to (32), implies the equality $\nu = 0$. Further, from the linear independence of $e_1$ and $v$ the inequality $\mu \neq 0$ follows. Thus $(e_1-e_2)^2 \in A(v)$ and $e_1, (e_1-e_2)^2$ are linearly independent. Hence and from the isomorphism between $A(v)$ and the complex field it follows that the subalgebra $A\big((e_1-e_2)^2\big)$ is of dimension two. Thus $A\big((e_1-e_2)^2\big) = A(v)$ and, consequently, $e_1 \in A\big((e_1-e_2)^2\big)$. By symmetry, we also have the relation $e_2 \in A\big((e_1-e_2)^2\big)$, which shows that the subalgebra $A\big((e_1-e_2)^2\big)$ contains two non-trivial idempotents. But this contradicts the isomorphism between $A\big((e_1-e_2)^2\big)$ and the complex field. The Theorem is thus proved.

## References


[1] A. A. Albert, *Absolute-valued algebraic algebras*, Bull. Amer. Math. Soc. 55 (1949), pp. 763-768.

[2] K. Urbanik, *Absolute-valued algebras with an involution*, Fund. Math. 49 (1961), pp. 247-258.

[3] — and F. B. Wright, *Absolute-valued algebras*, Proc. Amer. Math. Soc. 11 (1960), pp. 861-866.

[4] F. B. Wright, *Absolute-valued algebras*, Proc. Nat. Acad. Sci. 39 (1953), pp. 330-332.


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*Reçu par la Rédaction le 7. 9. 1961*

# A characterization of abelian groups of automorphisms of a simply ordering relation *

by

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A *binary relation* $S$ is a set of ordered pairs $\langle x, y\rangle$ of elements $x$ and $y$; the *field* of $S$, denoted by $F(S)$, is the set of all elements $x$ for which there exists an element $y$ such that either $\langle x, y\rangle \in S$ or $\langle y, x\rangle \in S$. A binary relation $S$ is a *simply ordering relation* if for any elements $x, y, z \in F(S)$,

(i) $\langle x, x\rangle \in S$,

(ii) for $x \neq y$, either $\langle x, y\rangle \in S$ or $\langle y, x\rangle \in S$, but not both,

and

(iii) if $\langle x, y\rangle \in S$ and $\langle y, z\rangle \in S$, then $\langle x, z\rangle \in S$.

A set $X$ is said to be *simply ordered by a relation* $S$, if $S$ is a simply ordering relation and $X \subseteq F(S)$. Two binary relations $S$ and $T$ are *isomorphic*, in symbols $S \cong T$, if there exists a one-to-one mapping $f$ of $F(S)$ onto $F(T)$ such that for $x, y \in F(S)$, $\langle x, y\rangle \in S$ if and only if $\langle f(x), f(y)\rangle \in T$. The mapping $f$ is called an *isomorphism of $S$ onto $T$*. If the range of $f$ is a proper subset of $F(T)$ then $f$ is an *isomorphism of $S$ into $T$*; if $S$ and $T$ are the same relation, then the isomorphism onto is called an *automorphism* of $S$. Given a binary relation $S$, the set of automorphisms of $S$, denoted by $G(S)$, is a group under the usual operations of functional composition and inverse. In this paper we are interested in those groups $G(S)$ which are groups of automorphisms of a simply ordering relation $S$. We shall prove the following theorem. Let $G$ be an abelian group. A necessary and sufficient condition that $G$ be isomorphic to a group $G(S)$, for some simply ordering relation $S$, is that $G$ be isomorphic to a direct (cartesian) product $\prod_{i \in I} G_i$ of groups $G_i$ each of which is a subgroup of the additive group of real numbers. This result will be provided as a consequence to several lemmas.

* This paper contains results announced by the authors in [1] and [2]. The first named author was supported by a grant from the National Science Foundation. Problems of the type we consider here were presented by Goffman [4].

If $S$ is a binary relation, we sometimes write $xSy$ for the expression $\langle x, y\rangle \in S$. We introduce the notion of a type $\sigma$ of a relation $S$ in such a manner that if $\sigma$ is the type of $S$ and $\tau$ is the type of $T$, then $\sigma = \tau$ if and only if $S \cong T$. The type of a simply ordering relation is called an *order type*. It is quite clear that if $S \cong T$, then $G(S)$ is isomorphic to $G(T)$. Thus no ambiguity is introduced if we let $G(\sigma)$ stand for the isomorphism class of groups $G(S)$ where $S$ has type $\sigma$.

