On classes of abelian groups

by

S. Balcerzyk (Toruń)

The present paper is concerned with the study of the structure of classes of abelian groups.

The notion of a class of abelian groups was introduced by J.-P. Serre (see [8]) for a generalization of the Hurewicz theorem on homotopy and homology groups of topological space.

In this paper we give a full description of all classes of abelian groups. Any class of abelian groups $\mathcal{C}$ which is not the class of all abelian groups is determined by:

(1) a cardinal number $\mathfrak{m} > \aleph_0$ or a set Θ of types of rational groups;

(2) a collection $\mathfrak{B}$ of functions defined on the cartesian product of the set of primes by the set of natural numbers, functions taking cardinal numbers as values;

(3) a collection $\mathfrak{D}$ of functions defined on the set of primes and taking cardinal numbers as values.

The number $\mathfrak{m}$ in (1) (if it exists) is characterized by the following property: *any torsion free abelian group A such that $|A| < \mathfrak{m}$ is in $\mathcal{C}$.* If such a number does not exist, the set of types Θ is characterized as the set of types of rational groups which are in $\mathcal{C}$.

The collection $\mathfrak{B}$ of functions $\mathfrak{b}(p,n)$ in (2) is characterized by the following properties: $\mathfrak{b}(p,n) = 0$ *for* $n \geqslant m(p)$ (with $m(p)$ depending on $\mathfrak{b}$) *and the group* $\sum_{p \in P} \sum_{n \in N} Z(p^n)^{\mathfrak{b}(p,n)}$ *is in* $\mathcal{C}$.

The number $\mathfrak{m}$ (or the set Θ) and the collections $\mathfrak{B}, \mathfrak{D}$ satisfy some conditions which follow from the properties of $\mathcal{C}$ as a collection of groups.

§ 1 contains definitions, notation and some information. For the proofs of the statements contained in this section we refer the reader to [2], [4].

In § 2 there are proved three lemmas on extensions of groups. Two of them establish a relation between invariants of a group, its subgroup and its factor group in the case of a primary group of bounded order.

In § 3 we show that any class $\mathcal{C}$ is completely determined by its subclass $\mathcal{T}(\mathcal{C})$ consisting of all torsion groups from $\mathcal{C}$ and by the collection

$\mathcal{F}(\mathcal{C})$ consisting of all torsion free groups from $\mathcal{C}$. Theorem 1 gives a description of all collections of type $\mathcal{F}(\mathcal{C})$ and the necessary and sufficient conditions for collections of abelian groups $\mathcal{F}$ and $\mathcal{T}$ to be represented as $\mathcal{F} = \mathcal{F}(\mathcal{C})$, $\mathcal{T} = \mathcal{T}(\mathcal{C})$ for some class $\mathcal{C}$. Thus, the study of classes of abelian groups is reduced to the case of classes consisting of torsion groups.

The results of § 4 are summarized in Theorem 7: any class $\mathcal{C}$ consisting of torsion groups is fully determined by the collection $\mathfrak{B}(\mathcal{C})$ of all invariants of groups which are in $\mathcal{C}$ and having primary components of bounded order and by the collection $\mathfrak{D}(\mathcal{C})$ of all invariants of divisible groups which are in $\mathcal{C}$. Theorems 4 and 6 contain characterizations of collections of the type $\mathfrak{B}(\mathcal{C})$, $\mathfrak{D}(\mathcal{C})$.

In § 5 we study classes which are weakly complete, i.e., classes which are closed with respect to functors $\otimes$ and Tor. Theorem 10 states that a class $\mathcal{C}$ is weakly complete if and only if the subclass $\mathcal{C}_0$ of $\mathcal{C}$ consiting of all torsion groups in $\mathcal{C}$, p-primary components of which satisfy the descending chain condition is weakly complete. In this section we are reminded also of an example of the class, given in [1], which is not weakly complete. All p-primary components of groups in this class are finite.

In § 6 the preceding results are applied to the description of all complete classes: it is given in terms of an ideal of subsets of the set of primes and a semigroup of functions defined on the set of primes and taking non negative integers as values.

In the last § 7 we prove that a class is perfect if and only if it is weakly complete.

§ 1. Notation, definitions and information. All groups considered in this paper are abelian groups.

(1.1) A non empty collection $\mathcal{C}$ of groups is called a *class* iff [1] the following conditions are satisfied:

(i) *If a group A is isomorphic to some group in $\mathcal{C}$, then A is in $\mathcal{C}$.*

(ii) *If a group A is a subgroup or a factor group of a group in $\mathcal{C}$, then A is in $\mathcal{C}$.*

(iii) *If a group A is an extension of a group in $\mathcal{C}$ by a group in $\mathcal{C}$, then A is in $\mathcal{C}$.*

A class $\mathcal{C}$ is called *weakly complete* iff

(iv) *if groups A and B are in $\mathcal{C}$ then the groups $A \otimes B$ and* $\mathrm{Tor}(A, B)$ *are in $\mathcal{C}$.*

A class $\mathcal{C}$ is called *complete* iff

(v) *if a group A is in $\mathcal{C}$ then the groups $A \otimes B$ and* $\mathrm{Tor}(A, B)$ *are in $\mathcal{C}$ for any arbitrary group B.*

[1] "iff" means "if and only if"

A class is called *strongly complete* iff

(vi) *any direct sum of groups from $\mathcal{C}$ is also in $\mathcal{C}$.*

If a group A operates as a group of automorphisms of a group B, then with the pair (A, B) one can connect groups $H_m(A, B)$, $m = 0, 1, 2, \ldots$, which are called the *homology groups* of the group A with coefficient group B (see [3]). If $B = Z$ (infinite cyclic group) and A operates trivially in B then the group $H_m(A, Z)$ is denoted by $H_m(A)$.

A class $\mathcal{C}$ is called *perfect* iff

(vii) *if a group A is in $\mathcal{C}$ then the groups $H_m(A)$ are in $\mathcal{C}$ for all $m > 0$.*

(1.2) If X is any set, then $|X|$ denotes the *cardinal number of* X. The following notation is used:

N is the set of all positive integers $N = (1, 2, \ldots)$,

P is the set of all primes,

R is the additive group of all rational numbers,

$Z(n)$ is the cyclic group of order n,

Z is the additive group of all integers,

$Z(p^\infty)$ is the generalized cyclic group of Prüfer.

If A is a group and $\mathfrak{m}$ is a cardinal number, then $A^{\mathfrak{m}}$ is the discrete direct sum of $\mathfrak{m}$ copies of a group A.

If A, B are groups, then $A \underset{\approx}{\subseteq} B$ means that the group B contains a subgroup isomorphic to A.

(1.3) If R_0 is a subgroup of R and $R_0 \neq 0$, then with each element $r \in R_0$, $r \neq 0$ we can connect a *characteristic* $\chi = \langle \chi_p \rangle$ (for $p \in P$) with χ_p taking values $0, 1, \ldots, \infty$. χ_p is the l.u.b. of the set of integers $n \geqslant 0$ such that an equation $p^n x = r$ admits a solution in R_0. It is easy to see that if $r' \neq 0$ is another element of R_0 and χ' is its characteristic, then

(i) $\chi'_p = \infty$ iff $\chi_p = \infty$,

(ii) $\chi'_p = \chi_p$ for almost all $p \in P$.

If any two characteristics satisfy the above conditions, then they are said to be *equivalent* and a class of equivalent characteristics is called a *type*. Consequently, with each group $R_0 \subset R$, $R_0 \neq 0$ we can connect the type $\tau(R_0)$, which is the equivalence class of a characteristic of any element $\neq 0$ in R_0.

Groups $R_0, R'_0 \subset R$ are isomorphic iff $\tau(R_0) = \tau(R'_0)$.

In the set of all types we can define a relation $\leqslant$ as follows: if $\chi \in \tau$ and $\chi' \in \tau'$ then $\tau \leqslant \tau'$ iff $\chi_p \leqslant \chi'_p$ for almost all $p \in P$.

The sum $\tau'' = \tau + \tau'$ of types τ and τ' is the type containing the characteristic χ'' determined by the condition $\chi''_p = \chi_p + \chi'_p$ with $\chi \in \tau$ and $\chi' \in \tau'$.

A group G is said to be a *rational* group if it is isomorphic with some subgroup R_0 of the group R, i.e. if $G \underset{\approx}{\subseteq} R$. The type $\tau(G)$ is the type $\tau(R_0)$.

(1.4) Any torsion free group G contains a free (abelian) group of rank $|G|$, or G is isomorphic with a subgroup of a finite direct sum of rational groups.

(1.5) If G is an arbitrary group and n is any integer then we write:

$$G[n] = \{g \in G;\ ng = 0\},$$
$$nG = \{g \in G;\ g = ng' \text{ for some } g' \in G\},$$
$$T(G) = \bigcup_{n=1}^{\infty} G[n],$$
$$G^p = \bigcup_{n=1}^{\infty} G[p^n]\ (^2).$$

If G is a torsion group, i.e. $T(G) = G$, then it may be represented as $G = \sum_{p \in P} G^p$, G^p being a p-primary component of G.

(1.6) A group G is said to be *divisible* iff $nG = G$ for all integers $n \neq 0$. A torsion divisible group G may be represented as

$$G = \sum_{p \in P} Z(p^\infty)^{\mathfrak{b}(p)},$$

$\mathfrak{b}(p)$ being a cardinal number for each $p \in P$. The function $\mathfrak{b}$ defined on P and taking cardinal numbers $\mathfrak{b}(p)$ as values is said to be the *invariant* of the group G.

(1.7) If a torsion group G is a direct sum of cyclic groups

$$G = \sum_{p \in P} \sum_{n \in N} Z(p^n)^{\mathfrak{a}(p,n)}$$

($\mathfrak{a}(p,n)$ being cardinal numbers), then the function $\mathfrak{a}$ defined on the set $P \times N$ and taking cardinal numbers $\mathfrak{a}(p,n)$ as values is called the *invariant* of the group G.

(1.8) If for a p-primary group G^p holds $p^r G^p = 0$, i.e., if G^p is of bounded order, then G^p is isomorphic with a direct sum of cyclic p-primary groups of orders $\leqslant p^r$.

Any p-primary group G^p may be isomorphically embedded in a group $Z(p^\infty)^m$ if $G^p[p] \approx Z(p)^m$.

(1.9) If a p-primary group G^p is a direct sum of cyclic groups with unbounded orders, then there exists a homomorphic mapping of G^p onto $Z(p^\infty)$.

(1.10) A subgroup B of a group G is said to be *pure* in G iff $nB = B \cap nG$ for all n.

For every torsion group G there exists a subgroup B which satisfies the following conditions:

(2) This notation does not cause any confusion with the notation G^m of (1.2).

(i) *B is a pure subgroup of G,*

(ii) *B is a direct sum of cyclic groups,*

(iii) *the factor group G/B is a divisible group.*

Each subgroup B with all the above properties is called a (Kulikov's) *basic subgroup* of G. Any two basic subgroups B, B' of the group G are isomorphic; the groups G/B, G/B' are, in general, not isomorphic.

(1.11) A group G is an *extension* of a group K by a group L iff $K \subset G$ and $G/K \approx L$. By (1.10), it follows that each torsion group is an extension of a direct sum of cyclic groups by a divisible torsion group. Any group G is an extension of torsion group $T(G)$ by torsion free group $G/T(G)$.

(1.12) Any group G is isomorphic with the injective limit of the system consisting of all finitely generated subgroups of G with natural embeddings (as the mappings of the system).

(1.13) If a sequence $0 \to G' \to G \to G'' \to 0$ is exact and H is an arbitrary group, then the sequence

$$0 \to \mathrm{Tor}(G', H) \to \mathrm{Tor}(G, H) \to \mathrm{Tor}(G'', H) \to G' \otimes H \to G \otimes H \to G'' \otimes H \to 0$$

is exact (for the definition of functor $\otimes$ and Tor see [2]). If H is a torsion free group, or the image of G' in G is a pure subgroup of G, then the sequence $0 \to G' \otimes H \to G \otimes H \to G'' \otimes H \to 0$ is exact.

The following relations hold

(1.13.1) $A^p \otimes A^q = 0$, $\mathrm{Tor}(A^p, A^q) = 0$ *if p, q are different primes.*

(1.13.2) $A^p \otimes D = 0$ *if D is a divisible group.*

(1.13.3) $Z(n) \otimes A = A/nA$, $\mathrm{Tor}(Z(n), A) = A[n]$.

(1.13.4) $A \otimes B = B \otimes A$, $\mathrm{Tor}(A, B) = \mathrm{Tor}(B, A)$.

(1.13.5) $Z \otimes A = A$.

(1.13.6) $\left(\sum_\gamma A_\gamma\right) \otimes B = \sum_\gamma A_\gamma \otimes B$, $\mathrm{Tor}\left(\sum_\gamma A_\gamma, B\right) = \sum_\gamma \mathrm{Tor}(A_\gamma, B)$.

(1.13.7) $\mathrm{Tor}(A, B) = \mathrm{Tor}(T(A), T(B))$.

(1.13.8) $\mathrm{Tor}(A, B) = 0$ *if A or B is torsion free group.*

(1.13.9) *If B, B_1 are basic subgroups of G, G_1 respectively, then* $G \otimes G_1 = B \otimes B_1$.

(1.13.10) $\mathrm{Tor}(Z(p^\infty), A) = A^p$.

Proof. By (1.13.7) and (1.13.1) we can restrict ourselves to the case $A = A^p$. Let R_p be the rational group generated by numbers p^{-n}, $n = 1, 2, \ldots$; thus we have an exact sequence $0 \to Z \to R_p \to Z(p^\infty) \to 0$ and by (1.13.8) we get another exact sequence $0 \to \mathrm{Tor}(Z(p^\infty), A^p) \to Z \otimes A^p \to R_p \otimes A^p \to Z(p^\infty) \otimes A^p \to 0$. Since $p^n R_p = R_p$ for all n, we have $R_p \otimes A^p = 0$, and consequently $\mathrm{Tor}(Z(p^\infty), A^p) = Z \otimes A^p = A^p$.

