closely related to the problem of whether $|\mathfrak{A}^I_{\mathcal{D}}|$ can be a singular cardinal. The authors have been unable to find any facts that would solve these problems even under the assumption of the generalized continuum hypothesis.

THEOREM 1.29. *If $\mathcal{D}$ is a prime dual ideal over I and $|\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i: i \in I\rangle| < 2^{\aleph_0}$, then for some $j \in I$, $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i: i \in I\rangle \cong \mathfrak{A}_j$.*

Proof. By 1.28 it follows that either there is an $\mathfrak{n} < \aleph_0$ such that

$$I' = \{i \in I: |\mathfrak{A}_i| = \mathfrak{n}\} \in \mathcal{D},$$

or $\mathcal{D}$ is countably complete. In the first case notice that there are only finitely many isomorphism types of relational systems with $\mathfrak{n}$ elements. In other words, there must exist a finite set $J \subset I'$ such that

$$I' = \bigcup \big\{\{i \in I': \mathfrak{A}_i \cong \mathfrak{A}_j\}: j \in J\big\}.$$

Now $\mathcal{D}$ is prime and the union is finite, and so there is an index $j \in J$ where

$$\{i \in I': \mathfrak{A}_i \cong \mathfrak{A}_j\} \in \mathcal{D}.$$

Applying 1.3, 1.12, and 1.20, we obtain the conclusion. In the second case, where $\mathcal{D}$ is countably complete, we invoke e.g. Theorem 3.7 in Smith-Tarski [36], p. 251, to conclude that $\mathcal{D}$ is $\mathfrak{m}$-complete for all cardinals $\mathfrak{m}$ weakly accessible from $\aleph_0$. Let

$$I'' = \{i \in I: |\mathfrak{A}_i| < 2^{\aleph_0}\}.$$

If $I'' \in \mathcal{D}$, then, as in the case with finite cardinals, we note that there is a set $J \subseteq I''$ such that $|J| \leqslant 2^{2^{\aleph_0}}$ and

$$I'' = \bigcup \big\{\{i \in I'': \mathfrak{A}_i \cong \mathfrak{A}_j\}: j \in J\big\}.$$

In as much as $\mathcal{D}$ is $\mathfrak{m}$-complete for some $\mathfrak{m} > 2^{2^{\aleph_0}}$, the argument may proceed as before. On the other hand, if $I \sim I'' \in \mathcal{D}$, then $\{i \in I: |\mathfrak{A}_i| \geqslant 2^{\aleph_0}\} \in \mathcal{D}$. Using 1.3 and 1.19, we would have $|\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i: i \in I\rangle| \geqslant 2^{\aleph_0}$, contradicting the hypothesis.

The final result of this section is the proof of the analogue of 1.29 for arbitrary dual ideals. To carry out the argument we need first, however, to distinguish a class of dual ideals that are remote from the prime dual ideals. As is well known, a characteristic feature of prime dual ideals $\mathcal{D}$ over I is that the quotient modulo $\mathcal{D}$ of the Boolean algebra of all subsets of I has just two elements. Another formulation of this property is that the reduced cartesian power $2^I_{\mathcal{D}}$ has two elements, and the connection with 1.20 is easily seen. We shall call a dual ideal $\mathcal{D}$ (not necessarily prime) an ideal of *finite index* if $2^I_{\mathcal{D}}$ is finite. Equivalently we could say that $\mathcal{D}$ is of finite index if and only if there is a finite partition $\langle J_k: k \in K\rangle$ of I such that each $\mathcal{D} \restriction J_k$ is a prime dual ideal over J_k. When $\mathcal{D}$ is not of finite index we can make an estimate of the cardinality of a reduced product.

THEOREM 1.30. *If $\mathcal{D}$ is a dual over I which is not of finite index, and if $|\mathfrak{A}_i| > 2$ for all $i \in I$, then $|\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i: i \in I\rangle| \geqslant 2^{\aleph_0}$.*

Proof. To say that $\mathcal{D}$ is not of finite index means that the quotient modulo $\mathcal{D}$ of the Boolean algebra of subsets of I is infinite. As is well known, every infinite Boolean algebra contains an infinite sequence of mutually disjoint elements. The proof of this fact can be easily modified to yield a partition $\langle J_\nu: \nu < \omega\rangle$ of I such that $I \sim J_\nu \notin \mathcal{D}$ for all $\nu < \omega$.

Let $\mathfrak{A}_i = \langle A_i, R_i \rangle$ and choose elements $a_{i,0}, a_{i,1} \in A_i$ with $a_{i,0} \neq a_{i,1}$ for $i \in I$. Corresponding to each function $\varphi \in 2^\omega$ let the I-termed sequence f_φ be defined by the equation

$$f_\varphi(i) = a_{i,\varphi(\nu)},$$

where $i \in I$ and ν is the unique integer such that $i \in J_\nu$. It is easy to show for $\varphi, \psi \in 2^\omega$ that

$$\{i \in I\colon f_\varphi(i) = f_\psi(i)\} = \bigcup \{J_\nu\colon \nu < \omega \text{ and } \varphi(\nu) = \psi(\nu)\}.$$

If $\varphi \neq \psi$, then it follows at once from our condition on the partition that $f_\varphi \neq_{\mathcal{D}} f_\psi$, which establishes the desired conclusion.

The above theorem is actually a consequence of Theorem 4.4 in Smith-Tarski [36], p. 254, which could have been used in the proof. However, the construction is so simple that we have preferred to give the direct proof.

THEOREM 1.31. *If $\mathcal{D}$ is any dual ideal over I and $|\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_i\colon i \in I\rangle| < 2^{\aleph_0}$, then there is a finite subset $J \subseteq I$ such that*

$$\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_i\colon i \in I\rangle \cong \mathfrak{P}\langle \mathfrak{A}_j\colon j \in J\rangle.$$

Proof. By the remark following 1.8 we see that the theorem reduces to the case where $|\mathfrak{A}_i| \geq 2$ for all $i \in I$. By 1.30 we conclude that $\mathcal{D}$ must be of finite index, that is, there is a finite partition $\langle J_k\colon k \in K\rangle$ of I such that $\mathcal{D} \restriction J_k$ is prime over J_k. By 1.2 the reduced product is isomorphic to a direct product of factors $\mathfrak{P}_{\mathcal{D}\restriction J_k}\langle \mathfrak{A}_i\colon i \in J_k\rangle$. But each of these factors must have cardinality less than $2^{\aleph_0}$. Hence by 1.29 the kth factor is isomorphic to $\mathfrak{A}_{j_k}$ for some $j_k \in J_k$. Let $J = \{j_k\colon k \in K\}$, and the conclusion follows.

2. Model theoretical properties of reduced products. Many new relational systems may be constructed from given systems by means of reduced products and powers. Several facts about the cardinalities of the resulting systems were given in the previous section, but these facts and this type of construction would be of not interest if it were not possible to discover something useful about the relational structure of the product systems. Two hints as to the possibilities are contained in 1.13 and 1.15, from which it is seen that a reduced product is related to its factors through the well-known operations of forming direct products and homomorphic images and contains in many cases a variety of subsystems. However, these hints do not tell the whole story, because the homomorphic images involved, for example, are only of a very special type. The way to understand more clearly the nature of the construction is to investigate which properties of relational systems are preserved under the formation of reduced products. In particular, to make the problem more precise, we shall discuss in this section the question of which first-order properties are preserved and then give some applications of the results obtained.

To be able to formulate first-order conditions on relational systems, we make use of a language $L^{(\alpha)}$, α an infinite ordinal number, involving a transfinite sequence $\langle v_\xi\colon \xi < \alpha\rangle$ of distinct *variables*. In addition, the *logical constants* $\wedge$, $\vee$, $\neg$, $\bigwedge$, $\bigvee$, and $=$, standing for conjunction, disjunction, negation, universal and existential quantification, and identity, are used, together with a ternary relation symbol R. The *formulas* of $L^{(\alpha)}$ are built from *atomic formulas* of the forms $\mathrm{R}(v_\xi, v_\eta, v_\zeta)$ and $v_\xi = v_\eta$ in the usual way by means of the sentential connectives and the quantifiers. *Free* and *bound* occurrences of variables in formulas are defined in the well-known way. A *sentence* is a formula without free variables. Upper case Greek letters will range over formulas, while bold-face Greek letters will denote sets of formulas. Let $\mathfrak{A} = \langle A, R\rangle$ be a relational system, and let $f \in A^\alpha$ be an α-termed sequence of elements of the system. We assume as known the meaning of the phrase *the sequence f satisfies the formula Φ in the system $\mathfrak{A}$* (cf. [42]). A sentence Φ is *true* in $\mathfrak{A}$ if it is satisfied by all (or at least one) of the sequences in A^α; in this case $\mathfrak{A}$ is also said to be a *model* of Φ. Two formulas are *logically equivalent* if in every relational system they are satisfied by exactly the same sequences.

For our purposes we need to distinguish a special class of formulas, now generally called *Horn formulas*, which were first investigated in [10] only in connection with direct products. These are formulas in prenex normal form (with arbitrarily many changes of quantifiers) with a matrix that is a conjunction of (quantifier free) formulas of the form $\Theta_0 \vee \ldots \vee \Theta_{\nu-1}$ where each Θ_μ, $\mu < \nu$, is either an atomic formula or a negation of an atomic formula, *but* where at most one atomic formula occurs unnegated. Such disjunctions are called *basic Horn formulas*. Notice that any formula obtained by conjunctions and quantifications (in any order) of basic Horn formulas is logically equivalent to a Horn formula.