We shall assume that ordinals have been defined in such a way that each ordinal coincides with the set of smaller ordinals. We also assume that cardinals have been defined as those ordinals which are not set-theoretically equivalent with any smaller ordinal. The Greek letters $\alpha, \beta, \gamma, \varkappa$ (with appropriate subscripts) shall denote ordinals and cardinals. It is clear that the membership relation restricted to each ordinal determines a simply ordering relation; furthermore, the membership relation is a well-ordering relation. If $\alpha$ is an ordinal we also let the symbol $\alpha$ stand for the order type of the well-ordering relation determined by $\alpha$. If $\alpha$ is a well-ordered type, we let $\alpha^*$ denote the inversaly well-ordered type obtained from $\alpha$.

A binary relation $S$ is a *subrelation* of a relation $T$ if $S \subseteq T$. In a similar manner, we can speak of a type $\sigma$ being a *subtype* of a type $\tau$. A simply ordering relation $S$ is *densely ordered*, if

(i) $F(S)$ contains at least two elements,

(ii) if $xSy$ and $x \neq y$, then there exists a $z \in F(S)$ such that $x \neq z$, $y \neq z$, $xSz$ and $zSy$.

A simply ordering relation is *scattered* if no subrelation of $S$ is densely ordered (1). Since the notion of isomorphism among relations preserves the properties of being densely ordered and being scattered, we can extend the definitions to order types.

We assume that the reader is familiar with the notions of an ordered sum $S+T$ of relations $S$ and $T$, an ordered sum $\sigma+\tau$ of types $\sigma$ and $\tau$, an ordered sum $\sum_{i,T} S_i$ of relations $S_i$ over a simply ordering relation $T$, and an ordered sum $\sum_{i,T} \sigma_i$ of types $\sigma_i$ over a simply ordering relation $T$ (2). In this note, we shall use these notions only for simply ordering relations and types. Using the notion of $+$, we can speak of the *initial*, *middle*, or *final segments* of a simply ordering relation or type (3). Clearly, each segment of a type $\sigma$ is a subtype of $\sigma$; however, a subtype of an order type $\sigma$ need not be a segment of $\sigma$. If $f \in G(S)$ and $R$ is a segment of $S$, then we say that $f$ is *fixed* on $R$ whenever $f$ maps $F(R)$ onto $F(R)$. It is

(1) These definitions are found in [5].
(2) For a comprehensive treatment of these notions, we refer the reader to [6].
(3) See also [6].



clear that if $R$ is a middle segment of $S$ such that $S = T+R+U$ and if $f$ is fixed on $R$, then $f$ must also be fixed on $T$ and on $U$. It is also clear that if $f$ is fixed on $R$, then the mapping $f$ restricted to $F(R)$ must be in $G(R)$.

Whenever no ambiguity can arise, we shall speak of order preserving mappings which map order types (or segments thereof) onto other order types (or segments thereof) rather than the mappings which map the corresponding relations on relations. In terms of the notions introduced so far, we state in what follows several well-known facts concerning densely ordered types, scattered types, and well-ordered types.

(I) *Every finite type is a well-ordered type.*

(II) *Every well-ordered (inversely well-ordered) type is a scattered type.*

(III) *Every well-ordered (inversely well-ordered) type admits only the trivial automorphism.*

(IV) *Every segment (containing more than one point) of a densely ordered type is also a densely ordered type.*

(V) *No well-ordered (inversely well-ordered) type contains as a subtype any infinite inversely well-ordered (well-ordered) type.*

(VI) *If $T$ is a densely ordered type, $\sigma_i$ are non-zero scattered types, then every segment $\sigma$ of the ordered sum $\sum_{i,T} \sigma_i$ which contains points from more than one of the segments $\sigma_i$ is not scattered.*

We begin by constructing some order types which admit only the identity mapping as an automorphism. Let $R$ be the usual ordering relation over the set of all rationals. Thus, $F(R)$ is the set of all rationals and $R$ is clearly densely ordered. Let $\varkappa$ be an enumeration of the rationals, i.e. $\varkappa$ is a one-to-one mapping whose domain is $F(R)$ and whose range is the set of finite non-zero oridnals. According to our convention, for each $i \in F(R)$, $\varkappa_i$ also denotes the finite order type determined by $\varkappa_i$. Let $\tau = \sum_{i,R} \varkappa_i$.