(1.14) Künneth formulae. If G_1, G_2 are any groups, then the homology groups of the direct sum G_1+G_2 are given by the formula

$$H_n(G_1+G_2) = \sum_{i+j=n} H_i(G_1)\otimes H_j(G_2) + \sum_{i+j=n-1} \mathrm{Tor}\big(H_i(G_1),\, H_j(G_2)\big).$$

If a group G operates trivially in a coefficient group A, then

$$H_n(G, A) = H_n(G)\otimes A + \mathrm{Tor}\big(H_{n-1}(G), A\big).$$

(1.15) If a group G is an injective limit $G = \underrightarrow{\mathrm{Lim}}\, G_\gamma$, then the group $H_n(G)$ is an injective limit $H_n(G) = \underrightarrow{\mathrm{Lim}}\, H_n(G_\gamma)$ with respect to homomorphisms $H_n(G_\gamma)\to H_n(G_{\gamma'})$ induced by homomorphisms $G_\gamma \to G_{\gamma'}$.

(1.16) If G is any group, then

(1.16.1) $H_0(G) = Z,\ H_1(G) = G$.

(1.16.2) $H_n(Z) = 0$ for $n > 1$.

(1.16.3) $H_{2n}\big(Z(m)\big) = 0,\ H_{2n-1}\big(Z(m)\big) = Z(m)$ for $n > 0$.

(1.16.4) $H_{2n}\big(Z(p^\infty)\big) = 0,\ H_{2n-1}\big(Z(p^\infty)\big) = Z(p^\infty)$ for $n > 0$.

This last formula follows by (1.15) since the group $Z(p^\infty)$ is an injective limit of groups $Z(p^k)$, $k = 1, 2, \ldots$, with respect to natural embeddings, which induce natural embeddings of groups $H_{2n-1}\big(Z(p^k)\big) = Z(p^k)$.

(1.16.5) $H_n\big(Z(p^k)^{\mathfrak{m}}\big) = Z(p^k)^{\mathfrak{m}}$ if $\mathfrak{m} \geqslant \aleph_0$ and $n > 0$.

(1.16.6) $H_{2n}\big(Z(p^\infty)^{\mathfrak{m}}\big) = 0$, if $n > 0$.
$H_{2n-1}\big(Z(p^\infty)^{\mathfrak{m}}\big) = Z(p^\infty)^{\mathfrak{m}}$ if $\mathfrak{m} \geqslant \aleph_0$.

(1.16.7) If G^p is a finite p-primary group of dimension $r = \dim G^p$, then the groups $H_n(G^p)$, $n > 0$, are finite and p-primary, and their dimensions $s(n, r) = \dim H_n(G^p)$ satisfy relations:

(i) $s(1, r) = r$,

(ii) $s(n, r+1) = s(n, r) + s(n-1, r) + \ldots + s(1, r) + \frac{1}{2}\big(1-(-1)^n\big)$,

(iii) $s(n, r) \leqslant r^n$,

(iv) $s(n, r+1) - s(n, r) \leqslant (r+1)^n - r^n$.

Formulae (i), (ii) follow by the Künneth formulae and (1.16.1); formula (iii) follows by (ii) inductively with respect to r; (iv) follows by (iii).

(1.16.8) If $n > 0$ and r is finite, then $H_n\big(Z(p^\infty)^r\big) = Z(p^\infty)^{t(n,r)}$ and numbers $t(n, r)$ satisfy the conditions:

(i) $t(n, r) = 0$ for even n.

(ii) $t(n, r) = \binom{r + \frac{n-1}{2}}{\frac{n+1}{2}}$ for odd n.

(iii) $t(n, r) \leqslant r^n$.

The proof of these statements is similar to that of (1.16.7).

(1.17) If G is a torsion group $G = \sum_{p\in P} G^p$ then for $n > 0$ the groups $H_n(G)$ are torsion groups and their p-primary components are isomorphic with $H_n(G^p)$: $H_n(G) = \sum_{p\in P} H_n(G^p)$. If a group G^p satisfies $p^r G^p = 0$, then $p^r H_n(G^p) = 0$ for $n > 0$.

This proposition follows by (1.12), (1.14), (1.15) and (1.16).

§ 2. Lemmas on group extensions. In this section we present three lemmas, which will be used in § 4.

Let G be a p-primary group of bounded order, i.e., $p^M G = 0$ for some natural number M. If A is a subgroup of G and $B = G/A$, then $p^M A = p^M B = 0$ and there exist direct decompositions

(2.1) $$G = G_1 + G_2 + \ldots + G_M,$$

(2.2) $$A = A_1 + A_2 + \ldots + A_M,$$

(2.3) $$B = B_1 + B_2 + \ldots + B_M$$

such that the groups G_n, A_n, B_n are direct sums of groups $Z(p^n)$, $n = 1, 2, \ldots, M$.

Lemma 1. *If G is a p-primary group, $p^M G = 0$, A is a subgroup of G, $B = G/A$, the groups G, A, B are decomposed as in* (2.1), (2.2), (2.3) *and*

$$\mathfrak{g}(n) = \dim G_n, \quad \mathfrak{a}(n) = \dim A_n, \quad \mathfrak{b}(n) = \dim B_n,$$

then there exist cardinal numbers $\mathfrak{h}_n(k)$, $n = 1, 2, \ldots, M$, $k = 1, 2, \ldots, n$, such that

(2.4) $$\mathfrak{g}(n) = \sum_{k=0}^{n} \mathfrak{h}_n(k),$$

(2.5) $$\mathfrak{a}(n) = \sum_{k=n}^{M} \mathfrak{h}_k(n),$$

(2.6) $$\mathfrak{b}(n) = \sum_{k=0}^{M-n} \mathfrak{h}_{n+k}(k)$$

for $n = 1, 2, \ldots, M$.

Proof. To prove the lemma it is sufficient to prove the following statement:

(S) *There exist direct decompositions $G = \sum_{\lambda\in\Lambda}\{g_\lambda\}$, $A = \sum_{\lambda\in\Lambda'}\{a_\lambda\}$ and a decomposition of the set of indices $\Lambda = \bigcup_{n=1}^{M}\bigcup_{k=0}^{n}\Lambda_{nk}$ such that*

(S.1) $$\Lambda' = \bigcup_{n=1}^{M}\bigcup_{k=1}^{n}\Lambda_{nk},$$

(S.2) *If* $\lambda \in \Lambda_{nk}$ *then* g_λ *is of order* p^n (*for* $k = 0, 1, \ldots, n$) *and* a_λ *is of order* p^k (*for* $k = 1, 2, \ldots, n$).

(S.3) *If* $\lambda \in \Lambda_{nn}$ *then* $g_\lambda = a_\lambda$ *for* $n = 1, 2, \ldots, M$; *if* $\lambda \in \Lambda_{nk}$ *then* $p^{n-1}g_\lambda = p^{k-1}a_\lambda$ *for* $n = 1, 2, \ldots, M$, $k = 1, 2, \ldots, (n-1)$.

(S.4) *If* $a = \sum_{\lambda \in \Lambda} m_\lambda g_\lambda$ *belongs to the group* A *and*

(i) $m_\lambda = 0$ *for* $\lambda \in \Lambda_{nn}$,

(ii) $0 \leqslant m_\lambda < p^{n-k}$ *for* $\lambda \in \Lambda_{nk}$ $n > k \geqslant 0$

then $m_\lambda = 0$ *for all* $\lambda \in \Lambda$.

(S.5) *Any element* $g \in G$ *may be represented as* $g = a' + \sum_{\lambda \in \Lambda} m_\lambda g_\lambda$ *with* $a' \in A$ *and* m_λ *satisfying* (i), (ii) *above*.

To obtain (2.4)-(2.6) it is sufficient to put $\mathfrak{h}_n(k) = |\Lambda_{nk}|$.

In the group $G[p]$, any subgroup is also a direct summand. Then, for some subgroup $G' \subset G[p]$ we have $G[p] = A[p] + G'$. In the group $A[p]$, we have the following chain of subgroups

$$A[p] \cap p^{M-1}G \subset A[p] \cap p^{M-2}G \subset \ldots \subset A[p] \cap pG \subset A[p];$$

then $A[p]$ may be decomposed

$$A[p] = H_1 + H_2 + \ldots + H_M$$

in such a way that

$$A[p] \cap p^{n-1}G = H_n + H_{n+1} + \ldots + H_M \quad \text{for} \quad n = 1, 2, \ldots, M.$$

In each of the groups H_n we have the chain of subgroups

$$H_n \cap p^{n-1}A \subset H_n \cap p^{n-2}A \subset \ldots \subset H_n \cap pA \subset H_n;$$

then H_n may be decomposed

$$H_n = H_{n1} + H_{n2} + \ldots + H_{nn}$$

in such a way that

$$H_n \cap p^{k-1}A = H_{nk} + H_{n(k+1)} + \ldots + H_{nn} \quad \text{for} \quad n = 1, 2, \ldots, M,$$

$k = 1, 2, \ldots, n$ (the group $H_n \cap p^n A$ is the zero group).

In the group G' we have the chain of subgroups

$$G' \cap p^{M-1}G \subset G' \cap p^{M-2}G \subset \ldots \subset G' \cap pG \subset G';$$

then G' may be decomposed

$$G' = H_{10} + H_{20} + \ldots + H_{M0}$$

in such a way that

$$G' \cap p^{n-1}G = H_{n0} + H_{(n+1)0} + \ldots + H_{M0} \quad \text{for} \quad n = 1, 2, \ldots, M.$$

Since any group H_{nk} is a direct sum of cyclic groups $Z(p)$ and $H_{nk} \subset p^{n-1}G \cap p^{k-1}A$ for $n = 1, 2, \ldots, M$, $k = 1, 2, \ldots, n$ and $H_{n0} \subset p^{n-1}G$, there exist disjoint sets of indices Λ_{nk}, $n = 1, 2, \ldots, M$, $k = 0, 1, \ldots, n$, elements $g_\lambda \in G$ for $\lambda \in \Lambda = \bigcup_{n=1}^{M} \bigcup_{k=0}^{n} \Lambda_{nk}$ and elements $a_\lambda \in A$ for $\lambda \in \Lambda' = \bigcup_{n=1}^{M} \bigcup_{k=1}^{n} \Lambda_{nk}$ such that

(2.7) $p^{k-1}a_\lambda = p^{n-1}g_\lambda$ *for* $\lambda \in \Lambda_{nk}$, $k > 0$, *and* $a_\lambda = g_\lambda$ *if* $n = k$,

(2.8) *elements* $p^{n-1}g_\lambda$ (for $\lambda \in \Lambda_{nk}$) *form a basis of the group* H_{nk}.

It is easy to see that elements g_λ for $\lambda \in \Lambda$ form a basis of the group G. In fact, elements g_λ are linearly independent, and the whole group $G[p]$ is contained in the group $\bar{G} = \sum_{\lambda \in \Lambda} \{g_\lambda\}$. Moreover, any element of $G[p]$ has the same height in G as in $\bar{G}$, whence $\bar{G}$ is a direct summand of G (see [7]), which contains all $G[p]$, consequently $\bar{G} = G$. By similar arguments the elements a_λ for $\lambda \in \Lambda'$ form a basis of the group A. Thus all of the conditions (S.1)-(S.3) are satisfied.

Let us suppose that all assumptions of (S.4) are satisfied. If not all m_λ are 0 then $a \neq 0$ and for some $\lambda_0 \in \Lambda$ the order of a is the same, say p^α, as that of $m_{\lambda_0}g_{\lambda_0}$. Since $p^{\alpha-1}a \in A[p]$, we have $\lambda_0 \in \Lambda'$ and $\lambda_0 \in \Lambda_{nk}$ for some $n > k > 0$. Let $a = \sum_{\lambda \in \Lambda'} r_\lambda a_\lambda$ and $m_{\lambda_0} = p^\beta m'_{\lambda_0}$ with $p \nmid m'_{\lambda_0}$. Obviously $\alpha + \beta = n$ and $p^{\alpha-1}m_{\lambda_0}g_{\lambda_0} = p^{\alpha+\beta-1}m'_{\lambda_0}g_{\lambda_0} = p^{n-1}m'_{\lambda_0}g_{\lambda_0} = p^{k-1}m'_{\lambda_0}a_{\lambda_0}$; this last element, being λ_0-component of $p^{\alpha-1}a$, is equal to $p^{\alpha-1}r_{\lambda_0}a_{\lambda_0}$. Hence $p^{k-1}m'_{\lambda_0} \equiv p^{\alpha-1}r_{\lambda_0} (\operatorname{mod} p^k)$, $\alpha \leqslant n$ and $0 \leqslant p^{n-\alpha}m'_{\lambda_0} < p^{n-k}$. From this last relation it follows that $m'_{\lambda_0} = 0$ or $\alpha - 1 > k$; in the second case, from the above congruence it follows that $p \mid m'_{\lambda_0}$, then also $m'_{\lambda_0} = 0$ and we get a contradiction.

To prove (S.5) let g be any element of G and $g = \sum_{\lambda \in \Lambda} m_\lambda g_\lambda$. For any $\lambda \in \Lambda_{nk}$ an integer m_λ may be uniquely represented as $m_\lambda = p^{n-k}q_\lambda + m'_\lambda$ with $0 \leqslant m'_\lambda < p^{n-k}$. If we write

$$g_{nk} = \sum_{\lambda \in \Lambda_{nk}} p^{n-k}q_\lambda g_\lambda \tag{2.9}$$

and

$$G_0 = \sum_{n=1}^{M} \sum_{\lambda \in \Lambda_{n0}} \{g_\lambda\},$$

then, for the proof of (S.5), it is sufficient to show that $g_{nk} \in A + G_0$ for $n = 1, 2, \ldots, M$, $k = 0, 1, \ldots, n$. We shall prove it inductively with respect to the order of g_{nk}.

If g_{nk} is of order p, then it belongs to $G[p]$, which is the same as $A[p] + G_0[p] \subset A + G_0$.

Let us suppose that any element of the form (2.9) and of an order $< p^\alpha$ belongs to $A + G_0$. Let g_{nk} be of the form (2.9) and of the order p^α.

This implies that $a \leqslant k$ and $p^n | p^{a+n-k} q_\lambda$; consequently $q_\lambda = p^{k-a} q'_\lambda$ for some integer q'_λ. Moreover,

$$p^{a-1}\Big(g_{nk} - \sum_{\lambda\in\Lambda_{nk}} p^{k-a} q'_\lambda a_\lambda\Big) = p^{a-1} \sum_{\lambda\in\Lambda_{nk}} (p^{n-k} q_\lambda g_\lambda - p^{k-a} q'_\lambda a_\lambda)$$

$$= \sum_{\lambda\in\Lambda_{nk}} q'_\lambda (p^{n-1} g_\lambda - p^{k-1} a_\lambda) = 0$$

then, by our assumption, element g_{nk} belongs to $A + G_0$.