Instead of properties of relational systems we shall speak of classes of systems. Bold-face capital Gothic letters will denote such classes. A class $\mathbf{K}$ of systems is called an *elementary class* (in symbols, $\mathbf{K} \in \mathbf{EC}$) if there is a sentence Φ such that $\mathbf{K}$ is the class of all models of Φ. Inasmuch as each sentence contains only finitely many variables, it makes no difference in this definition if we restrict attention to sentences of $L^{(\omega)}$. An arbitrary intersection of elementary classes is an *elementary class in the wider sense* (or a class in $\mathbf{EC}_\Delta$); in other words, $\mathbf{K} \in \mathbf{EC}_\Delta$ means that there is a set $\mathbf{\Sigma}$ of sentences (of $L^{(\omega)}$) such that $\mathbf{K}$ consists of all those systems that are models of all sentences of $\mathbf{\Sigma}$. A class $\mathbf{K}$ is a *Horn class* $\mathbf{HC}$($\mathbf{K}$ $\in$ or $\mathbf{K} \in \mathbf{HC}_\Delta$) if analogous conditions hold involving sentences or sets of sentences that are all Horn formulas. A class $\mathbf{K}$ is an *α-quasi-elementary class* ($\mathbf{K} \in \mathbf{QC}_\Delta^{(\alpha)}$) if there is a set Σ of formulas of the language $L^{(\alpha)}$ such that $\mathbf{K}$ consists of all systems $\mathfrak{A} = \langle A, R \rangle$ for which there is a sequence $f \in A^\alpha$ simultaneously satisfying all the formulas in Σ. This definition depends essentially on the ordinal α. We say $\mathbf{K} \in \mathbf{QC}_\Delta$ if $\mathbf{K} \in \mathbf{QC}_\Delta^{(\alpha)}$ for at least one $\alpha \geqslant \omega$.

Let $\mathfrak{A} = \langle A, R \rangle$ and $\mathfrak{B} = \langle B, S \rangle$ be relational systems. The systems $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* ($\mathfrak{A} \equiv \mathfrak{B}$) if they belong to the same elementary classes; or, in other words, if every sentence true in $\mathfrak{A}$ is also true in $\mathfrak{B}$ and conversely. The intersection of all elementary classes containing $\mathfrak{A}$ is the *elementary type* of $\mathfrak{A}$ and is just the equivalence class of $\mathfrak{A}$ under the relation $\equiv$. Isomorphic systems have the same elementary type, but the converse is not always true. We say that $\mathfrak{B}$ is an *elementary extension* of $\mathfrak{A}$ (or $\mathfrak{A}$ is an *elementary subsystem* of $\mathfrak{B}$) if $\mathfrak{A}$ is a subsystem of $\mathfrak{B}$ and whenever $f \in A^\omega$ and Φ is a formula of $L^{(\omega)}$ satisfied by f in $\mathfrak{A}$, then f also satisfies Φ in $\mathfrak{B}$. If $\mathfrak{A}$ is isomorphic to an elementary subsystem of $\mathfrak{B}$, then $\mathfrak{A}$ is *elementarily embeddable* in $\mathfrak{B}$. Of course elementary extensions of a system belong to the same elementary type. More information on these notions can be found in [42]. The languages $L^{(\alpha)}$ can be used to provide another definition of elementary extension. Suppose that $\mathfrak{A}$ is a subsystem of $\mathfrak{B}$ and $|\alpha| \geqslant |\mathfrak{A}|$. Let $f \in A^\alpha$ be a sequence whose range is all of A. If this one sequence f satisfies exactly the same formulas in $\mathfrak{A}$ as it does in $\mathfrak{B}$, then $\mathfrak{B}$ is an elementary extension of $\mathfrak{A}$.

If f is an α-termed sequence and $\xi < \alpha$, then $f(\xi/a)$ is that α-termed sequence g such that $g(\xi) = a$, but for all $\eta < \alpha$ if $\eta \neq \xi$, then $g(\eta) = f(\eta)$. That is, $f(\xi/a)$ is the result of replacing the ξ-th term of the sequence f by the entry a. If f is an α-termed sequence, and if for each $\xi < \alpha$, $f(\xi)$ is an I-termed sequence, then f may be considered as an α by I matrix. The transpose of that matrix is denoted by $f^\vee$. In other words, $f^\vee$ is an I-termed sequence, such that for each $i \in I$, $f^\vee(i)$ is an α-termed sequence where $f^\vee(i)(\xi) = f(\xi)(i)$, for $\xi < \alpha$. Notice that if g is an I-termed sequence, then $f^\vee(i)\big(\xi/g(i)\big) = f(\xi/g)^\vee(i)$, for $i \in I$, $\xi < \alpha$.

The above notation is useful for investigating the properties of satisfaction. For example, if $\mathfrak{A} = \langle A, R \rangle$, $f \in A^\alpha$ and Φ is a formula of $L^{(\alpha)}$, then f satisfies $\bigvee v_\xi \Phi$ in $\mathfrak{A}$ if and only if $f(\xi/a)$ satisfies Φ in $\mathfrak{A}$ for some $a \in A$. Next, suppose that $\langle \mathfrak{A}_i \colon i \in I \rangle$ is an I-termed sequence of relational systems, where $\mathfrak{A}_i = \langle A_i, R_i \rangle$, for $i \in I$, and that $\mathcal{D}$ is a dual prime ideal over I. For our purposes, it is easier to express facts about satisfaction in terms of α-termed sequences of elements from the direct product rather than from the reduced product. If f is an α-termed sequence of elements from $\mathfrak{P}\langle \mathfrak{A}_i \colon i \in I \rangle$, then by f^* we shall denote the α-termed sequence characterized by the property that $f^*(\xi) = f(\xi)/{=_\mathcal{D}}$ for $\xi < \alpha$. Hence, f^* is the corresponding sequence of elements of $\mathfrak{P}_\mathcal{D}\langle \mathfrak{A}_i \colon i \in I \rangle$. To shorten writing let $\mathfrak{B}^* = \mathfrak{P}_\mathcal{D}\langle \mathfrak{A}_i \colon i \in I \rangle$ and $B = \mathrm{P}\langle A_i \colon i \in I \rangle$. For each formula Φ of $L^{(\alpha)}$ and each $f \in B^\alpha$ let

$$J_{f,\Phi} = \{i \in I \colon f^\vee(i) \text{ satisfies } \Phi \text{ in } \mathfrak{A}_i\}.$$

Finally let

$$\Gamma = \{\Phi \colon \text{for all } f \in B^\alpha,\ f^* \text{ satisfies } \Phi \text{ in } \mathfrak{B}^* \text{ if and only if } J_{f,\Phi} \in \mathcal{D}\};$$

and

$$\Gamma' = \{\Phi \colon \text{for all } f \in B^\alpha,\ \text{if } J_{f,\Phi} \in \mathcal{D} \text{ then } f^* \text{ satisfies } \Phi \text{ in } \mathfrak{B}^*\}.$$

Obviously $\Gamma \subseteq \Gamma'$, and each of the two sets of formulas is closed under logical equivalence. We shall refer to these sets in order to make clear the connection between formulas satisfied in the reduced product and formulas satisfied in the factors. In particular, the following lemma holds for the given reduced product, to which our notation refers.

LEMMA 2.1. *If $\mathcal{D}$ is a proper dual ideal, then every Horn formula of $L^{(\alpha)}$ belongs to Γ'; if in addition $\mathcal{D}$ is prime, then every formula belongs to Γ.*

Proof. The required conclusions will follow directly from these five statements:

(i) *every atomic formula belongs to Γ;*

(ii) *if $\Phi, \Psi \in \Gamma$, then $\Phi \wedge \Psi \in \Gamma$;*

(iii) *if $\Phi, \Psi \in \Gamma'$, then $\Phi \wedge \Psi$, $\bigvee v_\xi \Phi$, $\bigwedge v_\xi \Phi \in \Gamma'$;*

(iv) *every basic Horn formula belongs to Γ';*

(v) *if $\mathcal{D}$ is prime and $\Phi \in \Gamma$, then $\neg\Phi$, $\bigvee v_\xi \Phi \in \Gamma$.*

Statement (i) is a trivial consequence of the definitions of reduced products and the set Γ; while (ii) and the first part of the conclusion of (iii) follow from the facts that for all $X, Y \subseteq I$, $X \cap Y \in \mathcal{D}$ if and only if $X, Y \in \mathcal{D}$, and that $J_{f,\Phi\wedge\Psi} = J_{f,\Phi} \cap J_{f,\Psi}$, for all $f \in B^\alpha$, and all formulas Φ, Ψ. To establish the remaining parts of (iii), assume first that $\Phi \in \Gamma'$, and let Ψ be the formula $\bigvee v_\xi \Phi$, where $\xi < \alpha$. Suppose that $f \in B^\alpha$ and $J_{f,\Psi} \in \mathcal{D}$. Hence, if $i \in J_{f,\Psi}$, then $f^\vee(i)$ satisfies $\bigvee v_\xi \Phi$ in $\mathfrak{A}_i$, and so $f^\vee(i)(\xi/a)$ satisfies Φ in $\mathfrak{A}_i$ for some $a \in A_i$. Applying the axiom of choice, let $g \in B$ be chosen so that $f(i)\big(\xi/g(i)\big)$ satisfies Φ in $\mathfrak{A}_i$, for all $i \in J_{f,\Psi}$. It follows that $J_{f,\Psi} \subseteq J_{f(\xi/g),\Phi}$; hence, this latter set is in $\mathcal{D}$. In view of the assumption on Φ, the sequence $f(\xi/g)^*$ satisfies Φ in $\mathfrak{B}^*$, from which we easily conclude that f^* satisfies $\bigvee v_\xi \Phi$ in $\mathfrak{B}^*$. Thus we have shown that $\bigvee v_\xi \Phi \in \Gamma'$. Next assume that $\Phi \in \Gamma'$, and let Ψ now be the formula $\bigvee v_\xi \Phi$. Suppose that $f \in B^\alpha$ and $J_{f,\Psi} \in \mathcal{D}$. To show that f^* satisfies Ψ in $\mathfrak{B}^*$, it is sufficient to establish that $f(\xi/g)^*$ satisfies Φ in $\mathfrak{B}^*$, for all $g \in B$. If $g \in B$, it is easily checked that $J_{f,\Psi} \subseteq J_{f(\xi/g),\Phi}$, and hence $J_{f(\xi/g),\Phi} \in \mathcal{D}$. The desired conclusion now follows from the assumption on Φ.