LEMMA 1 (4). *The order type $\tau$ admits only the trivial automorphism.*

Proof. Let $f$ be an automorphism of $\tau$. Aussume that $f$ is not the identity mapping. Then there exists an $i \in F(R)$ such that the segment $\varkappa_i$ is mapped by either $f$ or $f^{-1}$ onto a segment $\sigma$ of $\tau$ where $\sigma$ is not contained in any segment $\varkappa_j$ for $j \in F(R)$. By (I) and (VI) we see that while $\varkappa_i$ is scattered its image under $f$ (or $f^{-1}$) cannot be scattered. This is a contradiction. Thus $G(\tau)$ is the one element group.

LEMMA 2. *If $\alpha$ is a well-ordered type, then the type $\alpha^*+\tau+\alpha$ admits only the trivial automorphism.*

Proof. Let $f$ be an automorphism of $\alpha^*+\tau+\alpha$. By (IV) and (VI) every proper initial or final segment of $\tau$ is not scattered. By (II), $f$ must

(4) The type $\tau$ has also been defined in Morel [6], p. 70, and the fact that $\tau$ has no non-trivial automorphism is used in a proof therein.

leave each of the segments $a^*$, $\tau$, and $a$ fixed. By Lemma 1, $f$ is the identity on $\tau$; by (III), $f$ is the identity on $a$ and $a^*$. Thus $f$ is the identity mapping on $a^*+\tau+a$.

Lemma 3 [5]. *If for each $i \in I$ $G_i = G(S_i)$ for some simply ordering relation $S_i$, then there exists a simply ordering relation $S$ such that $G(S)$ is isomorphic to the direct product $\prod_{i\in I} G_i$.*

Proof. With no loss of generality, we assume that the index set $I$ is a cardinal $\beta$ ($\beta$ may be a finite cardinal). Let $T_\beta$ be the natural well-ordering relation whose field is $\beta$, and, for each $\gamma \in \beta$, let $T_\gamma$ be the initial segment of $T_\beta$ determined by $\gamma$. Let $\sigma_\gamma$ be the type of $S_\gamma$ and let $a$ be an infinite cardinal greater than each of the powers of $S_\gamma$. For each $\gamma \in \beta$, let $\nu_\gamma = \sigma_\gamma + a^* + \tau + a$ and let $\sigma = \sum_{\gamma,T_\beta} \nu_\gamma$.

Let $f$ be an automorphism of $\sigma$. We shall prove that

(1) *for each $\gamma \in \beta$, $f$ leaves the middle segments $\sigma_\gamma$ and $\nu_\gamma$ of $\sigma$ fixed.*

In order to prove (1), we first prove by transfinite induction that

(2) *for each $\gamma \in \beta$, $f$ leaves the initial segment $\sum_{\delta,T_\gamma} \nu_\delta$ of $\sigma$ fixed.*

If we consider the underlying pairwise disjoint relations, the definition of the ordered sum $\sum_{\delta,T_\gamma} \nu_\delta$ is determined by a union of sets. Thus, in order to prove (2), it is sufficient to show that

(3) *for each $\gamma \in \beta$, if $f$ leaves the initial segment $\sum_{\delta,T_\gamma} \nu_\delta$ of $\sigma$ fixed, then $f$ also leaves fixed the initial segment $\sum_{\delta,T_{\gamma+1}} \nu_\delta$ of $\sigma$.*

We write $\sum_{\delta,T_{\gamma+1}} \nu_\delta = \big(\sum_{\delta,T_\gamma} \nu_\delta\big) + \nu_\gamma$, and let us assume that $f$ is already fixed on $\sum_{\delta,T_\gamma} \nu_\delta$. Assume $\gamma+1 \leqslant \beta$, and we write

$$\sigma = \sum_{\delta,T_\gamma} \nu_\delta + (\sigma_\gamma + a^* + \tau + a) + \sigma' .$$

If $f$ does not leave the middle segment $\sigma_\gamma$ fixed, then by the fact that $f$ leaves $\sum_{\delta,T_\gamma} \nu_\delta$ fixed the image of $\sigma_\gamma$ under either $f$ or $f^{-1}$ must contain an initial segment of $a^*$. Since $a$ was so chosen that its cardinality exceeds the cardinality of each $S_\gamma$ and since every initial segment of $a^*$ has the same power as $a$, we see that this leads to a contradiction. Thus $f$ is fixed on $\sigma_\gamma$. By a similar argument, but this time appealing to (II), (IV), and (VI), we see that $f$ is also fixed on $\nu_\gamma$. Therefore (3) is proved. Our proof of (3) gives us immediately both (2) and (1).