LEMMA 2. *If cardinal numbers* $\mathfrak{g}(n)$, $\mathfrak{a}(n)$, $\mathfrak{b}(n)$ *satisfy all conditions of Lemma* 1, *then there exist a p-primary group* G *of bounded order, and its subgroup* A *such that invariants of* G, A *and* $B = G/A$ *are* $\mathfrak{g}$, $\mathfrak{a}$ *and* $\mathfrak{b}$ *respectively.*

Proof. By our assumption there exist a set of indices Λ and its decomposition $\Lambda = \bigcup_{n=1}^{M} \bigcup_{k=0}^{n} \Lambda_{nk}$ such that $|\Lambda_{nk}| = \mathfrak{h}_n(k)$, $n = 1, 2, \ldots, M$, $k = 0, 1, \ldots, n$. We define the group G as

$$G = \sum_{n=1}^{M} \sum_{k=0}^{n} G_{nk}, \quad \text{with} \quad G_{nk} = \sum_{\lambda\in\Lambda_{nk}} Z(p^n),$$

and

$$A = \sum_{n=1}^{M} \sum_{k=1}^{n} A_{nk}, \quad \text{with} \quad A_{nk} = p^{n-k} G_{nk},$$

for $n = 1, 2, \ldots, M$, $k = 1, 2, \ldots, n$. It is easy to see that all our conditions are satisfied.

LEMMA 3. *Let* G *be a torsion group,* $G_0 \subset G$, $G_1 = G/G_0$ *and* G_0 *be a direct sum of infinitely many cyclic groups. If* $|G_1| < |G_0|$ *then there exists a direct decomposition* $G = G' + G''$ *inducing a direct decomposition* $G_0 = G' \cap G_0 + G''$ *and such that* $|G'| \leqslant |G_1|$ *if* $|G_1|$ *is infinite and* G' *is finite if* G_1 *is finite.*

This lemma is an easy consequence of the fact that any infinite subgroup of G_0 is contained in a direct summand of G_0 of the same cardinality, and any finite subgroup of G_0 is contained in a finite direct summand of G_0.

§ 3. Torsion free part of a class. Let $\mathcal{C}$ be an arbitrary class; by $\mathcal{T}(\mathcal{C})$ we shall denote the collection of all torsion groups which are in $\mathcal{C}$ and by $\mathcal{F}(\mathcal{C})$ the collection of all torsion free groups which are in $\mathcal{C}$. It is easy to see that $\mathcal{T}(\mathcal{C})$ is a class. In this paragraph we shall study the structure of $\mathcal{F}(\mathcal{C})$, and then we shall reduce the problem of characterization of all classes $\mathcal{C}$ to the case of $\mathcal{C}$ consisting of torsion groups.

THEOREM 1. *If* $\mathcal{C}$ *is a class containing at least one torsion free group, then* $\mathcal{F}(\mathcal{C})$ *is one of the following collections:*

(3.1) *the collection* $\mathcal{F}_\infty$ *of all torsion free groups,*

(3.2) *the collection* $\mathcal{F}_\mathfrak{m}$ *of all torsion free groups of cardinality* $< \mathfrak{m}$ *with* $\mathfrak{m} > \aleph_0$,

(3.3) *the collection* $\mathcal{F}_\Theta$ *of all groups* G *which may be embedded into a finite direct sum* $\sum_{i=1}^{n} R_i$ *of rational groups* R_i *(n depending on* G*) such that types* $\tau(R_i)$ *belong to the set of types* Θ *which satisfies the following conditions:*

(3.4) *if* $\tau \in \Theta$ *and* $\tau_1 \leqslant \tau$ *then* $\tau_1 \in \Theta$,

(3.5) *if* $\tau_1, \tau_2 \in \Theta$ *then* $\tau_1 + \tau_2 \in \Theta$.

Proof. If in the collection $\mathcal{F}(\mathcal{C})$ one can find groups of cardinality $\mathfrak{n} > \aleph_0$, then by (1.4) one can find in $\mathcal{F}(\mathcal{C})$ also a free group of rank $\mathfrak{n}$, and consequently each torsion free group of cardinality $\leqslant \mathfrak{n}$ is in $\mathcal{F}(\mathcal{C})$ (being isomorphic with a factor group of a free group of rank $\mathfrak{n}$). These arguments lead to collections (3.4) or (3.5) in all cases when $\mathcal{F}(\mathcal{C})$ contains groups of infinite rank. Let us suppose that in $\mathcal{F}(\mathcal{C})$ there are only groups of finite rank and let Θ be the set of all types τ such that a rational group of type τ is in $\mathcal{F}(\mathcal{C})$.

If a group F is in $\mathcal{F}(\mathcal{C})$, then it can be embedded into the group $\sum_{i=1}^{n} R_i$, R_i being rational groups. Without any restriction we can suppose that R_i are homomorphic images of F and consequently are in $\mathcal{F}(\mathcal{C})$, thus $\tau(R_i) \in \Theta$.

On the other hand, if the types $\tau(R_i)$ belong to Θ, then R_i are in $\mathcal{F}(\mathcal{C})$ and consequently $\sum_{i=1}^{n} R_i$ is in $\mathcal{F}(\mathcal{C})$ and each subgroup of $\sum_{i=1}^{n} R_i$ is also in $\mathcal{F}(\mathcal{C})$.

It remains to prove that the set Θ satisfies conditions (3.4), (3.5).

Condition (3.4) is obviously satisfied, because any group of type $\tau_1 \leqslant \tau$ can be isomorphically embedded into the group of type τ, which is in $\mathcal{C}$.

Let τ_1, τ_2 be two types belonging to Θ and let R_1, R_2 be subgroups of the group R in which 1 has characteristics χ', χ'' belonging to τ_1, τ_2 respectively. Let us consider, at first, the case of all χ'_p, χ''_p being finite.

If R_3 is the subgroup of R in which 1 has characteristic $\chi''' = \chi' + \chi''$ then $\tau(R_3) = \tau_1 + \tau_2$, $R_3 \supset R_2$ and $R_1/\{1\} \approx R_3/R_2$ because each of these groups is isomorphic with the direct sum $\sum_{p\in P} Z(p^{\chi'_p})$ and $\chi'_p = \chi'''_p - \chi''_p$. Let φ be an isomorphic mapping φ: $R_1/\{1\} \to R_3/R_2$ and let F be the subgroup of the direct sum $R_1 + R_3$ defined as follows: $\langle r, r' \rangle$ belongs to F

iff $\varphi(r+\{1\}) = r'+R_2$. The projection $\langle r, r'\rangle \to r$ maps F homomorphically onto R_1 and the intersection of its kernel with F is isomorphic with R_2, consequently F as an extension of a group in $\mathcal{C}$ by a group in $\mathcal{C}$ is also in $\mathcal{C}$. On the other hand, the projection $\langle r, r'\rangle \to r'$ maps F onto R_3; thus R_3 is also in $\mathcal{C}$, and consequently $\tau_1+\tau_2 \in \Theta$.

Let us consider the case of quite arbitrary types τ_1, τ_2. Characteristics χ', χ'' can be represented as $\chi' = \overline{\chi}'+\overline{\overline{\chi}}'$, $\chi'' = \overline{\chi}''+\overline{\overline{\chi}}''$, $\overline{\chi}', \overline{\chi}''$ taking finite values only and $\overline{\overline{\chi}}', \overline{\overline{\chi}}''$ taking values 0 and ∞ only. By the preceding arguments the group $\overline{R}_3$ with characteristic of 1 equal to $\overline{\chi}'+\overline{\chi}''$ is in $\mathcal{C}$ and the groups $\overline{\overline{R}}_1, \overline{\overline{R}}_2$ with characteristics of 1 equal to $\overline{\overline{\chi}}'$ and $\overline{\overline{\chi}}''$ respectively, also are in $\mathcal{C}$. It is easy to see that the image of the group $R_1+R_2+R_3$ (being in $\mathcal{C}$) with respect to the mapping $\langle r_1, r_2, r_3\rangle \to r_1+ +r_2+r_3$ is of type $\tau_1+\tau_2$ and thus $\tau_1+\tau_2 \in \Theta$.

The following theorem gives another characteristic property of groups which are in $\mathcal{F}_\Theta$.

THEOREM 2. *If a set of types Θ satisfies conditions* (3.4) *and* (3.5) *then a torsion free group F of finite rank is in $\mathcal{F}_\Theta$ iff for each non-trivial homomorphic mapping φ: $F\to R$ we have $\tau(\varphi(F)) \in \Theta$.*

Proof. If a group F is in $\mathcal{F}_\Theta$ then $F \subsetapprox \sum_{i=1}^{n} R_i$ and $\tau(R_i) \in \Theta$; since R is divisible group, φ can be extended to $\overline{\varphi}$ defined on the group $\sum_{i=1}^{n} R_i$. The image of $\overline{\varphi}$ contains the image of φ and is of the type $\leqslant \tau(R_1)+\ldots+ +\tau(R_n)$; thus $\tau(\varphi(F)) \in \Theta$.

On the other hand, if F is a group of finite rank, then it may be embedded into a group of the form $\sum_{i=1}^{n} R_i$, R_i being rational groups and projections being onto. If F has the property mentioned in the theorem, then $\tau(R_i) \in \Theta$ and F is in $\mathcal{F}_\Theta$.

Theorem 1 presents only necessary conditions for a collection $\mathcal{F}$ of torsion free groups to be of the form $\mathcal{F} = \mathcal{F}(\mathcal{C})$ for some class $\mathcal{C}$. In Theorem 3 we state that the conditions of Theorem 1 are also sufficient for $\mathcal{F}$ to be of the form $\mathcal{F}(\mathcal{C})$, and, moreover, we give necessary and sufficient conditions for a pair $\mathcal{F}, \mathcal{T}$, $\mathcal{T}$ being a class of torsion groups, to be represented as $\mathcal{F} = \mathcal{F}(\mathcal{C})$ and $\mathcal{T} = \mathcal{T}(\mathcal{C})$ with the same class $\mathcal{C}$.

THEOREM 3. *If $\mathcal{F}$ is a non-empty collection of the form* (3.1), (3.2) *or* (3.3) *and $\mathcal{T}$ is a class of torsion groups, then the following conditions are necessary and sufficient for the existence of a class $\mathcal{C}$ such that $\mathcal{F} = \mathcal{F}(\mathcal{C})$ and $\mathcal{T} = \mathcal{T}(\mathcal{C})$:*

(3.1′) *if $\mathcal{F} = \mathcal{F}_\infty$, then $\mathcal{T}$ is the class of all torsion groups,*

(3.2′) *if $\mathcal{F} = \mathcal{F}_\mathfrak{m}$, then $\mathcal{T}$ contains all torsion groups of cardinality $<\mathfrak{m}$,*

(3.3′) *if $\mathcal{F} = \mathcal{F}_\Theta$ and Θ satisfies conditions* (3.4), (3.5), *then the type determined by the characteristic χ belongs to Θ iff the group $\sum_{p\in P} Z(p^{\chi_p})$ is in $\mathcal{T}$.*

If all these conditions are satisfied, then $\mathcal{C}$ is identical with the collection of groups G which are extensions of a group in $\mathcal{T}$ by a group in $\mathcal{F}$.

Proof. At first we prove that if $\mathcal{F} = \mathcal{F}(\mathcal{C})$ and $\mathcal{T} = \mathcal{T}(\mathcal{C})$ for some class $\mathcal{C}$, then all the conditions (3.1′)-(3.3′) are satisfied.

By the same arguments as those used in the proof of Theorem 1, we see that every torsion group (in the case 3.1′)) or every torsion group of cardinality $<\mathfrak{m}$ (in the case (3.2′)) must be in the class $\mathcal{C}$ and consequently in $\mathcal{T} = \mathcal{T}(\mathcal{C})$. In the last case $\mathcal{F} = \mathcal{F}_\Theta$, if a type τ is determined by a characteristic χ and 1 has a characteristic χ in the rational group R_0, then $R_0/\{1\} \approx \sum_{p\in P} Z(p^{\chi_p})$. By this relation it follows that R_0 is in $\mathcal{C}$ iff the group $\sum_{p\in P} Z(p^{\chi_p})$ is in $\mathcal{T}$ and this implies (3.3′).

It is obvious that the collection described in the last part of the theorem is the only one that can satisfy the relations $\mathcal{F} = \mathcal{F}(\mathcal{C})$, $\mathcal{T} = \mathcal{T}(\mathcal{C})$. All that we need to finish the proof is to verify that this collection is the class.

A group G is in $\mathcal{C}$ iff there exist a group T in $\mathcal{T}$ and F in $\mathcal{F}$ such that $T \subset G$ and $G/T = F$.

If $\mathcal{F} = \mathcal{F}_\infty$ then $\mathcal{C}$ consists of all groups and thus is a class.

Let us suppose that $\mathcal{F} = \mathcal{F}_\mathfrak{m}$ or $\mathcal{F} = \mathcal{F}_\Theta$.

If a group G is in $\mathcal{C}$ and G' is a subgroup of G, then the group $G/T(G) = F$ is in $\mathcal{F}$, $T(G)$ is in $\mathcal{T}$ and $T(G') = T(G) \cap G'$ is in $\mathcal{T}$. Moreover, $G'/T(G') = G'/[T(G)\cap G'] \approx [G'+T(G)]/T(G) \subset G/T(G) = F$ and $G'/T(G')$ is in $\mathcal{F}$; since $T(G')$ is in $\mathcal{T}$, G' is in $\mathcal{C}$.

If a group G is in $\mathcal{C}$ and $G' = G/H$ is its factor group, then let us denote by G_0 such a subgroup of G that $T(G) = G_0/H$. Since $G_0 \supset T(G)$, the torsion free group $G'/T(G') = G/H \big/ G_0/H \approx G/G_0$ is isomorphic with the factor group of $F = G/T(G)$ and thus $G'/T(G')$ is also in $\mathcal{F}$ (in the case $\mathcal{F} = \mathcal{F}_\Theta$ we use Theorem 2). Let us write $U = [T(G)+H]/H$; then $U \approx T(G)/[T(G)\cap H]$ and U is in $\mathcal{T}$ because $T(G)$ is in $\mathcal{T}$. On the other hand, $U \subset T(G')$ and

$$T(G')/U \approx G_0/[T(G)+H] \approx G_0/T(G) \big/ [T(G)+H]/T(G);$$

then the torsion group $T(G')/U$ is isomorphic with the factor group of the group $G_0/T(G)$ which is in $\mathcal{F}$. Conditions (3.2′) and (3.3′) imply that any torsion group which is a homomorphic image of a group in $\mathcal{F}$ is in $\mathcal{T}$. Consequently the group $T(G')/U$ is in $\mathcal{T}$ and the group $T(G')$, as an extension of the group U in $\mathcal{T}$ by a group in $\mathcal{T}$, also is in $\mathcal{T}$. Finally, the group G is an extension of the group $T(G')$ in $\mathcal{T}$ by the group G/G_0 in $\mathcal{F}$ and thus is in $\mathcal{C}$.

Let G' be an extension of a group G_1 by a group G_2, both being in $\mathcal{C}$: $G'/G_1 = G_2$. Hence $T(G_1)$, $T(G_2)$ are in $\mathcal{T}$ and $F_1 = G_1/T(G_1)$, $F_2 = G_2/T(G_2)$ are in $\mathcal{F}$. Obviously $T(G') \supset T(G_1)$ and $T(G')/T(G_1) \underset{\approx}{\subseteq} T(G_2)$, thus $T(G')$ is in $\mathcal{T}$. Let F' be the factor group $F' = G'/T(G')$. It is easy to see that the rank of G' is the sum of those of G_1 and G_2 (see [7]). Thus, if $\mathcal{F} = \mathcal{F}_{\mathfrak{m}}$ then $\mathcal{F}'$ is of cardinality $< \mathfrak{m}$ and is in $\mathcal{F}$. If $\mathcal{F} = \mathcal{F}_\Theta$ then F' is of finite rank. To prove that F' is in $\mathcal{F}_\Theta$ we shall study its homomorphisms into the group R. Let φ be a homomorphic mapping $\varphi\colon F' \to R$; φ induces the homomorphism $\varphi'\colon G' \to R$ with $\operatorname{Im}\varphi' = \operatorname{Im}\varphi$. If $G_1 \subset \operatorname{Ker}\varphi'$, then φ' induces homomorphism φ_2: $G'/G_1 = G_2 \to R$ such that $\operatorname{Im}\varphi_2 = \operatorname{Im}\varphi'$ and since F_2 is in F_Θ, $\tau(\operatorname{Im}\varphi) = \tau(\operatorname{Im}\varphi_2) \in \Theta$. If $G_1 \not\subset \operatorname{Ker}\varphi'$, then $\varphi(G_1) \neq 0$ and $\tau(\varphi'(G_1)) \in \Theta$. It is easy to see that the group $\varphi'(G)/\varphi'(G_1) \approx \sum_{p\in P} Z(p^{\chi_p})$ is a homomorphic image of $G_2 = G'/G_1$ and consequently the type τ_2 having a characteristic χ belongs to Θ. Since $\tau(\varphi'(G_1)) \in \Theta$ and $\tau(\varphi'(G')) = \tau(\varphi'(G_1)) + \tau_2$, $\tau(\varphi'(G')) \in \Theta$ and F' is in $\mathcal{F}_\Theta$.

§ 4. Classes consisting of torsion groups. In this paragraph we shall study the structure of classes consisting of torsion groups only. We shall call such a class a *torsion class*. With any torsion class $\mathcal{C}$ we connect two collections of functions.

DEFINITION 1. We shall denote by $\mathfrak{A}(\mathcal{C})$ the collection of all functions $\mathfrak{a}$ defined on the set $P \times N$, taking cardinal numbers as values and such that the group $\sum_{p\in P}\sum_{n\in N} Z(p^n)^{\mathfrak{a}(p,n)}$ is in $\mathcal{C}$.

DEFINITION 2. We shall denote by $\mathfrak{D}(\mathcal{C})$ the collection of all functions $\mathfrak{d}$ defined on the set P, taking cardinal numbers as values and such that the group $\sum_{p\in P} Z(p^\infty)^{\mathfrak{d}(p)}$ is in $\mathcal{C}$.

The collections $\mathfrak{A}(\mathcal{C})$ and $\mathfrak{D}(\mathcal{C})$ are uniquely determined by the class $\mathcal{C}$ and they describe groups of very special type, which are in $\mathcal{C}$. In the present paragraph we shall prove that, in fact, each class is uniquely determined by some collections $\mathfrak{A}$, $\mathfrak{D}$ of types described above, and we shall give necessary and sufficient conditions for such collections to be represented as $\mathfrak{A} = \mathfrak{A}(\mathcal{C})$, $\mathfrak{D} = \mathfrak{D}(\mathcal{C})$ for some class $\mathcal{C}$.

Now we give necessary and sufficient conditions for a collection $\mathfrak{D}$ of functions to be represented as $\mathfrak{D} = \mathfrak{D}(\mathcal{C})$ for some class $\mathcal{C}$.

THEOREM 4. *If $\mathfrak{D}$ is a collection of functions defined on the set P and taking cardinal numbers as values, then $\mathfrak{D}$ may be represented as $\mathfrak{D} = \mathfrak{D}(\mathcal{C})$ for some class $\mathcal{C}$ iff the following conditions are satisfied:*

(4.1) *if $\mathfrak{d}$ is in $\mathfrak{D}$ and $\mathfrak{d}'(p) \leqslant \mathfrak{d}(p)$ for all $p \in P$, then $\mathfrak{d}'$ is in $\mathfrak{D}$,*

(4.2) *if $\mathfrak{d}, \mathfrak{d}'$ are in $\mathfrak{D}$, then $\mathfrak{d} + \mathfrak{d}'$ is in $\mathfrak{D}$.*

If these conditions are satisfied, then a class $\mathcal{C}$ may be taken as the collection of all groups A which are isomorphic with some subgroups of the group $\sum_{p\in P} Z(p^\infty)^{\mathfrak{d}(p)}$ for $\mathfrak{d}$ in $\mathfrak{D}$.

Proof. If $\mathfrak{D} = \mathfrak{D}(\mathcal{C})$, then conditions (4.1), (4.2) are implied by properties (i), (ii) of § 1.

Let us suppose that conditions (4.1), (4.2) are satisfied. Then by (1.8) it follows that the collection $\mathcal{C}$ described in Theorem 4 is identical with the collection of groups $A = \sum_{p\in P} A^p$ such that $\dim A[p] \leqslant \mathfrak{d}(p)$ for all $p \in P$ and for some function $\mathfrak{d}$ in $\mathfrak{D}$. It is obvious that these groups form a class and $\mathfrak{D} = \mathfrak{D}(\mathcal{C})$.

Remark. The class $\mathcal{C}$ described in Theorem 4 is not the only one that satisfies $\mathfrak{D} = \mathfrak{D}(\mathcal{C})$.

We shall consider a subclass $\mathcal{C}_b$ of $\mathcal{C}$ consisting of all such groups $A = \sum_{p\in P} A^p$ that each group A^p is of bounded order, i.e. that there exist numbers $M(p)$ (depending on A) such that $p^{M(p)}A^p = 0$. Of course, $\mathcal{C}_b$ is a class; we shall denote by $\mathfrak{B}(\mathcal{C})$ the collection $\mathfrak{B}(\mathcal{C}) = \mathfrak{A}(\mathcal{C}_b)$. It is obvious that a function $\mathfrak{b}$ of $\mathfrak{A}(\mathcal{C})$ is in $\mathfrak{B}(\mathcal{C})$ iff $\mathfrak{b}(p, n) = 0$ for $n > M(p)$.

We shall study relations between collections $\mathfrak{A}(\mathcal{C})$, $\mathfrak{B}(\mathcal{C})$ and $\mathfrak{D}(\mathcal{C})$.

THEOREM 5. *A function $\mathfrak{a}$ defined on $P \times N$ and taking cardinal numbers as values is in $\mathfrak{A}(\mathcal{C})$ iff there exist functions $\mathfrak{b}$, $\mathfrak{d}$ such that*

(4.3) *$\mathfrak{b}$ is in $\mathfrak{B}(\mathcal{C})$ and $\mathfrak{d}$ is in $\mathfrak{D}(\mathcal{C})$,*

(4.4) *$\mathfrak{d}(p) \geqslant \aleph_0$ or $\mathfrak{d}(p) = 0$ for $p \in P$,*

(4.5) *$\mathfrak{a}(p, n) \leqslant \mathfrak{b}(p, n) + \mathfrak{d}(p)$ for $p \in P$, $n \in N$.*

Proof. If the conditions (4.3)-(4.5) are satisfied, then

$$\sum_{p\in P}\sum_{n\in N} Z(p^n)^{\mathfrak{a}(p,n)} \underset{\approx}{\subseteq} \sum_{p\in P}\sum_{n\in N} [Z(p^n)^{\mathfrak{b}(p,n)} + Z(p^\infty)^{\mathfrak{d}(p)}]$$

and the last group is in $\mathcal{C}$; consequently $\mathfrak{a}$ is in $\mathfrak{A}(\mathcal{C})$.

Let $\mathfrak{a}$ be any function in $\mathfrak{A}(\mathcal{C})$ and let us write $\mathfrak{s}(p, n) = \sum_{k=n}^{\infty} \mathfrak{a}(p, k)$. Since inequalities $\mathfrak{s}(p, 1) \geqslant \mathfrak{s}(p, 2) \geqslant \ldots$ hold, there exist such $M(p)$ that $\mathfrak{s}(p, n)$ are all identical for $n \geqslant M(p)$. We define $\mathfrak{d}(p) = \mathfrak{s}(p, M(p))$ and

$$\mathfrak{b}(p, n) = \begin{cases} \mathfrak{a}(p, n) & \text{for} \quad n < M(p), \\ 0 & \text{for} \quad n \leqslant M(p). \end{cases}$$

Conditions (4.4), (4.5) are obviously satisfied and $\mathfrak{b}$ is in $\mathfrak{B}(\mathcal{C})$; thus we need only to prove that $\mathfrak{d}$ is in $\mathfrak{D}(\mathcal{C})$, or that for each $p \in P$ the group $Z(p^\infty)^{\mathfrak{d}(p)}$ is a homomorphic image of the group $A^p = \sum_{n=1}^{\infty} Z(p^n)^{\mathfrak{a}(p,n)}$.

If $\mathfrak{d}(p) = \aleph_0$ then $\mathfrak{a}(p, n) \neq 0$ for infinitely many n's, and then the group A^p may be homomorphically mapped onto $\sum_{m=1}^{\infty} Z(p^m)$, and by (1.8) also onto $Z(p^\infty)^{\aleph_0} = Z(p^\infty)^{\mathfrak{d}(p)}$.

If $\mathfrak{d}(p) > \aleph_0$ then the set N_0 of indices n such that $\mathfrak{a}(p, n) > \aleph_0$ is infinite and consequently we have a sequence of homomorphic mappings (all being onto)

$$A^p \to \sum_{n \in N_0} Z(p^n)^{\mathfrak{a}(p,n)} \approx \sum_{n \in N_0} Z(p^n)^{n\mathfrak{a}(p,n)} \to \sum_{n \in N_0} [Z(p) + Z(p^2) + \ldots + Z(p^n)]^{\mathfrak{a}(p,n)}$$
$$\approx Z(p)^{\mathfrak{d}(p)} + Z(p^2)^{\mathfrak{d}(p)} + \ldots$$
$$\approx [Z(p) + Z(p^2) + \ldots]^{\mathfrak{d}(p)} \to Z(p^\infty)^{\mathfrak{d}(p)},$$

and the last group is in $\mathcal{C}$; consequently $\mathfrak{d}$ is in $\mathfrak{D}(\mathcal{C})$.

By the above theorem, the collection $\mathfrak{A}(\mathcal{C})$ is completely determined by the collections $\mathfrak{B}(\mathcal{C})$ and $\mathfrak{D}(\mathcal{C})$, and for further study we need to have necessary and sufficient conditions for a collection $\mathfrak{B}$ to be represented as $\mathfrak{B} = \mathfrak{B}(\mathcal{C})$ for some class $\mathcal{C}$. Moreover we must also know the relations between $\mathfrak{B}$ and $\mathfrak{D}$ necessary and sufficient for the common representation $\mathfrak{B} = \mathfrak{B}(\mathcal{C})$, $\mathfrak{D} = \mathfrak{D}(\mathcal{C})$.

To give a statement of Theorem 6 in an abbrieviated form we introduce a relation E which holds between functions $\mathfrak{g}, \mathfrak{a}, \mathfrak{b}$ defined on $P \times N$ and taking cardinal numbers as values.

Definition 3. The relation $E(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ holds iff the following conditions are satisfied.

(4.6) *There exist integers* $M(p)$ *such that* $\mathfrak{g}(p, n) = \mathfrak{a}(p, n) = \mathfrak{b}(p, n) = 0$ *for* $n > M(p)$.

(4.7) *There exist cardinal numbers* $\mathfrak{h}_n(p, k)$ *for* $0 \leqslant k \leqslant n$, $n \in N$, $p \in P$ *such that*

$$\mathfrak{g}(p, n) = \sum_{k=0}^{n} \mathfrak{h}_n(p, k), \quad \mathfrak{a}(p, n) = \sum_{k=n}^{\infty} \mathfrak{h}_k(p, n), \quad \mathfrak{b}(p, n) = \sum_{k=0}^{\infty} \mathfrak{h}_{n+k}(p, n).$$

By conditions (4.6), (4.7) it follows that $\mathfrak{h}_n(p, k) = 0$ for $n \geqslant M(p)$ and then all sums in (4.7) are finite.

The relation E establishes the connection between invariants of a group, its subgroup and its factor group. More precisely:

Lemma 4. *If* G, A, B *are torsion groups having primary components of bounded orders and* $\mathfrak{g}, \mathfrak{a}, \mathfrak{b}$ *are their invariants, then the relation* $E(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ *holds iff there exists a subgroup* $A' \subset G$ *such that* $A \approx A'$ *and* $G/A' \approx B$.

This lemma follows by Lemmas 1 and 2 when applied to primary components of groups G, A, B.

Theorem 6. *If* $\mathfrak{B}$ *is a collection of functions* $\mathfrak{b}$ *defined on the set* $P \times N$ *and taking cardinal numbers as values, then* $\mathfrak{B}$ *may be represented as* $\mathfrak{B} = \mathfrak{B}(\mathcal{C})$ *iff the following conditions are satisfied:*

(4.8) *if* $\mathfrak{b}$ *is in* $\mathfrak{B}$ *then there exist integers* $M(p)$ *such that* $\mathfrak{b}(p, n) = 0$ *for* $n \geqslant M(p)$,

(4.9) *if* $\mathfrak{g}$ *is in* $\mathfrak{B}$ *and* $E(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ *then* $\mathfrak{a}$ *is in* $\mathfrak{B}$,

(4.10) *if* $\mathfrak{g}$ *is in* $\mathfrak{B}$ *and* $E(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ *then* $\mathfrak{b}$ *is in* $\mathfrak{B}$,

(4.11) *if* $\mathfrak{a}, \mathfrak{b}$ *are in* $\mathfrak{B}$ *and* $E(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ *then* $\mathfrak{g}$ *is in* $\mathfrak{B}$.

If conditions (4.8)-(4.11) *are satisfied, then a class* $\mathcal{C}$ *may be taken as the collection of all such groups* A *that*

$$A \approx \sum_{p \in P} \sum_{n \in N} Z(p^n)^{\mathfrak{b}(p,n)} \quad \textit{for some } \mathfrak{b} \textit{ in } \mathfrak{B}.$$

Proof. If a collection $\mathfrak{B}$ is represented as $\mathfrak{B} = \mathfrak{B}(\mathcal{C})$, then (4.8) holds by the definition of $\mathfrak{B}(\mathcal{C})$. Let us suppose that $E(\mathfrak{g}, \mathfrak{a}, \mathfrak{b})$ holds and G, A, B are groups with invariants $\mathfrak{g}, \mathfrak{a}, \mathfrak{b}$ respectively. By the definition of $\mathfrak{B}(\mathcal{C})$ it follows that if $\mathfrak{g}$ is in $\mathfrak{B}$, then G is in $\mathcal{C}$ and by Lemma 4 the group A may be embedded isomorphically into G and B is isomorphic with the factor group of G; consequently A, B are in $\mathcal{C}$ and thus $\mathfrak{a}, \mathfrak{b}$ are in $\mathfrak{B}$. If $\mathfrak{a}, \mathfrak{b}$ are in $\mathfrak{B}$ then A, B are in $\mathcal{C}$ and by Lemma 4 the group G is an extension of a group isomorphic with A by a group isomorphic with B and then G is in $\mathcal{C}$; consequently $\mathfrak{g}$ is in $\mathfrak{B}$.

Let us suppose that all conditions (4.8)-(4.11) are satisfied. It is sufficient to show that the collection $\mathcal{C}$ defined in our theorem is a class. Conditions (i)-(iii) of (1.1) follow by Lemma 4.

Remark. It is easy to prove that conditions (4.9) and (4.10) are equivalent since the relation E is symmetric with respect to variables $\mathfrak{a}, \mathfrak{b}$.

Theorem 7. *If* $\mathfrak{B}$ *is a collection of functions defined on the set* $P \times N$ *and taking cardinal numbers as values and* $\mathfrak{D}$ *is a collection of functions defined on the set* P *and taking cardinal numbers as values, then* $\mathfrak{B}$ *and* $\mathfrak{D}$ *may be represented as* $\mathfrak{B} = \mathfrak{B}(\mathcal{C})$, $\mathfrak{D} = \mathfrak{D}(\mathcal{C})$ *iff* $\mathfrak{B}$ *satisfies the conditions of Theorem* 6, $\mathfrak{D}$ *satisfies the conditions of Theorem* 4 *and, moreover,*

(4.12) *if* $\mathfrak{d}$ *is in* $\mathfrak{D}$ *and* $\mathfrak{b}(p, n) = 0$ *for* $n \geqslant M(p)$ *and* $\sum_{n \in N} \mathfrak{b}(p, n) \leqslant \mathfrak{d}(p)$ *for all* $p \in P$, *then* $\mathfrak{b}$ *is in* $\mathfrak{B}$.

If all conditions mentioned above are satisfied and $\mathfrak{A}$ *is the collection defined by conditions* (4.3)-(4.5) *in Theorem* 5, *then the class* $\mathcal{C}$ *consists of all such torsion groups* G *that invariants of their basic subgroups* A *are in* $\mathfrak{A}$ *and invariants of factor groups* G/A *are in* $\mathfrak{D}$.

The class $\mathcal{C}$ *is uniquely determined by the collections* $\mathfrak{B}$ *and* $\mathfrak{D}$.

Before beginning of the proof we give another characterization of groups which are in the collection $\mathcal{C}$ defined in Theorem 7.

LEMMA 5. *A group* $G = \sum_{p\in P} G^p$ *is in* $\mathcal{C}$ *iff* G *may be represented as* $G = B + \bar{G}$ *in such a way that*

(4.13) B *is a direct sum of cyclic groups and its invariant is in* $\mathfrak{B}$;

(4.14) *there exists a function* $\mathfrak{d}$ *in* $\mathfrak{D}$ *such that*

$$|\bar{G}^p| \leqslant \mathfrak{d}(p) \quad \textit{if} \quad \mathfrak{d}(p) \geqslant \aleph_0,$$

$$\bar{G}^p \approx Z(p^\infty)^{\mathfrak{d}(n)} \quad \textit{if} \quad \mathfrak{d}(p) < \aleph_0.$$

Proof. It is easy to see that if G satisfies all these conditions then it is in the collection $\mathcal{C}$.

If a group G is in $\mathcal{C}$, A is its basic subgroup with invariant $\mathfrak{a}$ and invariant $\mathfrak{d}_1$ of $D = G/A$ is in $\mathfrak{D}$, then by Theorem 5 there exists a function $\mathfrak{b}$ in $\mathfrak{B}$ and a function $\mathfrak{d}_2$ in $\mathfrak{D}$ such that $\mathfrak{a}(p, n) \leqslant \mathfrak{b}(p, n) + \mathfrak{d}_2(p)$ and $\mathfrak{d}_2(p) \geqslant \aleph_0$ or $\mathfrak{d}_2(p) = 0$. By this relation it follows that the group A contains a direct summand B with invariant $\mathfrak{b}$ which is in $\mathfrak{B}$ and invariant $\mathfrak{a}_1$ of factor group A/B (which is a direct sum of cyclic groups) satisfies $\mathfrak{a}_1(p, n) \leqslant \mathfrak{d}_2(p)$ for all $p \in P$, $n \in N$. Since A is pure in G and B is its direct summand with primary components of bounded order, B is a direct summand of G: $G = B + \bar{G}$. Relations (4.14) follow with $\mathfrak{d} = \mathfrak{d}_1 + \mathfrak{d}_2$.

Proof of Theorem 7. By the properties of basic subgroups and Theorem 5 it follows that the collection $\mathcal{C}$ described in our theorem is the only possible one that can satisfy relations $\mathfrak{B} = \mathfrak{B}(\mathcal{C})$, $\mathfrak{D} = \mathfrak{D}(\mathcal{C})$. All that we must prove is that $\mathcal{C}$ is a class.

Let us suppose, that a group G is an extension of a group G_1 by a group G_2, both being in $\mathcal{C}$: $G/G_1 = G_2$. By Lemma 5, the groups G_1, G_2 may be represented as $G_1 = B_1 + \bar{G}_1$, $G_2 = B_2 + \bar{G}_2$ in such a way that invariants of the groups B_1, B_2 are in $\mathfrak{B}$, and for some function $\mathfrak{d}$ in $\mathfrak{D}$ we have

(4.15) $|\bar{G}_1^p| + |\bar{G}_2^p| \leqslant \mathfrak{d}(p)$ if $\mathfrak{d}(p) \geqslant \aleph_0$,

(4.16) $\bar{G}_1^p + \bar{G}_2^p \approx Z(p^\infty)^{\mathfrak{d}(p)}$ if $\mathfrak{d}(p) < \aleph_0$.

We divide the set P into two disjoint subsets P_1, P_2 in such a way that $p \in P_1$ iff $\mathfrak{d}(p) \geqslant \aleph_0$. At first we prove that G is in $\mathcal{C}$ if one of the following conditions holds:

(4.17) $B_1 = 0$,

(4.18) $\bar{G}_1 = 0$.

If (4.17) holds then $G_1 = \bar{G}_1$ and $G/\bar{G}_1 = G_2$. Let us denote by ν the natural homomorphism of G onto G_2 and $G_0 = \nu^{-1}(G_2)$. If $p \in P_1$ then $|G_0^p| \leqslant |G_2^p| \cdot |G_1^p| \leqslant \mathfrak{d}(p)$ and $G^p/G_0^p \approx B_2^p$. Let A^p be a basic subgroup of G^p and $D^p = G^p/A^p$. Since for some positive integers a_p we have $p^{a_p} B_2^p = 0$, $p^{a_p} G^p \subset G_0^p$ and there exists a homomorphism of the group $G^p/p^{a_p} G^p \approx A^p/p^{a_p} A^p$ (see [4]), there exists a homomorphism h_p: $A^p/p^{a_p} A^p \to B_2^p$

onto B_2^p such that $|\mathrm{Ker}\, h_p| \leqslant \mathfrak{d}(p)$. By (4.12) and (4.13) it follows that invariant $\mathfrak{a}'$ of the group $\sum_{p\in P_1} A^p/p^{a_p} A^p$ is in $\mathfrak{B}$. If $\mathfrak{a}$ is the invariant of A then, since $|p^{a_p} A^p| \leqslant |p^{a_p} G^p| \leqslant |G_0^p| \leqslant \mathfrak{d}(p)$, we have $\sum_{n > a_p} \mathfrak{a}(p, n) \leqslant \aleph_0 \cdot \mathfrak{d}(p) = \mathfrak{d}(p)$ and for $n \leqslant a_p$ we have $\mathfrak{a}(p, n) \leqslant \mathfrak{a}'(p, n)$. Thus, by Theorem 5, invariant $\mathfrak{a}$ is in $\mathfrak{A}$. Since the divisible group D^p is a homomorphic image of G^p, also the group $p^{a_p} G^p \subset G_0^p$ may be mapped onto D^p. Consequently $|D^p| \leqslant |G_0^p| \leqslant \mathfrak{d}(p)$ for $p \in P_1$ and finally, by the definition of $\mathcal{C}$, the group $\sum_{p\in P_1} G^p$ is in $\mathcal{C}$.

If $p \in P_2$ then G_1^p, $\bar{G}_2^p$ are divisible and G^p is a direct sum $G^p \approx G_1^p + G_2^p = G_1^p + \bar{G}_2^p + B_2^p$. By Lemma 5, the group $\sum_{p\in P_2} G^p$ is in $\mathcal{C}$. By the additive properties of $\mathfrak{B}$ and $\mathfrak{D}$ it follows that the group G is in $\mathcal{C}$.

If (4.18) holds then $G_1 = B_1$ and $G/B_1 = G_2$. Let us denote by ν, as before, the natural homomorphism of G onto G_2 and $G_* = \nu^{-1}(B_2)$. Consequently we have $G_*/B_1 = B_2$ and by (4.11) the invariant of G_* is in $\mathfrak{B}$; moreover, $G/G_* \approx \bar{G}_2$.

If $p \in P_1$ then $|\bar{G}_2^p| \leqslant \mathfrak{d}(p)$ and, since G_*^p is a direct sum of cyclic groups, by Lemma 3 we can represent the group G^p as $G^p = G_3^p + G_4^p$ in such a way that $|G_3^p| \leqslant \mathfrak{d}(p)$ and $G_4^p \subset G_*^p$. By (4.9) the invariant of the group $\sum_{p\in P_1} G_4^p$ is in $\mathfrak{B}$, then by Lemma 5 the group $\sum_{p\in P_1} G^p$ is in $\mathcal{C}$.

If $p \in P_2$ then $G_2 \approx Z(p^\infty)^{\mathfrak{d}(p)}$; moreover, there exist integers a_p such that $p^{a_p} G_*^p = 0$. These two statements imply that the group $D^p = p^{a_p} G^p \approx \bar{G}_2^p$ is the maximal divisible subgroup of G^p and the factor group G^p/D^p is a homomorphic image of G_*^p. Consequently the group $\sum_{p\in P_2} G^p$ may be represented as

$$\sum_{p\in P_2} G^p \approx \sum_{p\in P_2} \bar{G}_2^p + \sum_{p\in P_2} G_3^p$$

and by (4.10) the invariant of the group $\sum_{p\in P_2} G_3^p$ is in $\mathfrak{B}$; finally, the group $\sum_{p\in P_2} G^p$ is in $\mathcal{C}$. By the additive properties of $\mathfrak{B}$ and $\mathfrak{D}$ it follows that the group G is in $\mathcal{C}$.

Now let G be an extension of an arbitrary group $G_1 = B_1 + \bar{G}_1$ from $\mathcal{C}$ by a group G_2 from $\mathcal{C}$. Since $G/B_1/G_1/B_1 \approx G_2$ and $G_1/B_1 \approx \bar{G}_1$, G/B_1 may be considered as an extension of the group $\bar{G}_1$ by the group G_2. By our previous considerations, the group $G_3 = G/B_1$ is in $\mathcal{C}$. Now we can apply our previous considerations to the group B_1 and G_3; consequently the group G is in $\mathcal{C}$. Thus $\mathcal{C}$ satisfies (iii) of (1.1).

Let us suppose that a group G is in $\mathcal{C}$, G_1 is a factor group $G_1 = G/H$ and ν is the natural homomorphism of G onto G_1. By Lemma 5, the group G

may be represented as $G = B + \bar{G}$, B, $\bar{G}$ satisfying (4.13) and (4.14). We divide the set P into two disjoint subsets P_1, P_2 in such a way that $p \in P_1$ iff $\mathfrak{d}(p) \geqslant \aleph_0$.

If $p \in P_1$ then $|G_1^p/\nu(B^p)| \leqslant |\nu(\bar{G}^p)| \leqslant \mathfrak{d}(p)$ and by (4.10) the group $\sum_{p\in P_1} \nu(B^p)$ is in $\mathcal{C}$; since the group $\sum_{p\in P_1} G_1^p$ is an extension of this last group by the group $\sum_{p\in P_1} G_1^p/\nu(B^p)$, which is in $\mathcal{C}$, then, using our preceding proposition concerning extensions, we see that $\sum_{p\in P_1} G_1^p$ is in $\mathcal{C}$.

If $p \in P_2$ then $\sum_{p\in P_2} \nu(\bar{G}^p)$ is a divisible group and its invariant is in $\mathfrak{D}$. Consequently $G_1^p = B_1^p + \nu(\bar{G}^p)$ and B_1^p is isomorphic with the factor group of B^p. By (4.10) the invariant of the group $\sum_{p\in P_2} B_1^p$ is in $\mathfrak{B}$ and then the group $\sum_{p\in P_2} G_1^p$ is in $\mathcal{C}$. By the additive properties of $\mathfrak{B}$ and $\mathfrak{D}$ it follows that the group G is in $\mathcal{C}$, thus $\mathcal{C}$ satisfies the first part of (ii) of (1.1).

Let us suppose that a group G is in $\mathcal{C}$ and G_1 is a subgroup of G. We shall consider a decomposition $G = B + \bar{G}$ given by Lemma 5 and the same subsets P_1, P_2 as defined above.

If $p \in P_1$ then $G_1^p/[G_1^p \cap B^p] \subset G^p$ and consequently $|G_1^p/G_1^p \cap B^p| \leqslant \mathfrak{d}(p)$. On the other hand, by (4.9) the invariant of the group $\sum_{p\in P_1} G_1^p \cap B^p \subset \sum_{p\in P_1} B^p$ is in $\mathfrak{B}$. By our preceding considerations concerning extensions it follows that $\sum_{p\in P_1} G_1^p$ is in $\mathcal{C}$.

If $p \in P_2$ and $\bar{G}_1^p$ is the maximal divisible subgroup of G_1^p, then $\bar{G}_1^p \subset \bar{G}^p$. Moreover, $G_1^p/\bar{G}_1^p \subset G^p/\bar{G}_1^p \subset [B^p + \bar{G}^p]/\bar{G}_1^p$ and by the maximality of $\bar{G}_1^p$ and (4.5) it follows that the invariant of $\sum_{p\in P_2} G_1^p/\bar{G}_1^p$ is in $\mathfrak{B}$ and then the group $\sum_{p\in P_2} G_1^p$ is in $\mathcal{C}$. By the additive properties of $\mathfrak{B}$ and $\mathfrak{D}$ it follows that the group G_1 is in $\mathcal{C}$, thus $\mathcal{C}$ satisfies the second part of (ii) of (1.1) and the proof of the theorem is finished.

§ 5. Weakly complete classes. One of the most important properties of a class is weak completeness (see (iv) of (1.1)).

Let us remark, at first, that condition (iv) may be restricted to the case $A = B$ as follows by

LEMMA 6. *For any groups A, B we have*

$$A \otimes B \subset (A+B) \otimes (A+B),$$
$$\mathrm{Tor}(A, B) \subset \mathrm{Tor}(A+B, A+B).$$

The lemma follows by the additivity of the functors $\otimes$ and Tor.

LEMMA 7. *A class $\mathcal{C}$ is weakly complete iff its torsion subclass $\mathcal{T}(\mathcal{C})$ is weakly complete.*

Proof. Since the tensor and torsion product of torsion groups is also a torsion group, if $\mathcal{C}$ is weakly complete, then $\mathcal{T}(\mathcal{C})$ is also a class of this type.

Let us suppose that the class $\mathcal{T}(\mathcal{C})$ is weakly complete. Since by (1.13.7) the group $\mathrm{Tor}(A, A)$ is a torsion group and is isomorphic with $\mathrm{Tor}(T(A), T(A))$, we need to consider the functor $\otimes$ only.

Let A be any group in $\mathcal{C}$ and let us write $T = T(A)$, $F = A/T(A)$; thus we have an exact sequence $0 \to T \to A \to F \to 0$. By (1.13) we get exact sequences

(5.1) $\quad 0 \to T \otimes A \to A \otimes A \to F \otimes A \to 0,$

(5.2) $\quad 0 \to T \otimes T \to T \otimes A \to T \otimes F \to 0,$

(5.3) $\quad 0 \to F \otimes T \to F \otimes A \to F \otimes F \to 0.$

By our assumption the group $T \otimes T$ is in $\mathcal{C}$ and it is sufficient to prove that the groups $T \otimes F$ and $F \otimes F$ are in $\mathcal{C}$ for arbitrary T in $\mathcal{T}(\mathcal{C})$ and F in $\mathcal{F}(\mathcal{C})$, because this implies that the middle terms of (5.2) and (5.3) are in $\mathcal{C}$, and by (5.1) also the group $A \otimes A$ is in $\mathcal{C}$.

We shall consider two cases:

(1) Let us suppose $\mathcal{F}(\mathcal{C}) = \mathcal{F}_{\mathfrak{m}}$ with $\mathfrak{m} > \aleph_0$; then $\mathcal{C}$ contains all groups of cardinality $< \mathfrak{m}$. Since $|F| < \mathfrak{m}$, we have $|F \otimes F| < \mathfrak{m}$ and $F \otimes F$ is in $\mathcal{C}$. Moreover, there exist free groups W, W_0 in $\mathcal{C}$ and an exact sequence $0 \to W_0 \to W \to F \to 0$. Since the image of W_0 is pure in W, by (1.13) we get another exact sequence $0 \to T \otimes W_0 \to T \otimes W \to T \otimes F \to 0$, and it is sufficient to prove that $T \otimes W$ is in $\mathcal{C}$, W being a free group of rank $\mathfrak{n} < \mathfrak{m}$. Let B be a basic subgroup of T; then the sequence $0 \to B \to T \to D \to 0$ is exact, D being a divisible group. Consequently, the sequence

(5.4) $\quad 0 \to B \otimes W \to T \otimes W \to D \otimes W \to 0$

is exact. The group B is the direct sum of cyclic groups $B = \sum_{p\in P} \sum_{n\in N} Z(p^n)^{\mathfrak{b}(p,n)}$ and then

$$B \otimes W \approx \sum_{p\in P} \sum_{n\in N} Z(p^n)^{\mathfrak{b}_1(p,n)} \quad \text{for} \quad \mathfrak{b}_1(p,n) = \mathfrak{n}\cdot\mathfrak{b}(p,n).$$

Thus the group $B \otimes W$ may be represented up to an isomorphism as $B + B_1$ with $|B_1| \leqslant \max(\mathfrak{n}, \aleph_0) < \mathfrak{m}$ and finally $B \otimes W$ is in $\mathcal{C}$. Similar arguments applied to the group D lead to the conclusion that $D \otimes W$ is in $\mathcal{C}$. By the exactness of (5.4) it follows that the group $T \otimes W$ is in $\mathcal{C}$. Finally, the group $T \otimes F$ as a homomorphic image of $T \otimes W$ also is in $\mathcal{C}$.

(2) Let us suppose $\mathcal{F}(\mathcal{C}) = \mathcal{F}_{\Theta}$; if W_1 is a free subgroup of F generated by the maximal set of independent elements in F, then we have an exact sequence

(5.5) $\quad 0 \to W_1 \to F \to T_1 \to 0.$

T_1 being a torsion group in $\mathcal{T}(\mathcal{C})$. By (1.13) we get an exact sequence $W_1 \otimes T \to F \otimes T \to T_1 \otimes T \to 0$ and since $W_1 \otimes T$ is isomorphic with the finite direct sum of copies of T, its image in $F \otimes T$ is in $\mathcal{C}$. The group $T_1 \otimes T$ is in $\mathcal{C}$ by our assumption; thus $F \otimes T$, as an extension of a group in $\mathcal{C}$ by a group in $\mathcal{C}$, is also in $\mathcal{C}$.

To finish the proof let us multiply the sequence (5.5) by F. We get an exact sequence $0 \to W_1 \otimes F \to F \otimes F \to T_1 \otimes F \to 0$. The group $T_1 \otimes F$ is in $\mathcal{C}$ (as just proved) and $W_1 \otimes F$ is isomorphic with the finite direct sum of copies of F, thus $F \otimes F$ is in $\mathcal{C}$ and the proof is finished.

Definition 4. If $\mathcal{C}$ is an arbitrary class, then we shall denote by $\mathcal{C}_0$ the subclass of $\mathcal{C}$ consisting of all torsion groups A such that any of its primary components A^p satisfies the descending chain condition for subgroups.

This last condition of the definition may be alternatively expressed as the finiteness of groups $A[p]$ for any $p \in P$, or that any group A^p is a direct sum of a finite group and a finite number of copies of Prüfer's groups.

Theorem 8. *For an arbitrary class $\mathcal{C}$ the following conditions are equivalent:*

(i) *if groups A, B are in $\mathcal{C}$ then $A \otimes B$ is in $\mathcal{C}$,*

(ii) *if a group A is in $\mathcal{C}_0$, then $A \otimes A$ is in $\mathcal{C}_0$.*

Proof. Let us suppose that a class $\mathcal{C}$ satisfies (i) and let A be any group in $\mathcal{C}_0$. By (1.13.1) and (1.13.6) it follows that $A \otimes A = \sum_{p \in P} A^p \otimes A^p$. Since all groups $A^p \otimes A^p$ are finite and $A \otimes A$ is in $\mathcal{C}$, $A \otimes A$ is in $\mathcal{C}_0$ and (i) implies (ii).

Let us suppose that a class $\mathcal{C}$ satisfies condition (ii). By Lemma 6, by the proof of Lemma 7 and by (1.13.9) it follows that we need only to prove that if a torsion group A is a direct sum of cyclic groups and is in $\mathcal{C}$, then $A \otimes A$ is in $\mathcal{C}$. Let $\mathfrak{a}$ be the invariant of the group A. Since $\mathfrak{a}$ is in $\mathfrak{A}(\mathcal{C})$, by Theorem 5 there exist functions $\mathfrak{b}$ in $\mathfrak{B}(\mathcal{C})$, $\mathfrak{d}$ in $\mathfrak{D}(\mathcal{C})$, such that $\mathfrak{a}(p, n) \leqslant \mathfrak{b}(p, n) + \mathfrak{d}(p)$ for all $p \in P$, $n \in N$, $\mathfrak{d}(p) \geqslant \aleph_0$ or $\mathfrak{d}(p) = 0$. Thus the group A may be represented as $A = A_0 + A_1$, A_0 having invariant $\mathfrak{b}$ and invariant $\mathfrak{a}_1$ of A_1 satisfying $\mathfrak{a}_1(p, n) \leqslant \mathfrak{d}(p)$. Since the p-components of $A_1 \otimes A_1$ are $A_1^p \otimes A_1^p$ and $|A_1^p \otimes A_1^p| \leqslant \mathfrak{d}^2(p) = \mathfrak{d}(p)$, the group $A_1 \otimes A_1$ is isomorphic with a subgroup of a group $\sum_{p \in P} Z(p^\infty)^{\mathfrak{d}(p)}$ which is in $\mathcal{C}$; hence $A_1 \otimes A_1$ is in $\mathcal{C}$.

Without any restriction we can assume that $\mathfrak{b}(p, n) > \mathfrak{d}(p)$ or $\mathfrak{b}(p, n) = 0$ because in the opposite case we can adjoin a group $Z(p^n)^{\mathfrak{b}(p,n)}$ to the group A_1. Consequently, the group $Z(p^n)^{\mathfrak{b}(p,n)} \otimes A_1^p$ is isomorphic with a subgroup of $Z(p^n)^{\mathfrak{b}(p,n)}$ because it is of cardinality $\leqslant \mathfrak{b}(p, n)$ and

each element of this group is of order $\leqslant p^n$. Thus $A_0^p \otimes A_1^p \underset{\approx}{\subseteq} A_0^p$ and $A_0 \otimes A_1 \underset{\approx}{\subseteq} A_0$. Consequently the group $A_0 \otimes A_1$ is in $\mathcal{C}$.

The group A_0 may be represented as $A_0 = A_2 + A_3$ in such a way, that all values of the invariant of A_2 are finite, and all values of the invariant of A_3 are infinite or 0. Then by similar arguments as above we can prove that the group $A_2 \otimes A_3 + A_3 \otimes A_2 + A_3 \otimes A_3$ is in $\mathcal{C}$. Since $A_2 \otimes A_2$ is in $\mathcal{C}$ by our assumption, $A_0 \otimes A_0$ and $A \otimes A$ are in $\mathcal{C}$ and the proof is finished.

Since $Z(p^\infty) \otimes Z(p^n) = 0$, then by Theorem 8 it follows that a class $\mathcal{C}$ is closed with respect to functor $\otimes$ iff the subclass of $\mathcal{C}$ consisting of all torsion groups with finite primary components is closed with respect to functor $\otimes$.

Any torsion group with finite primary components is fully characterized by its invariant in $\mathfrak{B}(\mathcal{C}_0)$. Therefore it is useful to have a formula for invariant $\mathfrak{a}_2$ of a group $A \otimes A$ in terms of invariant $\mathfrak{a}$ of a group A. It is easy to see that this formula is

$$\mathfrak{a}_2(p, n) = [\mathfrak{a}(p, n)]^2 + 2 \sum_{m=n+1}^{\infty} \mathfrak{a}(p, n) \cdot \mathfrak{a}(p, m) .$$

The next theorem will concern functor Tor:

Theorem 9. *For an arbitrary class $\mathcal{C}$, the following two conditions are equivalent:*

(iii) *if groups A, B are in $\mathcal{C}$ then* $\operatorname{Tor}(A, B)$ *is in $\mathcal{C}$,*

(iv) *if a group A is in $\mathcal{C}_0$ then $A \otimes A$ is in $\mathcal{C}_0$ and if $\mathfrak{d}$ is in $\mathfrak{D}(\mathcal{C})$ then $\mathfrak{d}^2$ is in $\mathfrak{D}(\mathcal{C})$.*

Proof. Let us suppose that condition (iii) holds and A is any group in $\mathcal{C}_0$. If E is a basic subgroup of A, then $A \otimes A \approx E \otimes E \approx \operatorname{Tor}(E, E)$ and consequently $A \otimes A$ is in $\mathcal{C}$ and then in $\mathcal{C}_0$.

If $\mathfrak{d}$ is any function in $\mathfrak{D}(\mathcal{C})$, then the group $\mathfrak{D} = \sum_{p \in P} Z(p^\infty)^{\mathfrak{d}(p)}$ is in $\mathcal{C}$. Since by (1.13.6) and (1.13.10) the invariant of the group $\operatorname{Tor}(D, D)$ from $\mathcal{C}$ is equal to $\mathfrak{d}^2$, then $\mathfrak{d}^2$ is in $\mathfrak{D}(\mathcal{C})$ and condition (iv) is satisfied.

Let us suppose that condition (iv) holds, let A be any group in $\mathcal{C}$ and E its basic subgroup. By Lemma 6 it is sufficient to prove that $\operatorname{Tor}(A, A)$ is in $\mathcal{C}$. We have an exact sequence $0 \to E \to A \to D \to 0$ (D being divisible) and, moreover,

$$0 \to \operatorname{Tor}(E, A) \to \operatorname{Tor}(A, A) \to \operatorname{Tor}(D, A) \to 0 ,$$

$$0 \to \operatorname{Tor}(E, E) \to \operatorname{Tor}(E, A) \to \operatorname{Tor}(E, D) \to 0 ,$$

$$0 \to \operatorname{Tor}(D, E) \to \operatorname{Tor}(D, A) \to \operatorname{Tor}(D, D) \to 0;$$

all sequences being exact because of the purity of E in A. Let us denote by $\mathfrak{d}$ the invariant of the group D. The invariant of the group $\operatorname{Tor}(D, D)$

is $\mathfrak{d}^2$; thus this last group is in $\mathcal{C}$. Since $\mathrm{Tor}(E, E) \approx E \otimes E$ and E is in $\mathcal{C}$, by Theorem 8 it is sufficient to prove that the group $\mathrm{Tor}(E, D)$ is in $\mathcal{C}$. By (1.13.6) and (1.13.10) it follows that $\mathrm{Tor}(D, E) = \sum_{p \in P} \mathrm{Tor}(D^p, E^p) \approx \sum_{p \in P} (E^p)^{\mathfrak{d}(p)}$. By the Theorem 5, the group E may be represented as $E = E_0 + E_1$ in such a way that the invariant of E_0 is in $\mathfrak{B}(\mathcal{C})$ and invariant $\mathfrak{a}_1$ of E_1 satisfies relation $\mathfrak{a}_1(p, n) \leqslant \mathfrak{d}_1(p)$ for some $\mathfrak{d}_1$ in $\mathfrak{D}(\mathcal{C})$ such that $\mathfrak{d}_1(p) \geqslant \aleph_0$ or $\mathfrak{d}_1(p) = 0$. Since $\mathfrak{a}_1(p, n) \cdot \mathfrak{d}(p) \leqslant \mathfrak{d}_1(p) \cdot \mathfrak{d}(p) \leqslant \mathfrak{d}(p) + \mathfrak{d}_1(p) = \mathfrak{d}_2(p)$, we have $(E_1^p)^{\mathfrak{d}(p)} \subset Z(p^\infty)^{\mathfrak{d}_2(p)}$ and the group $\mathrm{Tor}(D, E_1) \approx \sum_{p \in P} (E_1^p)^{\mathfrak{d}(p)}$ is in $\mathcal{C}$ because $\mathfrak{d}_2$ is in $\mathfrak{D}(\mathcal{C})$.

On the other hand, if $\mathfrak{a}_0$ is the invariant of E_0, then there exist numbers $M(p)$ such that $\mathfrak{a}_0(p, n) = 0$ for $n \geqslant M(p)$. If we write $G = \sum_{p \in P} Z(p^{M(p)})^{\mathfrak{d}(p)}$, then $G \subsetneqq D$ and thus G is in $\mathcal{C}$. It is easy to see that the group $\mathrm{Tor}(D, E_0)$ is isomorphic with $E_0 \otimes G$, which is in $\mathcal{C}$ by our assumption and Theorem 8. Consequently the group $\mathrm{Tor}(D, E)$ is in $\mathcal{C}$.

The last two theorems imply

THEOREM 10. *A class $\mathcal{C}$ is weakly complete iff a class $\mathcal{C}_0$ is weakly complete.*

Proof. In fact, it is sufficient to prove that if $\mathcal{C}_0$ is weakly complete and $\mathfrak{d}$ is in $\mathfrak{D}(\mathcal{C})$ then $\mathfrak{d}^2$ is in $\mathfrak{D}(\mathcal{C})$. The function $\mathfrak{d}$ may be represented as $\mathfrak{d} = \mathfrak{d}_1 + \mathfrak{d}_2$ where $\mathfrak{d}_1$ takes only finite values and $\mathfrak{d}_2(p)$ is $\geqslant \aleph_0$ or equal to 0. Then $\mathfrak{d}^2(p) = \mathfrak{d}_1^2(p) + \mathfrak{d}_2(p)$ for all $p \in P$ and since $\mathfrak{d}_1^2$ is an invariant of a group $\mathrm{Tor}(D, D)$ with D having invariant $\mathfrak{d}_1$, then $\mathfrak{d}_1^2$ is in $\mathfrak{D}(\mathcal{C})$ and, by (4.2), also $\mathfrak{d}^2$ is in $\mathfrak{D}(\mathcal{C})$.

Let us consider more closely a case of a class $\mathcal{C}$ consiting of groups which are p_0-primary for some prime p_0. If $\mathfrak{d}$ is in $\mathfrak{D}(\mathcal{C}_0)$ then $\mathfrak{d}(p) = 0$ for $p \neq p_0$. Consequently, by the additivity of $\mathfrak{D}(\mathcal{C}_0)$, also $\mathfrak{d}^2$ is in $\mathfrak{D}(\mathcal{C}_0)$ since both values $\mathfrak{d}(p_0)$, $\mathfrak{d}^2(p_0)$ are finite and $\mathfrak{d}(p) = 0$ for $p \neq p_0$. If the group $Z(p_0)$ is in $\mathcal{C}_0$ then also each finitely generated p_0-primary group is in $\mathcal{C}_0$ and the preceding theorems imply

THEOREM 11. *If $\mathcal{C}$ is a class and each torsion group in $\mathcal{C}$ possesses only a finite number of non trivial primary components, then $\mathcal{C}$ is weakly complete.*

COROLLARY 1. *If a class $\mathcal{C}$ satisfies the condition*

(v) *if A^p are p-primary groups in $\mathcal{C}_0$, then the group $A = \sum_{p \in P} A^p$ is in $\mathcal{C}$, then $\mathcal{C}$ is weakly complete.*

COROLLARY 2. *If a class $\mathcal{C}$ contains a free group of infinite rank then $\mathcal{C}$ is weakly complete.*

In paper [1] there was given an example of a class which is not weakly complete. It may be described in our present terminology as follows:

Let $P = \{p_1, p_2, \ldots\}$ be any ordering of the set P. The collection $\mathfrak{D}(\mathcal{C})$ consists of the zero function only and a function $\mathfrak{a}$ is in $\mathfrak{A}(\mathcal{C}) = \mathfrak{B}(\mathcal{C})$ iff $\mathfrak{a}(p, n)$ is finite for all $p \in P$, $n \in N$ and

(5.6) *the sequence* $\left\{\frac{1}{k} \sum_{n \in N} \mathfrak{a}(p_k, n)\right\}_{k=1,2,\ldots}$ *is bounded.*

All of the conditions (i)-(iii) of Theorem 7 are satisfied. If the group A has invariant $\mathfrak{a}$ in $\mathfrak{A}(\mathcal{C})$ defined by

$$\mathfrak{a}(p_k, 1) = k, \quad \mathfrak{a}(p_k, n) = 0 \quad \text{for} \quad n \geqslant 2$$

then its tensor product $A \otimes A$ has invariant $\mathfrak{a}_2$ defined by

$$\mathfrak{a}_2(p_k, 1) = k^2, \quad \mathfrak{a}_2(p_k, n) = 0 \quad \text{for} \quad n \geqslant 2,$$

which does not satisfy (5.6), and thus $\mathcal{C}$ is not weakly complete.

§ 6. Complete classes. In this paragraph we give a characterization of all complete classes. It was proved in [8] that a class $\mathcal{C}$ is complete iff it has the following property

(6.1) *if a group A is in $\mathcal{C}$ then a group $A^{\mathfrak{m}}$ is in $\mathcal{C}$ for each cardinal number $\mathfrak{m}$.*

For any function $\mathfrak{d}$ defined on the set P and taking cardinal numbers as values, let us write

$$s(\mathfrak{d}) = \{p \in P;\ \mathfrak{d}(p) \neq 0\}.$$

It is easy to see that the family $\mathcal{I}(\mathcal{C})$ of subsets of P defined by the relation

$$\mathcal{I}(\mathcal{C}) = \{s(\mathfrak{d});\ \mathfrak{d} \text{ is in } \mathfrak{D}(\mathcal{C})\}$$

is an ideal of sets.

LEMMA 8. *If a class $\mathcal{C}$ is complete then a function $\mathfrak{d}$ is in $\mathfrak{D}(\mathcal{C})$ iff $s(\mathfrak{d})$ belongs to $\mathcal{I}(\mathcal{C})$.*

Proof. If $\mathfrak{d}$ is in $\mathfrak{D}(\mathcal{C})$ then $s(\mathfrak{d}) \in \mathcal{I}(\mathcal{I})$ by the definition of $\mathcal{I}(\mathcal{C})$. If $s(\mathfrak{d}) \in \mathcal{I}(\mathcal{C})$ then there exists a function $\mathfrak{d}_1$ in $\mathfrak{D}(\mathcal{C})$ such that $s(\mathfrak{d}) = s(\mathfrak{d}_1)$; hence for some cardinal number $\mathfrak{m}$ we have $\mathfrak{d}(p) \leqslant \mathfrak{m}\mathfrak{d}_1(p)$ for all $p \in P$. Since $\mathcal{C}$ is complete, $\mathfrak{m}\mathfrak{d}_1$ is in $\mathfrak{D}(\mathcal{C})$ and hence $\mathfrak{d}$ is also in $\mathfrak{D}(\mathcal{C})$.

With each function $\mathfrak{b}$ in $\mathfrak{B}(\mathcal{C})$ we connect a function $r_{\mathfrak{b}}$ which maps the set P into the set of all non negative integers:

$$(6.2)\qquad \begin{aligned} r_{\mathfrak{b}}(p) &= 0 \quad \text{if} \quad \mathfrak{b}(p, n) = 0 \text{ for all } n \in N, \\ r_{\mathfrak{b}}(p) &= m \quad \text{if} \quad \mathfrak{b}(p, m) \neq 0 \text{ and } \mathfrak{b}(p, n) = 0 \text{ for all } n > m. \end{aligned}$$

Of course p^m with $m = r_{\mathfrak{b}}(p)$ is the least upper bound of orders of elements in the p-primary component of the group having invariant $\mathfrak{b}$.

It is easy to see that by property (iii) of (1.1) the set

$$\mathfrak{S}(\mathcal{C}) = \{r_{\mathfrak{d}};\ \mathfrak{d} \text{ is in } \mathfrak{B}(\mathcal{C})\}$$

satisfies the following condition:

(6.3) *if* $r', r'' \in \mathfrak{S}(\mathcal{C})$ *then* $r'+r'' \in \mathfrak{S}(\mathcal{C})$.

LEMMA 9. *If a class* $\mathcal{C}$ *is complete then a function* $\mathfrak{b}$ *is in* $\mathfrak{B}(\mathcal{C})$ *iff* $r_{\mathfrak{b}}$ *belongs to* $\mathfrak{S}(\mathcal{C})$.

Proof. If $r_{\mathfrak{b}} \in \mathfrak{S}(\mathcal{C})$ then there exists $\mathfrak{b}_1$ in $\mathfrak{B}(\mathcal{C})$ such that $r_{\mathfrak{b}} = r_{\mathfrak{b}_1}$. By relations (6.2) it follows that there exist non negative integers $M(p)$ and cardinal number $\mathfrak{m}$ such that $\mathfrak{b}(p, n) = 0$ for $n \geqslant M(p)$ and $\mathfrak{m}\mathfrak{b}_1(p, M(p)) \geqslant \sum_{n\in N} \mathfrak{b}(p, n)$ for all $p \in P$.

Since the class $\mathcal{C}$ is complete, $\mathfrak{m}\mathfrak{b}_1$ is in $\mathfrak{B}(\mathcal{C})$ and by property (ii) of (1.1) it follows that $\mathfrak{b}$ is in $\mathfrak{B}(\mathcal{C})$.

THEOREM 12. *If a complete class* $\mathcal{C}$ *contains a torsion free group, then* $\mathcal{C}$ *is the class of all groups.*

If a class $\mathcal{C}$ *consists of torsion groups and is determined (in the sense of Theorem 7) by collections* $\mathfrak{D} = \mathfrak{D}(\mathcal{C})$ *and* $\mathfrak{B} = \mathfrak{B}(\mathcal{C})$, *then* $\mathcal{C}$ *is complete iff the following conditions are satisfied:*

(i) *there exists an ideal* $\mathcal{I}$ *of subsets of the set* P *such that the function* $\mathfrak{d}$ *is in* $\mathfrak{D}$ *iff* $\mathfrak{s}(\mathfrak{d}) \in \mathcal{I}$,

(ii) *there exists a set* $\mathfrak{S}$ *of functions mapping the set* P *into the set of non-negative integers, the set* $\mathfrak{S}$ *being closed with respect to addition and such that a function* $\mathfrak{b}$ *is in* $\mathfrak{B}$ *iff* $r_{\mathfrak{b}} \in \mathfrak{S}$.

Proof. The first part of the theorem is trivial.

If a class $\mathcal{C}$ is complete and consists of torsion groups, then conditions (i), (ii) are implied by Lemmas 8 and 9.

Let us suppose that conditions (i) and (ii) hold. Since by Lemma 5 each group G in $\mathcal{C}$ may be represented as $G = B + \bar{G}$, B having invariant $\mathfrak{b}$ in $\mathfrak{B}(\mathcal{C})$ and $\bar{G}$ being such that for some function $\mathfrak{d}$ in $\mathfrak{D}(\mathcal{C})$, we have

$$|\bar{G}^p| \leqslant \mathfrak{d}(p) \quad \text{if} \quad \mathfrak{d}(p) \geqslant \aleph_0,$$
$$\bar{G}^p \approx Z(p^\infty)^{\mathfrak{d}(p)} \quad \text{if} \quad \mathfrak{d}(p) < \aleph_0.$$

The group $G_1 = G^{\mathfrak{m}}$ admits a decomposition $G_1 \approx B^{\mathfrak{m}} + \bar{G}^{\mathfrak{m}}$. The invariant $\mathfrak{b}_1 = \mathfrak{m}\mathfrak{b}$ of $B^{\mathfrak{m}}$ is in $\mathfrak{B}(\mathcal{C})$ since $r_{\mathfrak{b}_1} = r_{\mathfrak{b}}$. If $\mathfrak{d}(p) \geqslant \aleph_0$ then $|\bar{G}_1^p| \leqslant \mathfrak{m}\mathfrak{d}(p)$ and if $\mathfrak{d}(p) < \aleph_0$ then $(G^p)^{\mathfrak{m}} \approx Z(p^\infty)^{\mathfrak{m}\mathfrak{d}(p)}$. It follows that G_1 is in $\mathcal{C}$ since $\mathfrak{s}(\mathfrak{m}\mathfrak{d}) = \mathfrak{s}(\mathfrak{d})$.

The condition (4.12) can be expressed in terms of the ideal $\mathcal{I}$ and the set $\mathfrak{S}$ as follows:

(iii) *If a set* P_0 *belongs to* $\mathcal{I}$ *then each function* r *such that* $r(p) = 0$ *for* $p \notin P_0$ *belongs to* $\mathfrak{S}$.

For completeness we present here one theorem from [8]:

THEOREM 13. *A class* $\mathcal{C}$ *is strongly complete iff it is one of the following classes:*

1) *the class of all groups,*

2) *a class of all torsion groups* A *such that* $A^p = 0$ *for* p *belonging to some fixed subset* P_1 *of the set* P.

In terms of the preceding theorem, classes in 2) are characterized by the principal ideal of all subsets of the set $P_0 = P \setminus P_1$ and the set $\mathfrak{S}$ consisting of all functions r such that $r(p) = 0$ for $p \notin P_0$.

§ 7. Perfect classes. In this paragraph we prove that a class is perfect iff it is weakly complete (Theorem 14). The most important tool in the proof of Theorem 14 is Lemma 10, which reduces the study of the structure of a group $H_n(G)$ to that of homology groups of groups A and B such that $G/A = B$. This lemma is based on the results of [5] expressed in terms of homology groups.

LEMMA 10. *If* $\mathcal{C}$ *is a class and a group* G *is an extension of a group* A *by a group* B *and the groups* $H_n(B, H_q(A))$ *are in* $\mathcal{C}$ *for* $p, q \geqslant 0, p+q > 0$ *then the groups* $H_n(G)$ *are in* $\mathcal{C}$ *for* $n > 0$.

Proof. By the results of [5] (for homology groups), there exists a spectral sequence $\{E^r\}$ such that $E^2_{p,q} \approx H_p(B, H_q(A))$ and in the group $H_n(G)$ there exists a sequence of subgroups

(7.1) $$H_n(G) = H_{n,0}(G) \supset H_{n-1,1}(G) \supset \ldots \supset H_{-1,n+1}(G) = 0$$

such that

(7.2) $$H_{p,q}(G)/H_{p-1,q+1}(G) \approx E^\infty_{p,q}.$$

Since $E^\infty_{p,q} = E^m_{p,q}$ for $m > \max(p, q+1)$ and the group $E^m_{p,q}$ (for $m > 1$) is a homomorphic image of some subgroup of $E^2_{p,q}$, by our assumption $E^\infty_{p,q}$ are in $\mathcal{C}$ for $p+q > 0$, and consequently the groups $H_n(G)$ are in $\mathcal{C}$ for $n > 0$.

LEMMA 11. *If* B *is a finite group, then* $B \otimes B \approx H_2(B) + H_2(B) + B$.

Proof. We shall proceed inductively with respect to the dimension of the group B.

If the group B is of dimension 1 then $B \approx Z(p^n)$ and $H_2(B) = 0$, $B \otimes B \approx B$ and the lemma holds.

Let us suppose that the lemma holds for all groups of dimension r and let B be any group of dimension $r+1$. Then there exists a direct decomposition $B = B_1 + Z(p^n)$, B_1 being a group of dimension r. Using the Künneth formula we get $H_2(B) \approx H_2(B_1) + Z(p^n) \otimes B_1$ and consequently

$$B\otimes B\approx B_1\otimes B_1+Z(p^n)\otimes B_1+Z(p^n)\otimes B_1+Z(p^n)$$
$$\approx H_2(B_1)+H_2(B_1)+B_1+Z(p^n)\otimes B_1+Z(p^n)\otimes B_1+Z(p^n)$$
$$\approx H_2(B)+H_2(B)+B;$$

thus the proof is finished.

LEMMA 12. *If B is a finite group, then $H_n(B)\underset{\approx}{\subseteq} B\otimes\ldots\otimes B$ (n-fold product) for any $n>0$.*

Proof. As a result of properties (1.17) and (1.13.1) we can restrict our considerations to the case of p-primary groups B.

If the group B is of dimension 1 then $B\approx Z(p^m)$ and formula (1.16.3) implies the lemma.

Let us suppose, that the lemma holds for all groups of dimension r and let B be any p-primary group of dimension $r+1$. Let p^m be the least of all orders of cyclic summands of B; then for some group B_1 of dimension r we have $B=Z(p^m)+B_1$. Since all groups $H_k(B_1)$, $k>0$ are direct sums of cyclic groups of orders $\geqslant p^m$, then using the Künneth formula we get

$$H_n(B)\approx H_m(B_1)+Z(p^m)^s,$$

where by (1.16.7) we have $s=s(n,r+1)-s(n,r)$. On the other hand, since the group $Z(p^m)\otimes B$ is a direct sum of groups $Z(p^m)$, the group $B\otimes\ldots\otimes B$ (n-fold product) is isomorphic with $B_1\otimes\ldots\otimes B_1+Z(p^m)^u$, where $u=(r+1)^n-r^n$. By inequality (1.16.7) (iv) we have $s\leqslant u$; thus by the induction hypothesis follows $H_n(B)\subset B\otimes\ldots\otimes B$.

THEOREM 14. *A class $\mathcal{C}$ is a perfect one iff it is weakly complete.*

Proof. Let $\mathcal{C}$ be any perfect class. Then by Lemma 11, by (1.17) and by Theorem 8 it follows that $\mathcal{C}$ is closed with respect to the tensor product.

If $\mathfrak{d}$ is any function in $\mathfrak{D}(\mathcal{C}_0)$ and D is a torsion divisible group having invariant $\mathfrak{d}$, then by (1.17) and (1.16.8) it follows that $H_3(D)$ is the torsion divisible group with invariant $\mathfrak{d}'$ defined by formula

$$\mathfrak{d}'(p)=\binom{\mathfrak{d}(p)+1}{2}.$$

Since $\mathcal{C}$ is a perfect class, $\mathfrak{d}'$ is in $\mathfrak{D}(\mathcal{C}_0)$, and by relation $2\mathfrak{d}'(p)\geqslant\mathfrak{d}(p)$ it follows that $\mathfrak{d}^2$ is in $\mathfrak{D}(\mathcal{C}_0)$. By Theorem 9, $\mathcal{C}$ is closed with respect to functor Tor; thus $\mathcal{C}$ is weakly complete.

Let $\mathcal{C}$ be any weakly complete class.

Let a group G be an extension of a group A by a group B; since all the groups considered are abelian ones, then by the description of operators in $H_q(A)$ given in [5] (when applied to homology groups) it follows that the group B operates trivially in $H_q(A)$. Hence we can apply the formula of (1.14) and we get

$$H_p(B,H_q(A))\approx H_p(B)\otimes H_q(A)+\mathrm{Tor}(H_{p-1}(B),H_q(A)).$$

By this last formula and Lemma 10 it follows, that if the groups $H_q(A)$, $H_p(B)$ are in $\mathcal{C}$ for $p,q>0$, then the groups $H_n(G)$ are in $\mathcal{C}$ for $n>0$. Consequently, to prove that $\mathcal{C}$ is a perfect class it is sufficient to prove that all groups $H(G)$, $n>0$, are in $\mathcal{C}$ if G is a group from $\mathcal{C}$ and if it is of one of the following types:

(1) G is a torsion free group,

(2) G is a direct sum of finite cyclic groups,

(3) G is a torsion divisible group.

Type (1). If $\mathcal{F}(\mathcal{C})=\mathcal{F}_{\mathfrak{m}}$ for some cardinal number $\mathfrak{m}>\aleph_0$, then the groups $H_n(G)$ are of cardinality $<\mathfrak{m}$ if G is in $\mathcal{F}_{\mathfrak{m}}$; consequently all groups $H_n(G)$ are in $\mathcal{C}$.

If $\mathcal{F}(\mathcal{C})=\mathcal{F}_\Theta$ then any group G which is in $\mathcal{F}_\Theta$ contains such an r-dimensional free subgroup G_0 that the sequence $0\to G_0\to G\to T\to 0$ is exact, T is a torsion group which can be represented as $T=\sum_{i=1}^{r}T_i$, and primary components of each group T_i are cyclic groups or $Z(p^\infty)$. By formulae (1.16) and (1.17) it follows that $H_n(T_i)\approx T_i$ or 0 for $n>0$; then $H_n(T_i)$ are in $\mathcal{C}$ and by the preceding remarks also $H_n(T)$ are in $\mathcal{C}$ for $n>0$. Since $H_n(G_0)$ are free groups of finite rank, $H_n(G_0)$ are in $\mathcal{C}$ and consequently $H_n(G)$ are in $\mathcal{C}$ for $n>0$.

Type (2). If G is a direct sum of cyclic groups, then it may be represented as $G=G_0+G_1+G_2$ where G_0^p are finite for all p, the invariant of G_1 is in $\mathfrak{B}(\mathcal{C})$ and takes only values $\geqslant\aleph_0$ or 0 and the invariant of G_2 is bounded by a function $\mathfrak{d}$ in $\mathfrak{D}(\mathcal{C})$ which takes values $\geqslant\aleph_0$ or 0.

Since each p-component G_2^p of G_2 is of cardinality $\leqslant\mathfrak{d}(p)$, we have

$$H_n(G_2)\approx\sum_{p\in P}H_n(G_2^p)\subset\sum_{p\in P}Z(p^\infty)^{\mathfrak{d}(p)}\quad\text{for}\quad n>0.$$

This last group is in $\mathcal{C}$; thus $H_n(G_2)$ are in $\mathcal{C}$ for $n>0$.

Since we know that $H_n(Z(p^m)^{\mathfrak{a}})\approx Z(p^m)^{\mathfrak{a}}$ for infinite $\mathfrak{a}$ and $n>0$ (see (1.16.5)), using the Künneth formula we can easily prove that $H_n(G_1)\underset{\approx}{\subseteq} G_1$, and thus $H_n(G_1)$ are in $\mathcal{C}$ for $n>0$.

Since, by Lemma 12, $H_n(G_0)\underset{\approx}{\subseteq} G_0\otimes\ldots\otimes G_0$ (n-fold product) and the class $\mathcal{C}$ is weakly complete, $H_n(G_0)$ are in $\mathcal{C}$ for $n>0$.

By the Künneth formula it follows that all groups $H_n(G)$, $n>0$, are in $\mathcal{C}$.

Type (3). If G is a divisible torsion group and $\mathfrak{d}$ is its invariant, then $H_n(G)=0$ for even positive n. Let n be an odd positive integer and $\mathfrak{d}(p)\geqslant\aleph_0$; then $H_n(G^p)\approx G^p$ by (1.16.6); if $\mathfrak{d}(p)<\aleph_0$ then by (1.16.8) $H_n(G^p)\approx Z(p^\infty)^{t(p)}$, where, by formula (iii) of (1.16.8), $t(p)\leqslant[\mathfrak{d}(p)]^n$. Since $\mathfrak{d}^n$ is in $\mathfrak{D}(\mathcal{C})$, the groups $H_n(G)$ are in $\mathcal{C}$ for all $n>0$ and the proof is finished.

Remark. Using the same method as in the proof of Lemma 11 we can prove that for each finite group B we have $H_3(B)+H_3(B)\supset B\otimes B$. By this relation and the first part of the proof of Theorem 14 it follows that a class is weakly complete even in the case when the group $H_3(A)$ is in $\mathcal{C}$ for any group A from $\mathcal{C}$. This last property is then equivalent to the perfectness of $\mathcal{C}$.

For any integer $n\geqslant 1$ and any (abelian if $n>1$) group A the groups $H_m(A,n)$ are defined as homology groups of the Eilenberg-MacLane complex $K(A,n)$. If $n=1$ then $H_m(A,1)=H_m(A)$. Theorem 14 and Proposition 6.11 of [6] (p. 304) imply

THEOREM 15. *If $\mathcal{C}$ is a weakly complete class and a group A is in $\mathcal{C}$, then all the groups $H_m(A,n)$, $m>0$, are in $\mathcal{C}$.*

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INSTYTUT MATEMATYCZNY POLSKIEJ AKADEMII NAUK
INSTITUTE OF MATHEMATICS OF THE POLISH ACADEMY OF SCIENCES

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A functional conception of snake-like continua

by

J. Mioduszewski (Wrocław)

It is known [5] that snake-like continua (in short SC) in the sense of Bing [2] may be regarded as inverse limit spaces of arcs (closed intervals) with projections which are continuous mappings onto.

This method of construction will be applied here to an important class of SC, viz. to the hereditarily indecomposable SC. The existence of hereditarily indecomposable SC was shown by Knaster [6]. Bing called them *pseudoarcs* and proved their homeomorphism to one another [3]. I prove that every SC is a continuous image of the pseudoarc; therefore, the pseudoarc will be called here the ***universal snake-like continuum*** (in short USC). This result seems to be a consequence of a certain theorem of Bing (see [2], Theorem 5 and Lehner [7], Theorem 1), but I intend to use this opportunity to exemplify how the method of inverse limits can be applied to this kind of problems. Therefore, my construction does not resort to Bing's geometrical method using crookedness. I use particularly the uniformization theorem of Sikorski and Zarankiewicz (see [9] and [11]) concerning continuous mappings of the closed interval onto itself.

Waraszkiewicz [12] showed that there exists no continuum of which an arbitrary continuum would be a continuous image, i.e. would be universal for the class of all continua. Henceforth, the following question seems to be interesting: how large is the class of continua for which USC is still universal?

§ 1. Preliminaries. We consider SC as inverse limit spaces $X=\varprojlim\{X_n,\pi_n^m\}$ of arcs X_n with projections $\pi_n^m\colon X_m\to X_n$, $m\geqslant n$, $m,n=1,2,\ldots$, which are continuous and onto (π_n^n are assumed to be identities). We assume, for convenience, that X_n are *closed unit intervals*, i.e. $X_n=\{x_n\colon 0\leqslant x_n\leqslant 1\}$. Consequently, SC are 1-dimensional metric continua (see theorems on inverse limits in [4]). It is also known that SC are imbeddable into the plane (see [2] and [5]) (¹).

(¹) A quite elementary proof of the last proposition is as follows.

The inverse limit does not change if we subtitute (even in infinitely many places) $\pi_n^{n+1}h_n$ for π_n^{n+1}, where h_n is a homeomorphism of X_{n+1} onto itself. Furthermore, every