The proof of (iv) divides into cases according to the type of basic Horn formula considered. If the formula is simply a single unnegated atomic formula, then it is in $\boldsymbol{\Gamma}'$ by virtue of (i) and the inclusion $\boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}'$. Suppose then the formula Φ is of the form $\neg\Theta_0 \vee \ldots \vee \neg\Theta_{\nu-1}$, where each Θ_μ, $\mu < \nu$, is atomic. By way of contradiction, assume that $f \in B^\alpha$, $J_{f,\Phi} \in \mathcal{D}$, but f^* does not satisfy Φ in $\mathfrak{B}^*$. Now $\neg\Phi$ is logically equivalent to a conjunction of atomic formulas and is in $\boldsymbol{\Gamma}$ in view of (i) and (iii). Hence, $J_{f,\neg\Phi} \in \mathcal{D}$, and so $0 = J_{f,\Phi} \cap J_{f,\neg\Phi} \in \mathcal{D}$, which contradicts the assumption that $\mathcal{D}$ is proper. Finally, consider the case where Φ has the form $\neg\Theta_0 \vee \ldots \vee \neg\Theta_{\nu-1} \vee \Theta_\nu$, where each Θ_μ, $\mu \leqslant \nu$, is atomic. Again assume that $f \in B^\alpha$, $J_{f,\Phi} \in \mathcal{D}$, but f^* does not satisfy Φ in $\mathfrak{B}^*$. Let Ψ be the formula $\Theta_0 \wedge \ldots \wedge \Theta_{\nu-1}$. Clearly, f^* must satisfy Ψ in $\mathfrak{B}^*$, and since $\Psi \in \boldsymbol{\Gamma}$, we conclude that $J_{f,\Psi} \in \mathcal{D}$. Notice that $J_{f,\Theta_\nu} \supseteq J_{f,\Phi} \cap J_{f,\Psi} \in \mathcal{D}$; hence, $J_{f,\Theta_\nu} \in \mathcal{D}$. But $\Theta_\nu \in \boldsymbol{\Gamma}$, and so f^* satisfies Θ_ν in $\mathfrak{B}^*$. This last statement obviously contradicts our assumption that f^* does not satisfy Φ in $\mathfrak{B}^*$, which completes the proof of (iv).

To prove (v), assume that $\mathcal{D}$ is prime and $\Phi \in \Gamma$. The equation $J_{f,\neg\Phi} = I \sim J_{f,\Phi}$ shows that if $\mathcal{D}$ is prime, then $J_{f,\neg\Phi} \in \mathcal{D}$ if and only if $J_{f,\Phi} \notin \mathcal{D}$. That $\neg\Phi \in \boldsymbol{\Gamma}$ now easily follows. It also follows from this equivalence that if Ψ is any formula such that $\Psi, \neg\Psi \in \boldsymbol{\Gamma}'$, then $\Psi \in \boldsymbol{\Gamma}$. Take the case where Ψ is the formula $\bigvee v_\xi \Phi$. From (iii) we know that $\Psi \in \boldsymbol{\Gamma}'$, but $\neg\Phi \in \boldsymbol{\Gamma} \subseteq \boldsymbol{\Gamma}'$, as we have just seen; hence $\bigwedge v_\xi \neg\Phi \in \boldsymbol{\Gamma}'$ by (iii) again. Since $\neg\Psi$ is logically equivalent to $\bigwedge v_\xi \neg\Phi$, we have $\neg\Psi \in \boldsymbol{\Gamma}'$ and $\Psi \in \boldsymbol{\Gamma}$, as was to be shown. The proof of the lemma is now complete.

As a direct consequence of Lemma 2.1, we have our first basic result that shows how the true sentences of a prime product are related to the sentences true in the factors.

THEOREM 2.2. *If $\mathcal{D}$ is a prime dual ideal over I and Φ is a sentence (of $L^{(\omega)}$), then Φ is true in $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$ if and only if $\{i \in I\colon \Phi$ is true in $\mathfrak{A}_i\} \in \mathcal{D}$.*

Clearly if a sentence is true in all the factors, then it is true in a prime product. Of course, the converse need not hold. From 2.2, we may easily derive two corollaries.

COROLLARY 2.3. *Every $\mathbf{EC}_\Delta$ is closed under the formation of prime products.*

COROLLARY 2.4. *If $\mathcal{D}$ is a prime dual ideal over I, and if $\mathfrak{A}_i \equiv \mathfrak{B}_i$ for $i \in I$, then $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle \equiv \mathfrak{P}_{\mathcal{D}}\langle\mathfrak{B}_i\colon i \in I\rangle$.*

In other words, prime products preserve elementary equivalence. Actually 2.4 is a special case of Theorem 5.1 in Feferman-Vaught [3]. From the argument given in [3], p. 77, it is easily seen that not only does 2.4 hold for arbitrary $\mathcal{D}$, but also *reduced products preserve elementary embeddability*. For the case of a prime $\mathcal{D}$, the argument via 2.1 seems to be the most direct; but for general $\mathcal{D}$, 2.1 is not quite strong enough and the method of [3] embodied in their Theorem 3.1, of which our 2.1 is a simplified version, would appear to yield the proper approach. Corollary 2.3 is due to Łoś and is stated by him without proof as (2.6) in [29], p. 105. Notice that the sense of the word "closed" in 2.3 can be rather freely interpreted; that is, if $\mathbf{K} \in \mathbf{EC}_\Delta$, then to conclude that $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle \in \mathbf{K}$ we need only assume that $\{i \in I\colon \mathfrak{A}_i \in \mathbf{K}\} \in \mathcal{D}$. This follows from 2.2 or from the stricter version of 2.3 together with 1.3. For the case of Horn classes, 2.1 has further consequences.

As a different kind of corollary of 2.2 we may derive a purely algebraic result about the finite subsystems of a prime reduced product which could have been easily proved in a direct way as was done in similar cases in Section 1.

COROLLARY 2.5. *If $\mathcal{D}$ is a prime dual ideal over I and $\mathfrak{B}$ is finite system, then $\mathfrak{B}$ is embeddable in $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$ if and only if $\{i \in I\colon \mathfrak{B}$ is embeddable $\mathfrak{A}_i\} \in \mathcal{D}$.*

Proof. Noting that with each finite system $\mathfrak{B}$ there is associated in an obvious way a sentence Φ such that Φ is true in a system $\mathfrak{C}$ if and only if $\mathfrak{B}$ is embeddable in $\mathfrak{C}$, we may apply 2.2 at once.

THEOREM 2.6. *If $\mathcal{D}$ is a proper dual ideal over I and if the sentence Φ is a Horn formula true in $\mathfrak{A}_i$ for each $i \in I$, then Φ is true in $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$.*

COROLLARY 2.7. *Every $\mathbf{HC}_\Delta$ is closed under the formation of proper reduced products.*

Inasmuch as direct products are special cases of reduced products, 2.6 and 2.7 generalize the results of Horn [10] as well as Theorem 1 of Chang-Morel [1]. 2.7 was first noted by Chang. Generalizations about Horn classes need be carried no further, for Keisler in [15] has proved (under the assumption of the continuum hypothesis) that if an $\mathbf{EC}_\Delta$ is closed proper reduced products, then it is necessarily an $\mathbf{HC}_\Delta$.

Corollary 2.3 may be strengthened as follows by going back to 2.1.

THEOREM 2.8. *Every $\mathbf{QC}_\Delta$ is closed under the formation of primeproducts.*

Proof. Let Δ be a set of formulas of the language $L^{(\alpha)}$ and suppose that Δ is simultaneously satisfiable in $\mathfrak{A}_i$ for each $i \in I$. Using the axiom of choice construct an α-termed sequence f of elements of the direct product of the $\mathfrak{A}_i$ such that $f^\vee(i)$ satisfies all formulas of Δ in $\mathfrak{A}_i$, for $i \in I$. If $\mathcal{D}$ is any prime dual ideal over I, then by 2.1 we see that f^* must satisfy all formulas of Δ in $\mathfrak{P}_{\mathcal{D}}\langle\mathfrak{A}_i\colon i \in I\rangle$.

As a corollary of 2.8 and 1.16 we may obtain the theorem of Henkin ([8], p. 414):

COROLLARY 2.9. *If $\mathbf{K} \in \mathbf{QC}_\Delta$ and each finite subsystem of $\mathfrak{A}$ is embeddable in some system $\mathbf{K}$, then $\mathfrak{A}$ is embeddable in some system in $\mathbf{K}$.*

The conclusions of 2.8 and 2.9 also hold for the classes $\mathbf{PC}_\Delta$ introduced in Tarski [40], p. 584. Since we shall not state any applications of these more general facts, and since the proof would require extending the formal language to include several relational symbols, we shall not give these theorems here. Instead we now turn our attention to the proof of the so-called *compactness theorem* for elementary classes (cf. [39]), which turns out to be very easy when prime products are employed. In terms of classes the theorem states that *if* $\mathbf{F} \subseteq \mathbf{EC}_\Delta$ *is a non-empty family with the finite intersection property, then* $\bigcap \mathbf{F} \neq 0$. To emphasize the role of the prime products, we shall state the equivalent version in terms of sentences.

THEOREM 2.10. *If* Σ *is a set of sentences such that every finite subset has a model, then* Σ *has a model.*

Proof. Without loss of generality we may suppose that Σ is closed under the formation of finite conjunctions. Let $\Sigma = \{\Phi_i: i \in I\}$. Suppose that systems $\mathfrak{A}_i$ are chosen so that $\mathfrak{A}_i$ is a model of Φ_i for each $i \in I$. We wish to construct a prime $\mathcal{D}$ over I such that $\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_i: i \in I\rangle$ is a model of Σ. To this end for each $j \in I$ let

$$J_j = \{i \in I: \mathfrak{A}_i \textit{ is a model of } \Phi_j\}.$$

Since $j \in J_j$, $J_j \neq 0$, and $J_j \cap J_k = J_l$, where l is chosen so that $\Phi_j \wedge \Phi_k$ is the sentence Φ_l. Whence, we see that there is a prime dual ideal $\mathcal{D}$ containing all the J_j. From 2.2 it follows at once that each Φ_j is true in $\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_i: i \in I\rangle$; hence $\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_i: i \in I\rangle$ is indeed a model of Σ.

Notice that in the above proof no other fact about the systems $\mathfrak{A}_i$ was used except that for each $i \in I$, Φ_i is true in $\mathfrak{A}_i$. In previous proofs of 2.10 there was no very obvious algebraic or structural connection to be seen relating the $\mathfrak{A}_i$ to the final model of Σ. From our proof it is apparent that formation of the prime product supplies just such an algebraic connection. In the case of denumerable sets of sentences one has even greater freedom in choosing a prime product than was indicated in the above proof. For if $\langle \Phi_\nu: \nu < \omega\rangle$ is any ω-sequence of sentences and if $\mathfrak{A}_\nu$ is a model of $\Phi_0 \wedge \ldots \wedge \Phi_\nu$, for each $\nu < \omega$, then $\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_\nu: \nu < \omega\rangle$ will be a model of $\{\Phi_\nu: \nu < \omega\}$ whenever $\mathcal{D}$ is *any* non principal dual prime ideal over ω. Notice also that in 2.10 if Σ is a class of Horn sentences, then instead of a prime ideal we may simply take the least ideal $\mathcal{D}'$ containing the J_j and the reduced product $\mathfrak{P}_{\mathcal{D}'}\langle \mathfrak{A}_i: i \in I\rangle$ is a model of Σ. This avoids one application of the axiom of choice, but the proof of 2.1, upon which the success of the construction depends, requires the axiom of choice.

The full strength of 2.1 still has not been used, for, with the exception of 2.8 and 2.9, the above results refer to the truth of sentences rather than the satisfaction of formulas. The extra information contained in 2.1 is best exposed by means of the notion of an *elementary extension.*

THEOREM 2.11. *If* $\mathcal{D}$ *is a prime dual ideal over* I, *then* $\mathfrak{A}$ *is elemenmentarily embeddable in* $\mathfrak{A}^I_{\mathcal{D}}$.

Proof. We shall show that the *canonical embedding* of $\mathfrak{A}$ into $\mathfrak{A}^I_{\mathcal{D}}$, mentioned in connection with 1.18, is the required embedding. Let $\mathfrak{A} = \langle A, R\rangle$, and let h be the function from A into the cartesian power such that $h(a) = \langle a: i \in I\rangle$ for $a \in A$. If $f \in A^\alpha$ is an α-termed sequence then $(hf)^*$ is the corresponding sequence of elements of $\mathfrak{A}^I_{\mathcal{D}}$, where hf represents the composition of the two functions. Assume now that $f \in A^\alpha$ and f satisfies Φ in $\mathfrak{A}$. We must show that $(hf)^*$ satisfies Φ in $\mathfrak{A}^I_{\mathcal{D}}$. In view of 2.1 it is sufficient to show that $J_{hf,\Phi} \in \mathcal{D}$. Notice that for each $i \in I$, $(hf)^\vee(i) = f$. Hence, $J_{hf,\Phi} = I$, and the conclusion follows.

THEOREM 2.12. *If* $\mathfrak{A} \equiv \mathfrak{B}$, *then* $\mathfrak{A}$ *is elementarily embeddable in some prime power of* $\mathfrak{B}$.

Proof. Let $\mathfrak{A} = \langle A, R\rangle$ and $\mathfrak{B} = \langle B, S\rangle$. Choose an ordinal $\alpha \geqslant \omega$ so that $|\alpha| \geqslant |\mathfrak{A}|$. Let $f \in A^\alpha$ be a sequence whose range is all of A. The set of all formulas of L^α has cardinality $|\alpha|$. Consider those formulas satisfied by f in $\mathfrak{A}$; they may be arranged in a sequence $\langle \Phi_i: i \in I\rangle$, where $|I| \leqslant |\alpha|$. Now $\mathfrak{A} \equiv \mathfrak{B}$, and it easily follows that corresponding to each $i \in I$ there is a sequence $g_i \in B^\alpha$ such that g_i satisfies Φ_i in $\mathfrak{B}$. For each $j \in I$, let $J_j = \{i \in I: g_i \textit{ satisfies } \Phi_j \textit{ in } \mathfrak{B}\}$. Clearly $j \in J_j$, and so $J_j \neq 0$. If $j, j' \in I$, then f satisfies $\Phi_j \wedge \Phi_{j'}$ in $\mathfrak{A}$; whence, for some $k \in I$, Φ_k is the formula $\Phi_j \wedge \Phi_{j'}$. From the definition we see that $J_k = J_j \cap J_{j'}$; thus the family $\{J_j: j \in I\}$ is closed under finite intersections. As usual, this family can be extended to a prime dual ideal $\mathcal{D}$ with $J_j \in \mathcal{D}$ for $j \in I$. It is to be shown that $\mathfrak{A}$ is isomorphic to an elementary subsystem of $\mathfrak{B}^I_{\mathcal{D}}$. Let h be the α-termed sequence of elements of B^I such that $h^\vee(i) = g_i$. Let h^* be the corresponding sequence of elements of $B^I_{\mathcal{D}}$. From 2.1 we know that for any formula Ψ, the sequence h^* satisfies Ψ in $\mathfrak{B}^I_{\mathcal{D}}$ if and only if $\{i \in I: h^\vee(i) \textit{ satisfies } \Psi \textit{ in } \mathfrak{B}\} \in \mathcal{D}$. It follows at once that for all $j \in I$, h^* satisfies Φ_j in $\mathfrak{B}^I_{\mathcal{D}}$. Among the formulas Φ_j, we find the formula $v_\xi = v_\eta$, if $f(\xi) = f(\eta)$, or the formula $\neg v_\xi = v_\eta$, if $f(\xi) \neq f(\eta)$. Hence, for $\xi, \eta < \alpha$, we have $f(\xi) = f(\eta)$ if and only if $h^*(\xi) = h^*(\eta)$. By a similar argument we show that, for $\xi, \eta, \zeta < \alpha$, $R(f(\xi), f(\eta), f(\zeta))$ if and only if $S^*(h^*(\xi), h^*(\eta), h^*(\zeta))$, where S^* is the relation of $\mathfrak{B}^I_{\mathcal{D}}$. In other words the correspondence of an element $f(\xi)$ of A with $h^*(\xi)$ yields an embedding of $\mathfrak{A}$ into $\mathfrak{B}^I_{\mathcal{D}}$. Let $\mathfrak{C}$ be the subsystem of $\mathfrak{B}^I_{\mathcal{D}}$ whose set of elements is the range of h^*. The isomorphism exhibited above means that h^* satisfies Ψ in $\mathfrak{C}$ if and only if f satisfies Ψ in $\mathfrak{A}$, for any formula Ψ. On the other hand, by construction we know that f satisfies Ψ in $\mathfrak{A}$ if and only if h^* satisfies Ψ in $\mathfrak{B}^I_{\mathcal{D}}$. Since h^* exhausts the elements of $\mathfrak{C}$, these equivalences prove that $\mathfrak{B}^I_{\mathcal{D}}$ is an elementary extension of $\mathfrak{C}$.

In the case that $\mathfrak{A}$ is an elementary extension of $\mathfrak{B}$, Kochen has pointed out that the conclusion of 2.12 can be strengthened. Namely, in the proof we can simply take $g_i(\xi) = f(\xi)$, whenever $f(\xi) \in B$, which has the consequence that the restriction of the embedding of $\mathfrak{A}$ in $\mathfrak{B}^I_{\mathcal{D}}$ to the subsystem $\mathfrak{B}$ is nothing more than the canonical embedding of $\mathfrak{B}$ in $\mathfrak{B}^I_{\mathcal{D}}$. Kochen has made use of this fact in connection with his "limit ultrapowers" introduced in [24].

Keisler has proved, with the aid of the generalized continuum hypothesis, a much stronger result than 2.12 in [21], where he shows that if $\mathfrak{A} \equiv \mathfrak{B}$, then for some I and for some $\mathcal{D}$ prime over I, $\mathfrak{A}^I_{\mathcal{D}} \cong \mathfrak{B}^I_{\mathcal{D}}$. As a consequence of this superior result, we see that $\mathfrak{A} \equiv \mathfrak{B}$ if and only if $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic prime powers; hence a purely algebraic model theoretic characterization of elementary equivalence is obtained. Unfortunately Keisler's interesting method seems to involve the generalized continuum hypothesis in an essential way, so the weaker result of 2.12 still has some interest. However, without use of the continuum hypothesis both Keisler in [12] and Kochen in [24] show that a suitable notion of "limit ultrapower" yields an algebraic characterization of elementary equivalence.

The next three results show how prime products can be used to give characterizations of various kinds of classes which were defined originally in a metamathematical way.

THEOREM 2.13. *In order that a class* $\mathbf{K}$ *of relational systems be in* $\mathbf{EC}_\Delta$ *it is necessary and sufficient that:*

(i) $\mathbf{K}$ *is closed under the formation of prime products*;

(ii) $\mathbf{K}$ *is closed under elementary equivalence.*

Proof. The necessity of (i) and (ii) follows from 2.3 and the definition of elementary equivalence. Assume now that $\mathbf{K}$ satisfies (i) and (ii), and let $\mathfrak{A}$ be a model of the set of sentences true of every system in $\mathbf{K}$. We must show that $\mathfrak{A} \in \mathbf{K}$. Let all the sentences true in $\mathfrak{A}$ be arranged in a sequence $\langle \Phi_i : i \in I \rangle$. For each $i \in I$, we may choose a system $\mathfrak{B}_i \in \mathbf{K}$ in which Φ_i is true, since otherwise $\neg \Phi_i$ would hold throughout $\mathbf{K}$ and also in $\mathfrak{A}$. For each $j \in I$, let $J_j = \{i \in I : \Phi_j \textit{ is true in } \mathfrak{B}_i\}$. As in our other proofs, it is easy to show that the family of sets $\{J_j : j \in I\}$ has the finite intersection property and is thus contained in a prime dual ideal $\mathcal{D}$ over I. Clearly $\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{B}_i : i \in I \rangle \in \mathbf{K}$ and each Φ_j is true in this prime product by 2.2. Whence, we conclude that $\mathfrak{A} \equiv \mathfrak{P}_{\mathcal{D}}\langle \mathfrak{B}_i : i \in I \rangle$, and so $\mathfrak{A} \in \mathbf{K}$, as was to be shown.

In view of 2.12, it should be noted that condition (ii) of 2.13 can be replaced by the following:

(ii′) *whenever* $\mathfrak{A}$ *is elementarily embeddable in a system in* $\mathbf{K}$, *then* $\mathfrak{A} \in \mathbf{K}$;

because (i) and (ii′) together imply (ii), which in turn implies (ii′). The result of Keisler [21] mentioned above (assuming the generalized continuum hypothesis) shows that (ii) may also be replaced by this condition:

(ii″) *whenever* $\mathcal{D}$ *is prime over* I *and* $\mathfrak{A}^I_{\mathcal{D}}$ *is isomorphic to a system in* $\mathbf{K}$, *then* $\mathfrak{A} \in \mathbf{K}$.

Using the notation $\widetilde{\mathbf{K}}$ for the complement of $\mathbf{K}$ in the class of all systems, this consequence of Keisler's result can easily be stated in words: $\mathbf{K} \in \mathbf{EC}_\Delta$ *if and only if* $\mathbf{K}$ *is closed under isomorphism and prime products, and* $\widetilde{\mathbf{K}}$ *is closed under prime powers.*

From the compactness theorem (2.10) it follows that $\mathbf{K} \in \mathbf{EC}$ if and only if both $\mathbf{K}, \widetilde{\mathbf{K}} \in \mathbf{EC}_\Delta$ (cf. Tarski [39], p. 714). Hence, 2.13 leads to a characterization of classes in $\mathbf{EC}$.

COROLLARY 2.14. *In order that a class* $\mathbf{K}$ *of relational systems be in* $\mathbf{EC}$ *it is necessary and sufficient that conditions* (i) *and* (ii) *of* 2.13 *as well as the following hold*:

(iii) *whenever* $\mathcal{D}$ *is prime over* I *and* $\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_i : i \in I \rangle \in \mathbf{K}$, *then for some* $i \in I$, $\mathfrak{A}_i \in \mathbf{K}$.

In other words, (iii) means that $\widetilde{\mathbf{K}}$ is closed under prime products. From Keisler's result mentioned above, we see that (under the continuum hypothesis) condition (ii) of 2.13 can be replaced by the condition that $\mathbf{K}$ is closed under isomorphism.

In [40], Tarski has introduced the notion of a *universal class* ($\mathbf{UC}$ and $\mathbf{UC}_\Delta$) meaning a class characterized by (prenex) universal sentences, and he has shown that $\mathbf{K} \in \mathbf{UC}_\Delta$ if and only if $\mathbf{K} \in \mathbf{EC}_\Delta$ and $\mathbf{K}$ is closed under the formation of subsystems. Combining 2.12 and 2.13, we derive the characterization of such classes first obtained by Łoś [29] for algebraic systems.

COROLLARY 2.15. *In order that a class* $\mathbf{K}$ *of relational systems be in* $\mathbf{UC}_\Delta$ *it is necessary and sufficient that* $\mathbf{K}$ *be closed under the formation of prime products, subsystems, and isomorphic images.*

It is also possible to characterize quasi-elementary classes in a style similar to the theorems just given.

THEOREM 2.16. *In order that a class* $\mathbf{K}$ *of relational systems be in* $\mathbf{QC}_\Delta$ *it is necessary and sufficient that:*

(i) $\mathbf{K}$ *is closed under isomorphism*;

(ii) $\mathbf{K}$ *is closed under the formation of prime products*;

(iii) $\mathbf{K}$ *is closed under the formation of elementary extensions*;

(iv) *there is a cardinal number* $\mathfrak{m}$ *such that every system in* $\mathbf{K}$ *has an elementary subsystem in* $\mathbf{K}$ *of cardinality at most* $\mathfrak{m}$.

Proof. That (i), (ii), and (iii) are necessary easily follows from the definition of $\mathbf{QC}_\Delta$ and 2.8. The necessity of (iv) is a consequence of the definition and the Löwenheim-Skolem theorem as given in Tarski-Vaught [42]. Note that $\mathfrak{m}$ may be taken as the cardinality of the set of formulas of $L^{(\alpha)}$, where $\mathbf{K} \in \mathbf{QC}_\Delta^{(\alpha)}$.

Now assume that $\mathbf{K}$ satisfies the stated conditions. Let $\langle \mathfrak{A}_i \colon i \in I\rangle$ be a sequence of systems in $\mathbf{K}$ such that any system of $\mathbf{K}$ of cardinality at most $\mathfrak{m}$ is isomorphic to some system in the sequence. Let $\mathfrak{A}_i = \langle A_i, R_i\rangle$ for $i \in I$, and choose $\alpha \geqslant \omega$ so that $|\alpha| \geqslant |\mathfrak{P}\langle \mathfrak{A}_i \colon i \in I\rangle|$. Let f be an α-termed sequence whose range is all of $\mathrm{P}\langle A_i \colon i \in I\rangle$. Let Σ be the set of those formulas Φ of $L^{(\alpha)}$ such that $f^\vee(i)$ satisfies Φ in A_i for all $i \in I$. We wish to show that Σ characterizes $\mathbf{K}$. Suppose then that $\mathfrak{B} = \langle B, S\rangle$ is a relational system, and that $g \in B^\alpha$ satisfies all formulas of Σ in $\mathfrak{B}$. Let Σ' be the set of all formulas satisfied by g in $\mathfrak{B}$. For each $\Phi \in \Sigma'$, let $J_\Phi = \{i \in I \colon f^\vee(i) \textit{ satisfies } \Phi \textit{ in } \mathfrak{A}_i\}$. If J_Φ were empty, then $\neg\Phi$ would be in Σ, which is impossible because $\Sigma \subseteq \Sigma'$. Also, if $\Phi, \Psi \in \Sigma'$, then $\Phi \wedge \Psi \in \Sigma'$ and $J_{\Phi\wedge\Psi} = J_\Phi \cap J_\Psi$. Hence, the family $\{J_\Phi \colon \Phi \in \Sigma'\}$ has the finite intersection property and can be extended to a prime dual ideal $\mathcal{D}$. Notice that by construction if $f(\xi) =_{\mathcal{D}} f(\eta)$ for $\xi, \eta < \alpha$, then $g(\xi) = g(\eta)$ because the formula $v_\xi = v_\eta$ must be in Σ'. Thus there is a well-defined function h on $\mathrm{P}_{\mathcal{D}}\langle A_i \colon i \in I\rangle$ such that $h(f(\xi)^*) = g(\xi)$. By an argument similar to that used in 2.12, it is easily checked that h yields an elementary embedding of $\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_i \colon i \in I\rangle$ into $\mathfrak{B}$. Therefore $\mathfrak{B} \in \mathbf{K}$. Finally, since by condition (iv) each system in $\mathbf{K}$ has an elementary subsystem isomorphic to one of the $\mathfrak{A}_i$, it follows that each system in $\mathbf{K}$ has a sequence satisfying all of the formulas in Σ, and the proof is complete.

At last it is time to indicate some examples of reduced products. In stating these examples we shall freely refer to relational systems with several relations, even though, strictly speaking, our results have only been proved for systems with one ternary relation. We may also use systems with operations. For a binary operation, say, may be identified with a ternary relation, and to say that R is an operation over A means that $\langle A, R\rangle$ is a model for the sentence

$$\bigwedge v_0 \bigwedge v_1 \bigvee v_2 \bigwedge v_3 [\mathrm{R}(v_0, v_1, v_3) \leftrightarrow v_2 = v_3]\,.$$

Now this sentence is (logically equivalent to) a Horn sentence; hence, the class of *'operational' systems* (or *algebras*) is closed under the formation of reduced products.

The class of *Boolean algebras* can be construed as a class of systems with one binary operation. Let $\mathfrak{A}$ be the two-element Boolean algebra.

As is well-known, the direct power $\mathfrak{A}^I$ is isomorphic to the Boolean algebra of all subsets of the set I. It is easy to check that if $\mathcal{D}$ is a dual ideal over I, then $\mathfrak{A}^I_{\mathcal{D}}$ is isomorphic to the quotient algebra of sets modulo $\mathcal{D}$ in the usual sense of ideal theory in Boolean algebras. Of course, if $\mathcal{D}$ is prime, then $\mathfrak{A}^I_{\mathcal{D}} \cong \mathfrak{A}$, as we have shown in general in 1.20. The method employed in Chang-Morel [1] was, in effect, to notice that if $\mathfrak{B}$ is an infinite atomistic Boolean algebra, then the reduced powers of $\mathfrak{B}$ need not be atomistic. Conclusion: *the class of atomistic Boolean algebras* (*an EC*) *though closed under direct products is not a Horn class.*

A *zero system* is a model of the Horn sentence

$$\bigwedge v_0 \bigwedge v_1 \bigwedge v_2 \neg \mathrm{R}(v_0, v_1, v_2)\,.$$

Hence, 1.6 is a special case of 2.7. We gave the direct argument in Section 1 because zero systems afforded a device for making deductions about the cardinalities of reduced powers, and we wished to illustrate some of the simple methods for handling them.

The class of *commutative fields* is an $\mathbf{EC}_\Delta$ which is not an $\mathbf{HC}_\Delta$, because it is not closed under direct products. The class is closed under prime products by 2.3, and in this case the prime product has an easy interpretation. Namely, if $\mathfrak{A}_i$, $i \in I$, are fields, then the prime products of the $\mathfrak{A}_i$ are exactly the quotients of the ring $\mathfrak{P}\langle \mathfrak{A}_i \colon i \in I\rangle$ by prime (= maximal, in this case) ideals. As indicated by Scott in [33], if the $\mathfrak{A}_i$ are merely integral domains, then the prime products are obtained by dividing the direct product by *minimal prime ideals.* Suppose that $\mathfrak{A}$ is the field of real numbers. The class of *real-closed fields* is an EC_Δ. Since $\mathfrak{A}$ is real-closed, so is every quotient field of $\mathfrak{A}^I$ for every index set I. This conclusion is also a special case of a theorem of Hewitt [9]. Suppose the $\mathfrak{A}_i$ run through all the finite fields and $\mathcal{D}$ is non-principal and prime over I; then $\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_i \colon i \in I\rangle$ is a non-denumerable field which satisfies all properties (which can be formulated in $L^{(\omega)}$) shared by all finite fields. For example, every element of this field must be the sum of two squares, but the field need not be algebraically closed. In this algebraic context the result of Keisler [21] mentioned after 2.12 has a striking consequence. Let us say that two fields are *equivalent* if some quotient field of a direct power of the first is isomorphic to a quotient of a direct power of the second. Keisler's metamathematical argument shows that the relation of being equivalent *is* an equivalence relation (in particular that it is transitive). This is a purely algebraic result discovered on metamathematical grounds. The metamathematical approach also gives additional information: from algebraic insight alone would it be clear that there are at most $2^{\aleph_0}$ equivalence classes under this relation?

Next consider the class of systems *simply ordered* by a (binary) relation. Again this class is an $\mathbf{EC}$ that is not an $\mathbf{HC}$. Let $\mathfrak{A}$ be the system of rational numbers with their natural ordering. The system $\mathfrak{A}$ satisfies the additional condition of being dense in its ordering. Since these conditions can be formulated in $L^{(\omega)}$, every prime power of $\mathfrak{A}$ is also densely ordered. Let $|I| = \aleph_0$ and $\mathcal{D}$ be a non-principal prime ideal over I. The densely ordered system $\mathfrak{A}^I_{\mathcal{D}}$ has the further (non-elementary) property of being an η_1-set in the sense of Hausdorff [7]. This phenomenon is not an isolated example, for Keisler in [19], [20] has defined a notion of $\aleph_\alpha$-*repleteness* for arbitrary relational systems, which can be viewed as a generalization of the notion of η_α-sets for ordered systems, and he has shown that large numbers of reduced products are $\aleph_\alpha$-replete. Thus Hausdorff's construction and some of the results of Erdös-Gillman-Hendriksen [2] then appear as special cases of a general procedure based on Keisler's method.

Finally, inasmuch as 2.13 gives a characterization of classes in $\mathbf{EC}_\Delta$, we shall demonstrate how prime products may be used to prove that certain classes are not in $\mathbf{EC}_\Delta$. For simplicity we shall restrict attention to subclasses of the class $\mathbf{G}$ of all groups. Let $\mathbf{FN}$ be the class of *finite groups*; $\mathbf{TF}$ the class of *torsion free groups* (i.e., no element of finite order); $\mathbf{TN}$ the class of *torsion groups* (every element of finite order); $\mathbf{SM}$ the class of *simple groups*; and $\mathbf{FR}$ the class of *free groups* with at least two generators. To show that $\mathbf{FN} \notin \mathbf{EC}_\Delta$, let groups $\mathfrak{A}_\nu$, $\nu < \omega$, be chosen so that $|\nu| \leqslant |\mathfrak{A}_i| < \aleph_0$. If $\mathcal{D}$ is a non-principal prime dual ideal over ω, then by 1.28, $\mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_\nu \colon \nu < \omega\rangle \notin \mathbf{FN}$. Hence, apply 2.3. This construction also shows $\mathbf{FN} \notin \mathbf{PC}_\Delta$. The class $\mathbf{TF}$ is actually in $\mathbf{EC}_\Delta$, but we shall show that $\mathbf{TF} \notin \mathbf{EC}$ by the same construction that we use to show that $\mathbf{TN} \notin \mathbf{EC}_\Delta$. Let π_ν be the νth prime number, and for each $\nu < \omega$, this time let $\mathfrak{A}_\nu$ be the cyclic group of order π_ν, a group in $\mathbf{TN}$. Let $\mathcal{D}$ be any non-principal prime dual ideal over ω. We wish to show that $\mathfrak{B} = \mathfrak{P}_{\mathcal{D}}\langle \mathfrak{A}_\nu \colon \nu < \omega\rangle \in \mathbf{TF}$. Let a be an ω-sequence of elements, with $a(\nu)$ chosen from $\mathfrak{A}_\nu$. Suppose in $\mathfrak{B}$ the element $a^* = a/{=_{\mathcal{D}}}$ has order $\mu > 0$. Clearly, $\{\nu < \omega \colon a(\nu)^\mu = 1_\nu\} \in \mathcal{D}$, where 1_ν is the unit element of $\mathfrak{A}_\nu$. All sets in $\mathcal{D}$ are infinite, so let $a(\nu')^\mu = 1_{\nu'}$, where ν' is chosen so that $\pi_{\nu'} > \mu$. It easily follows that $\pi_{\nu'}$ divides μ, which is impossible. Hence, $\mathfrak{B} \in \mathbf{TF}$, and obviously, since $|\mathfrak{B}| = 2^{\aleph_0}$, $\mathfrak{B} \notin \mathbf{TN}$. On the other hand, $|\mathfrak{A}_\nu| > 1$ for each ν, so $\mathfrak{A}_\nu \notin \mathbf{TF}$ for $\nu < \omega$. The two desired conclusions now follow from 2.3 and 2.14. These results have also been derived by Szmielew in [37], Chapter 6, as applications of her decision method for the theory of Abelian groups. We feel that our method is in some ways more direct, but of course it is a completely non-constructive approach, while Mrs. Szmielew's arguments are of a finitary character. The proofs for $\mathbf{FN}$, $\mathbf{TF}$, and $\mathbf{TN}$ can also be obtained in a direct way without prime products as shown by Tarski in [39], pp. 716 f.

Turning now to simple groups, we shall show that $\mathbf{SM}$ is not closed under elementary equivalence. This was first proved by Tarski [39], p. 717, footnote 17, using some rather deep results about groups. Our method of proof is to find a group in $\mathbf{SM}$ not all of whose prime powers are in $\mathbf{SM}$, and then to apply 2.11. The group needed is the *infinite alternating group* $\mathfrak{A}$ consisting of all finite even permutations of the set ω. That $\mathfrak{A}$ is simple is mentioned in van der Waerden [43], pp. 149-150. Let $\mathcal{D}$ be any non-principal prime ideal over ω. We want to show that $\mathfrak{A}^I_{\mathcal{D}} \notin \mathbf{SM}$. For this purpose, let $a_\nu = (0, 1, \ldots, 2\nu+2)$, and let $a^* = \langle a_\nu \colon \nu < \omega\rangle/{=_{\mathcal{D}}}$. The argument is completed by showing that a^* is not in the smallest normal subgroup of $\mathfrak{A}^I_{\mathcal{D}}$ generated by the image of $\mathfrak{A}$ under the canonical embedding of $\mathfrak{A}$ in $\mathfrak{A}^I_{\mathcal{D}}$.

In the case of free groups we shall show that if $\mathfrak{A}$ is the free group on two generators, then some prime power of $\mathfrak{A}$ is not free. Indeed almost no prime powers of $\mathfrak{A}$ are in $\mathbf{FR}$. Let the generators of $\mathfrak{A}$ be a and b. Let $\mathcal{D}$ be any non-principal prime ideal over ω. Let $a^* = \langle a \colon \nu < \omega\rangle/{=_{\mathcal{D}}}$ and $c^* = \langle a^\nu \colon \nu < \omega\rangle/{=_{\mathcal{D}}}$. In the group $\mathfrak{A}^I_{\mathcal{D}}$, the elements a^* and c^* commute but they do not lie in a cyclic subgroup of $\mathfrak{A}^I_{\mathcal{D}}$; hence, this group is not free. This same argument, which was noticed jointly by Keisler and the authors, would work for any index set and any prime ideal that was not countably complete.

Appendix. Boolean spaces. It was first noted by Tarski [39] that the space of elementary types is a topological space and that the compactness theorem for elementary classes corresponds exactly to the topological compactness of this space. The proof of the compactness theorem given in 2.10 by means of prime products also has topological significance, as will now be explained.

By a *Boolean space* we understand a totally disconnected compact Hausdorff space. In a Boolean space the Boolean algebra of closed and open subsets is a base for the topology. Boolean spaces are important because the Stone representation theorem shows that each abstract Boolean algebra is isomorphic to the algebra of closed and open subsets of a unique Boolean space (up to homeomorphism), and that every Boolean space is so obtained. Hence, the study of Boolean spaces is equivalent to the study of Boolean algebras (cf., e.g., Kelley [22], pp. 168 f.).

We wish to show that the study of Boolean spaces is equivalent to the study of spaces with an "ultralimit" operation (or "prime limit" to be consistent with the terminology of this paper.) To be specific, we say that an operation *lim* is an *ultralimit operation* on a set (space) X if for every index set I, every sequence $f \in X^I$, and every prime dual ideal $\mathcal{D}$ over I, $\lim_{\mathcal{D}} f$ is defined and $\lim_{\mathcal{D}} f \in X$. A subset $C \subseteq X$ is called *clopen* if for all sequences and ideals, $\lim_{\mathcal{D}} f \in C$ if and only if $\{i \in I \colon f(i) \in C\} \in \mathcal{D}$. For points $x, y \in X$, define $x \equiv y$ to mean that x and y belong to the same clopen subsets of X. It is easy to verify that the clopen sets form a Boolean algebra of sets; that $\equiv$ is an equivalence relation; and that each clopen set is the union of equivalence classes under $\equiv$. Notice also that if $f, g \in X^I$, and if for all $i \in I$, $f(i) \equiv g(i)$, then $\lim_{\mathcal{D}} f \equiv \lim_{\mathcal{D}} g$, for any prime $\mathcal{D}$ over I. In other words, the space $X/\equiv$ has an induced ultralimit operation, and the clopen subsets of $X/\equiv$ are exactly the sets $C/\equiv$, where C is a clopen subset of X.

THEOREM A. *If X has an ultralimit operation, then $X/\equiv$ with the clopen sets as a base for a topology is a Boolean space whose closed and open sets are exactly the original clopen sets.*

Proof. Without loss of generality we may assume that the equivalence relation $\equiv$ is the identity relation. Hence, if $x, y \in X$ are distinct, then there is a clopen C with $x \in C$ and $y \in X \sim C$. It follows at once that the topological space is Hausdorff and totally disconnected. To prove compactness, we need only consider families $\mathcal{F}$ of basic closed-open sets with the finite intersection property. Take the family $\mathcal{F}$ and extend it to a dual prime ideal $\mathcal{D}$ in the algebra of all subsets of X. Let $I = X$ and let $f \in X^I$ be the identity function. Let $x = \lim_{\mathcal{D}} f$; we wish to show that $x \in \bigcap \mathcal{F}$. Now if $C \in \mathcal{F}$, then $\{i \in I\colon f(i) \in C\} = C \in \mathcal{D}$; hence, $x \in C$, as was to be shown. Finally, if B is a subset of X which is both closed and open in the topology, then it is a union of clopen sets, which reduces to a finite union by compactness. However, a finite union of clopen sets is always clopen, and the proof is complete.

Next we wish to show that every Boolean space can be considered as a space with an ultralimit operation in a unique way. Let X be a Boolean space, and notice that every prime dual ideal in the algebra of all closed and open subsets of X has an intersection consisting of a single point. This remark allows us to define an ultralimit operation on X. For let $f \in X^I$ and $\mathcal{D}$ be prime over I. Let $\mathcal{E}$ be the class of closed and open subsets $C \subseteq X$ such that $\{i \in I\colon f(i) \in C\} \in \mathcal{D}$. Note that $\mathcal{E}$ is a prime dual ideal in the algebra of closed and open sets, and define the limit so that $\lim_{\mathcal{D}} f \in \bigcap \mathcal{E}$.

THEOREM B. *If X is a Boolean space, then there is a unique ultralimit operation over X whose clopen sets are exactly the closed and open subsets of X.*

Proof. The definition of the ultralimit operation on X given above implies at once that every $C \subseteq X$ that is both closed and open is clopen with respect to the limit. Could there be a clopen set that was not closed and open in the topology? No. The topology determined by the clopen sets by Theorem A is a compact Hausdorff topology with possibly more open sets than the original compact Hausdorff topology on X. In view of the maximality of such topologies (cf. Kelley [22], Theorem 5.8, p. 141), it follows that the two topologies are identical. Hence the defined limit

operation satisfies the condition of the theorem. Now suppose that lim′ were another operation satisfying the condition. We would have for $f \in X^I$, and $\mathcal{D}$ prime over I that

$$\{\lim'_{\mathcal{D}} f\} = \bigcap \{C \subseteq X\colon\ C \text{ clopen and } \{i \in I\colon f(i) \in C\} \in \mathcal{D}\}$$

and therefore, $\lim'_{\mathcal{D}} f = \lim_{\mathcal{D}} f$. Thus the two operations are the same.

We may now speak of *the* ultralimit operation on a given Boolean space X. It should be remarked that in the proof of Theorem B we have actually shown that any ultralimit operation on X whose clopen sets include the closed and open sets of the given topology is *the* ultralimit operation on X. Another useful fact is the characterization of the closed subsets of X.

THEOREM C. *If X is a Boolean space, then the closed subsets of X are exactly the sets closed under the ultralimit operation on X.*

Proof. By definition every clopen subset is closed under the ultralimit operation. Every closed set is an intersection of clopen sets; hence, closed sets are closed under the operation. Suppose now that $B \subseteq X$ is closed under the operation. Let x be a point in the topological closure of B, and let $\langle C_i\colon i \in I\rangle$ be a sequence constisting of all the clopen sets containing x. By hypothesis, $B \cap C_i \neq 0$ for $i \in I$. Applying the axiom of choice, let $f(i) \in B \cap C_i$ for each $i \in I$. Corresponding to each $j \in I$, let $J_j = \{i \in I\colon C_i \subseteq C_j\}$. The family of all the C_i is closed under finite intersections, and so it is easy to conclude that the family of all the J_j has the finite intersection property. Let $\mathcal{D}$ be prime over I with $J_j \in \mathcal{D}$ for $j \in I$. Let $y = \lim_{\mathcal{D}} f$, a point in B. If it were the case that $y \neq x$, then there would be a C_j with $x \in C_j$ and $y \notin C_j$. On the other hand $\{i \in I\colon f(i) \in C_j\}$ includes the set J_j and hence is in $\mathcal{D}$. This means that $\lim_{\mathcal{D}} f \in C_j$. The contradiction proves that $x = y$ and $x \in B$. In other words B is closed, as was to be shown.

EXAMPLE 1. Let A be a set and let X be the set of all subsets of A. We may easily define an ultralimit operation on X in a direct way. If $f \in X^I$ and $\mathcal{D}$ is prime over I, let

$$\lim_{\mathcal{D}} f = \{a \in A\colon\ \{i \in I\colon\ a \in f(i)\} \in \mathcal{D}\}.$$

It is a simple exercise to verify that this operation on X yields the topology obtained by identifying X with the space 2^A with the usual product topology.

EXAMPLE 2. For each relational systems, let $\tau(\mathfrak{A})$ denote the elementary type of $\mathfrak{A}$; whence $\tau(\mathfrak{A}) = \tau(\mathfrak{B})$ means that $\mathfrak{A}$ and $\mathfrak{B}$ are elementarily equivalent. Let X be the space of all elementary types. Define the limit operation as follows:

$$\lim_{\mathcal{D}} \langle \tau(\mathfrak{A}_i)\colon\ i \in I\rangle = \tau(\mathfrak{P}_{\mathcal{D}} \langle \mathfrak{A}_i\colon\ i \in I\rangle).$$

Corollary 2.4 shows that this operation is well-defined in X. The import of 2.2 is precisely that every class in **EC** corresponds to a clopen subset of X (the correspondence is obtained by taking the image of a class in **EC** under τ.) Notice that there are enough clopen sets corresponding to classes in **EC** to separate distinct points of X. It follows at once from Theorem A that X itself (without introducing an equivalence relation) is a Boolean space with the topology induced by the limit operation. Further, in any Boolean space, a subalgebra of the algebra of clopen sets which is adequate for separating points must be the whole algebra of clopen sets. This remark proves 2.14. From this point of view, 2.13 is now a corollary of Theorem C, and 2.10 is a corollary of Theorem A.

Example 3. Instead of elementary types, one might consider the space of isomorphism types of relational systems. By 1.12, there is a well-defined ultralimit operation in this space. When we apply Theorem A to this case it is necessary to introduce an equivalence relation to obtain a Boolean space. The result of Keisler [21] mentioned after 2.12 shows that this relation is nothing more than the relation of elementary equivalence, and so the same space is obtained as in Example 2.

References

[1] C. C. Chang and A. Morel, *On closure under direct product*, Journ. Symb. Logic 23 (1958), 149-154.

[2] P. Erdös, L. Gillman, and M. Hendriksen, *An isomorphism theorem for real closed fields*, Annals of Math. 61 (1955), pp. 542-554.

[3] S. Feferman and R. L. Vaught, *The first order properties of products of algebraic systems*, Fund. Math. 47 (1959), pp. 57-103.

[4] T. Frayne, A. C. Morel and D. S. Scott, *Set-theoretical properties of reduced products*, Amer. Math. Soc. Notices 5 (1958), p. 674.

[5] T. Frayne and D. S. Scott, *Model-theoretical properties of reduced products*, Amer. Math. Soc. Notices 5 (1958), p. 675.

[6] —, — and A. Tarski, *Reduced products*, Amer. Math. Soc. Notices 5 (1958), pp. 673-674.

[7] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914.

[8] L. Henkin, *Some interconnections between modern algebra and mathematical logic*, Trans. Amer. Math. Soc. 74 (1953), pp. 410-427.

[9] E. Hewitt, *Rings of real-valued continuous functions*. I, Trans. Amer. Math. Soc. 64 (1948), pp. 45-99.

[10] A. Horn, *On sentences which are true of direct unions of algebras*, Journ. Symb. Logic 16 (1951), pp. 14-21.

[11] H. J. Keisler, *Limit reduced products*, Amer. Math. Soc. Notices 6 (1959), p. 520.

[12] — *A mathematical characterization of elementary equivalence*, Amer. Math. Soc. Notices 6 (1959), p. 520.

[13] — *A characterization of Horn classes*, Amer. Math. Soc. Notices 6 (1959), p. 521.

[14] — *Isomorphism of ultraproducts*, Amer. Math. Soc. Notices 7 (1960), pp. 70-71.

[15] — *Properties preserved under reduced products*, Amer. Math. Soc. Nocites 7 (1960), p. 71.

[16] — *Complete theories of algebraically closed extensions of integral domains*, Amer. Math. Soc. Notices 7 (1960), pp. 267-268.

[17] — *On the class of limit ultrapowers of a relational system*, Amer. Math. Soc. Notices 7 (1960), pp. 878-879.

[18] — *Cardinalities of ultrapowers and a theorem of Rabin*, Amer. Math. Soc. Notices 7 (1960), p. 879.

[19] — *Replete relational systems*, Amer. Math. Soc. Notices 8 (1961), p. 63.

[20] — *The repleteness of ultraproducts*, Amer. Math. Soc. Notices 8 (1961), p. 63.

[21] — *Isomorphism of ultraproducts, II*, Amer. Math. Soc. Notices 8 (1961), pp. 63-64.

[21a] — *Properties preserved under reduced produits. II*, Math. Soc. Notices 8 (1961), p. 64.

[22] J. Kelley, *General topology*, New York 1955.

[23] S. Kochen, *Reduced powers and completeness, I, II*, Amer. Math. Soc. Notices 6 (1959), pp. 141-142.

[24] — *An algebraic characterization of arithmetical equivalence*, Amer. Math oc. Notices 6 (1959), p. 437.

[25] — *An algebraic characterization of arithmetical classes*, Amer. Math. Soc. Notices 6 (1959), p. 523.

[26] — *An algebraic characterization of arithmetical functions*, Amer. Math. Soc. Notices 6 (1959), p. 523.

[27] — *Ultrafiltered products and arithmetical extensions*, Doctoral dissertation, Princeton 1959.

[28] — *Ultraproducts in the theory of models*, Ithaca 1960, (mimeographed).

[29] J. Łoś, *Quelques remarques, théorèmes et problèmes sur les classes définissables d'algèbres*, In *Mathematical interpretations of formal systems*, by Th. Skolem et al., Amsterdam 1955, pp. 98-113.

[30] R. Lyndon, *Direct products of algebras*, Amer. Math. Soc. Notices 5 (1958), p. 848.

[31] A. Morel, D. S. Scott, and A. Tarski, *Reduced products and the compactness theorem*, Amer. Math. Soc. Notices 5 (1958), p. 674.

[32] M. Rabin, *Arithmetical extensions with prescribed cardinality*, Koninkl. Nederlandse Akademie van Wetenschappen. Proc., Ser. A, 21 (1959), pp. 439-446.

[33] D. Scott, *On constructing models of arithmetic*, In *Infinitistic Methods*, Proc. of the Symposium on Foundations of Mathematics, Warszawa, 1961, pp. 235-255.

[34] W. Sierpiński, *Sur la décomposition des ensembles en sousensembles presque disjoint*, Mathematica 14 (1938), pp. 15-17.

[35] Th. Skolem, *Peano's axioms and models of arithmetic*, In *Mathematical interpretations of formal systems*, by Th. Skolem and al., Amsterdam 1955, pp. 1-14.

[36] E. Smith and A. Tarski, *Higher degrees of distributivity and completeness in Boolean algebras*, Trans. Math. Amer. Soc. 84 (1957), pp. 230-257.

[37] W. Szmielew, *Elementary properties of Abelian groups*, Fund. Math. 41 (1955), pp. 203-271.

[38] A. Tarski, *Sur la décomposition des ensembles en sous-ensembles presque disjoints*, Fund. Math. 12 (1928), pp. 188-205.

[39] — *Some notions and methods on the borderline of algebra and metamathematics*, Proc. of the International Congress of Mathematicians, 1950, Providence 1952, pp. 705-720.

[40] — *Contributions to the theory of models. I, II*, Koninkl. Nederlandse Akademie van Wetenschappen. Proc., Ser. A, 16 (1954), pp. 572-588.

[41] — *Contributions to the theory of models. III*, Koninkl. Nederlandse Akademie van Wetenschappen. Proc., Ser. A, 17 (1955), pp. 56-64.

[42] — and R. L. Vaught, *Arithmetical extensions of relational systems*, Compositio Mathematica 13 (1957), pp. 81-102.

[43] B. Van der Waerden, *Modern algebra*, vol. 1, New York 1950.

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On ramification points in the classical sense

by

J. J. Charatonik (Wrocław)

Introduction. I call any point p of an arbitrary point set X a *point of order r in the classical sense*—or here briefly a *point of order r*—if p is a unique common end-point of every two of exactly r simple arcs contained in X. A point of order $r \geqslant 3$ will be called a *ramification point* (1).

Hilton and Wylie (see [1] (2), p. 380) constructed for every mapping $f\colon X \to Y$ a space Y_f called a *mapping cylinder* of f. We may understand Y_f as a cylinder with X as its top and with its base embedded in Y, the generators being segments connecting a point $x \in X$ with its image $f(x) \in Y$.

The first purpose of this paper is to prove that for each continuum Q and for each continuous mapping f of the Cantor set onto Q the mapping cylinder K_1 of f can be realized in the Euclidean space of dimension $2\dim Q+3$ as a continuum which is a union of straight segments, disjoint one from another out of Q and such that Q is the set of all ramification points of the continuum K_1. Namely one can place a cell I^k and a straight line L in the $(k+2)$-dimensional Euclidean space so that this straight line and the k-dimensional hyperplane containing the cell are *skew*, i.e. that there is no hyperplane of dimension $k+1$ which contains both those objects. Then the straight segments joining arbitrary points of the straight line L with arbitrary points of the cell I^k have at most the end-points in common. We then obtain the continuum K_1 by placing the Cantor set C in the straight line as well as the continuum $Q = f(C)$ in the cell I^k, and by joining every point $x \in C$ with its image $y = f(x) \in Q$ by the straight segment.

A further result is a construction of another two continua, namely the continuum K_2 having the same property, but in which the order of each ramification point is $2^{\aleph_0}$, and the continuum K_3, in which the

(1) A methodical investigation of sets of ramification points in the classical sense in the continua, i.e. in compact and connected metric spaces, was initiated by Professor B. Knaster in his Topological Seminar in Wrocław (Institute of Mathematics of the Polish Academy of Sciences). I am indebted to him for the project of this paper and for the idea of the proof of Theorem 1. He suggested also the existence of the singularity realized in the final part of this paper (see the dendroid Λ).

(2) The numbers in brackets denote the references, pp. 251 and 252.