[5] The lemma was also known to Anne Morel.



We now construct a one-to-one mapping $g$ of $G(\sigma)$ onto $\prod_{\gamma\in\beta} G_\gamma$ as follows: for $f \in G(\sigma)$, $g(f)$ is such an element of $\prod_{\gamma\in\beta} G_\gamma$ that the value of $g(f)$ on each $\gamma \in \beta$ is the automorphism induced by $f$ on $\sigma_\gamma$. The mapping $g$ is clearly a homomorphism onto $\prod_{\gamma\in\beta} G_\gamma$. By Lemma 2, $f$ is always trivial on each final segment of $\nu_\gamma$ of the form $a^*+\tau+a$. Thus the mapping $g$ is one-to-one.

Lemma 4. *Every proper subgroup $G$ of the additive group of real numbers is isomorphic to some $G(S)$.*

Proof. We shall construct an order type $\sigma$ which will be the type of $S$. Let $X$ be the set of all real numbers and let the cosets of $G$ in $X$ be denoted as usual by $t/G$. Let $h$ be any one-to-one mapping of the set of cosets $t/G$ into the set of infinite cardinals. For each real number $t \in X$, let

$$\varkappa(t) = h(t/G)^* + \tau + h(t/G) .$$

By Lemma 2, each $\varkappa(t)$ is an order type which admits only the trivial automorphism. Furthermore, for real numbers $t_1$ and $t_2$, $\varkappa(t_1) = \varkappa(t_2)$ if and only if $t_1/G = t_2/G$. It is also clear that if $t_1$ and $t_2$ belong to different cosets of $G$, then by (V) no initial segment of $\varkappa(t_1)$ is a final segment of $\varkappa(t_2)$. Let $T$ be the natural ordering relation among the real numbers and let $\sigma = \sum_{t,T} \varkappa(t)$.

For $t \in G$, let $f_t$ be that automorphism of $\sigma$ which maps each segment $\varkappa(s)$, $s \in X$ identically onto the segment $\varkappa(t+s)$. Since $s$ and $t+s$ belong to a single coset of $G$, $\varkappa(s) = \varkappa(t+s)$ and we see that $f_t$ is a well-defined automorphism of $\sigma$. We now prove that

(1) *every automorphism of $\sigma$ is equal to some $f_t$, $t \in G$.*

Let $f$ be an automorphism of $\sigma$. From our construction of $\sigma$ we see that

(2) *no segment $\varkappa(s)$ of $\sigma$ can be such that its image under $f$ or $f^{-1}$ contains points from more than one segment $\varkappa(t)$, $t \in X$.*

It follows from (2) that under $f$ each segment $\varkappa(s)$ must be mapped onto another segment $\varkappa(t)$ where $\varkappa(t) = \varkappa(s)$, i.e. $t/G = s/G$. We define a one-to-one order preserving mapping $g$ induced by $f$ on the elements of $X$ as follows: for each $s \in X$, $g(s)$ is that element $t$ of $X$ such that the segment $\varkappa(s)$ is mapped by $f$ onto the segment $\varkappa(t)$. It is clear from the definition of $g$ that $s$ and $g(s)$ belong to the same coset of $G$ and that $g$ is an automorphism of $X$. Furthermore, since $s$ and $g(s)$ belong to the same coset of $G$, for each $s \in X$, there exists an element $t_s \in G$ such that $g(s) = t_s + s$.

In order to complete the proof of (1), we show that

(3) *there exists a* $t \in G$ *such that for each* $s \in X$, $g(s) = t+s$.

Assume (3) does not hold, then the following two cases exhaust all the possibilities. Either

(4) *there exist* $s_1, s_2 \in X$ *and* $t_1, t_2 \in G$ *such that* $s_1 < s_2$, $t_1 < t_2$, $g(s_1) = t_1+s_1$ *and* $g(s_2) = t_2+s_2$,

or else

(5) *there exist* $s_1, s_2 \in X$ *and* $t_1, t_2 \in G$ *such that* $s_2 < s_1$, $t_1 < t_2$, $g(s_1) = t_1+s_1$ *and* $g(s_2) = t_2+s_2$.

Let us consider (4) first. Since $G$ is a proper subgroup of $X$, there exists an element $s_3 \in X-G$ such that $t_1 < s_3 < t_2$. We define the set

$$Y = \{s \in X;\ s < s_2 \text{ and } g(s) = t_s+s \text{ where } t_s \in G \text{ and } t_s < s_3\}.$$

Clearly $s_1 \in Y$ and $s_2$ is an upper bound for the set $Y$. Therefore there exists a least upper bound for $Y$, let it be $s_4$. Suppose that $g(s_4)$ is such that $g(s_4) > s_3+s_4$. Since $g$ is an automorphism of $X$, there exists some $s_5 \in X$ such that $g(s_5) = s_3+s_4$ and $s_5 < s_4$. Now, for each $s \in Y$, $s \leqslant s_4$, $g(s) = t_s+s < s_3+s_4 = g(s_5)$. Hence, for each $s \in Y$, $s < s_5$ and $s_5$ is another upper bound for $Y$ smaller than $s_4$. This is a contradiction. Suppose, on the other hand, $g(s_4) < s_3+s_4$. Then there exists an $s_5$ such that $g(s_5) = s_3+s_4$ and $g(s_5) = t_{s_5}+s_5$. Clearly $s_4 < s_5$ and, by the equality $s_3+s_4 = t_{s_5}+s_5$, we have $t_{s_5} < s_3$. Since $s_4 \leqslant s_2$, we have $g(s_5) = s_3+s_4 < t_2+s_2 = g(s_2)$ and $s_5 < s_2$. Thus $s_5 \in Y$ and $s_4 < s_5$ which again leads to a contradiction. The last possibility that $g(s_4) = s_3+s_4$ contradicts the fact that $s_3$ must not be a member of $G$. Thus (4) can not hold.

Assume that (5) holds. We consider the functions $f^{-1}$ and $g^{-1}$. If (5) holds for $f$ and $g$, then (4) holds for $f^{-1}$ and $g^{-1}$. By what we have already shown, (5) can not hold. Thus (4) and (5) fail, hence (3) holds. The mapping from $t$ to $f_t$ is the required isomorphism of $G$ onto $G(\sigma)$.

LEMMA 5. *Every subgroup of the group of real numbers is isomorphic to some* $G(S)$.

Proof. By Lemma 4, it is sufficient to show that the group of all real numbers is isomorphic to some proper subgroup of itself. As it is well known that the group of all reals is a continuum weak direct product of the group of rationals, it is easy to construct an isomorphism of the group of real numbers onto a proper subgroup of itself.

LEMMA 6. *Let an abelian group* $G$ *be isomorphic to some* $G(S)$. *Then* $G$ *is isomorphic to a direct product of groups* $G_i$ *each of which is a subgroup of the reals* [6].

[6] Originally, the authors discovered this result independently of those results of Cohn [3]. Since our proof is very closely related to those ideas in [3], we have decided to rely entirely on the results already published.



Proof. From a result of Cohn ([3], Theorem 1, p. 43) we see that $G$ can be represented as a direct product of groups $G_i$ each of which is the group of automorphisms on $S_i$, a segment of $S$. Since $G$ is abelian, each $G_i$ is abelian. Thus by Theorem 3 on p. 47 of [3], each $G_i$ is orderable. If we now examine the last part of the proof for Theorem 3 ([3], p. 49), we see that Cohn proved there that if each $G_i$ is orderable, then each $G_i$ is archimedean with respect to the ordering. This fact together with the classical result on archimedean ordered groups prove that each $G_i$ is isomorphic to a subgroup of the reals. Thus, the lemma is proved.

THEOREM. *Let* $G$ *be an abelian group.* $G$ *is the automorphism group of some simply ordering relation* $S$ *if and only if* $G$ *is isomorphic to a direct product of groups* $G_i$ *each of which is a subgroup of the reals.*

Proof. By Lemmas 3, 5 and 6.

## References


[1] C. C. Chang, *Groups which are automorphism groups of a simply ordered set*, Notices of the American Mathematical Society 5 (1958), p. 491.

[2] — and A. Ehrenfeucht, *A characterization of Abelian groups which are automorphism groups of simply ordered sets*, Notices of the American Mathematical Society 6 (1959), p. 170.

[3] P. M. Cohn, *Groups of order automorphisms of ordered sets*, Mathematika 4 (1957), pp. 41-50.

[4] C. Goffman, *The group of similarity transforms of a simply ordered set*, Bull. Amer. Math. Soc. 60 (1954), p. 289.

[5] F. Hausdorff, *Grundzüge der Mengenlehre*, Leipzig 1914. (Reprinted New York 1949.)

[6] Anne C. