On the recursiveness of sets of presentations of 3-manifold groups

by

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This note shows the impossibility of finding an algorithmic solution to any of the following problems:

$P_{\mathfrak{M}}$: Let $\mathfrak{M}$ denote any nonempty class of conected 3-manifolds (a 3-manifold may be bounded or not, compact or not, orientable or not). To decide: *Whether a finite presentation of a group defines a group isomorphic to the fundamental group of an element of* $\mathfrak{M}$.

To understand more precisely what will be proved, the notation of Rabin [3] will be used. Thus, Q denotes the set of all finite presentations; for $\Pi \in Q$, G_Π denotes the group which Π defines; $\approx$ denotes isomorphism. Since Q can be effectively enumerated, one may speak of recursive subsets of Q; to say that a subset $S \subset Q$ is recursive, means (according to the metamathematical belief called Church's Thesis) exactly that there exists an effective way of determining whether an element of Q belongs to S or not.

If $\mathfrak{M}$ is a class of 3-manifolds, define $S(\mathfrak{M}) = \{\Pi \in Q \mid$ there exists $M \in \mathfrak{M}$ such that $\pi_1(M) \approx G_\Pi\}$.

THEOREM. *If $S(\mathfrak{M})$ is not empty, then $S(\mathfrak{M})$ is not a recursive subset of Q.*

It has been communicated to me that G. Baumslag and R. H. Fox have proved this theorem for various special cases, including the cases $\mathfrak{M}$ = the class of closed 3-manifolds, and $\mathfrak{M}$ = the class of complements of knots in 3-space. Their proof utilizes Rabin's theorem, but requires other constructions than that used here. In more mystical language, one may state one of these cases as follows: *In general, one cannot tell whether a group, given by a presentation, is a knot group or not.*

The proof makes use of Theorem 1.1 of Rabin [3]. The essential step which is necessary before Rabin's Theorem can be applied is to find a finitely presented group A which is not isomorphic to a subgroup of the

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fundamental group of any 3-manifold. The proof of this fact about some group A is perhaps interesting in itself.

LEMMA 1. *If a finitely presented group G is isomorphic to a subgroup of $\pi_1(M)$, where M is a 3-manifold, then G or a subgroup of index 2 in G is isomorphic to a subgroup of $\pi_1(N)$, where N is a closed orientable 3-manifold.*

Proof. Let K be a finite 2-dimensional complex such that $\pi_1(K) \approx G$; let $f\colon K \to M$ be a map inducing an inclusion of G into $\pi_1(M)$:

$$\begin{array}{ccc} \pi_1(K) & \xrightarrow{f_*} & \pi_1(M) \\ \Big|\approx & \nearrow_{\iota} & \\ G & & \end{array}$$

This diagram, where ι is a monomorphism (= homomorphism with trivial kernel), is consistent.

K is compact; hence $f(K)$ is compact. Therefore, there is a compact connected 3-manifold T with boundary, such that $f(K) \subset T \subset M$. The following diagram is consistent:

$$\begin{array}{ccc} & \pi_1(T) & \\ \nearrow & & \searrow \\ \pi_1(K) & \longrightarrow & \pi_1(M) \end{array}$$

Since $\pi_1(K) \to \pi_1(M)$ is a monomorphism, $\pi_1(K) \to \pi_1(T)$ is also a monomorphism.

Let U be the *double* of T: That is, U is obtained from $T \cup (T \times 0)$ by identifying x and $(x, 0)$ for all $x \in \mathrm{Bd}\,T$. U is a closed 3-manifold containing T. There is a retraction $r\colon U \to T$, defined thus: $r(x) = x$, $r(x, 0) = x$ for all $x \in T$. The existence of this retraction shows that the inclusion $T \subset U$ induces a monomorphism $\pi_1(T) \to \pi_1(U)$. Hence $f^*\colon \pi_1(K) \to \pi_1(U)$ is a monomorphism.

If U is orientable, define $N = U$; if U is non-orientable let N be the orientable two-sheeted covering space of U. In the first case, f_* embeds $G \approx \pi_1(K)$ in $\pi_1(N)$; in the second case, f_* embeds G in $\pi_1(U)$ which contains $\pi_1(N)$ as a subgroup of index 2, so that either G or a subgroup of index 2 in G is embedded in $\pi_1(N)$.

I first saw something like Lemma 2, which follows, in an unpublished manuscript of J. Milnor on sums of 3-manifolds.

LEMMA 2. *If a group G can be embedded in $\pi_1(N)$ where N is a closed, orientable 3-manifold, and if G is neither finite, nor infinite cyclic, nor a nontrivial free product, then G can be embedded in $\pi_1(R)$, where R is a closed, orientable, aspherical 2-manifold.*

Proof. Suppose $N = N_1 \# N_2$, where N_1 and N_2 are non simply connected 3-manifolds and $\#$ is the operation "*Summenbildung*" ([2], p. 218). Then $\pi_1(N) \approx \pi_1(N_1) * \pi_1(N_2)$, where $*$ denotes free product. By

Kurosh's Subgroup Theorem ([1], p. 17), G, as a subgroup of this free product, is a free product of a free group and conjugates of subgroups of $\pi_1(N_1)$ and $\pi_1(N_2)$; it has been assumed that G is neither infinite cyclic nor a nontrivial free product. Hence G is conjugate to a subgroup of $\pi_1(N_i)$, for i either 1 or 2; thus G can be embedded in $\pi_1(N_i)$. Repeat this argument with N_i in place of N, and repeat it again, etc. Since the finitely generated group $\pi_1(N)$ cannot be expressed as a free product of more factors than the rank of $\pi_1(N)$ (a consequence of Grushko's Theorem [1], p. 57), one will eventually obtain a manifold $N_{ijk\ldots} = R$, such that G can be embedded in $\pi_1(R)$, and such that it is impossible to have $R = R_1 \# R_2$ where R_1 and R_1 are both non simply connected. Since G is neither finite nor infinite cyclic, it follows that $\pi_1(R)$ is infinite and not infinite cyclic. Whitehead [4] has remarked that when R is a closed orientable manifold, if $\pi_2(R) \neq 0$, then $\pi_1(R)$ is either infinite cyclic or a nontrivial free product; and he has shown that if $\pi_1(R)$ is a nontrivial free product, then $R = R_1 \# R_2$ where neither R_1 nor R_2 is simply connected. This implies, in the case at hand, that $\pi_2(R) = 0$; $\pi_1(R)$ being infinite, it follows that R is aspherical.

By virtue of these lemmas, one can find many groups which cannot be embedded in a 3-manifold group. In particular, let A be a free abelian group of rank 4.

LEMMA 3. *A is not isomorphic to a subgroup of the fundamental group of any 3-manifold.*

Proof. Since every subgroup of index two in A is isomorphic to A, by Lemma 1 if A could be embedded in $\pi_1(M)$, where M is a 3-manifold, then A could be embedded in $\pi_1(N)$, where N is a closed orientable 3-manifold. By Lemma 2, since A is not finite, not infinite cyclic, and not a nontrivial free product, A can be embedded in $\pi_1(R)$, where R is an aspherical 3-manifold. Hence A is isomorphic to $\pi_1(Я)$ where $Я$ is a covering space of R ($Я$ is therefore aspherical). Hence the homology groups of A and $Я$ are isomorphic; however, with coefficients ω, $H_4(A) \approx \omega$, whereas, $Я$ being 3-dimensional, $H_4(Я) = 0$. This contradiction completes the proof of the lemma.

Proof of theorem. Since $\mathfrak{M}$ is not empty, $S(\mathfrak{M})$ is not empty. From the definition of $S(\mathfrak{M})$, it is seen that the property of belonging to $S(\mathfrak{M})$ is what Rabin calls an algebraic property. Finally the group A is not isomorphic to a subgroup of the group defined by any element of $S(\mathfrak{M})$. Rabin's Theorem 1.1 now gives the direct result that $S(\mathfrak{M})$ is not recursive.

References

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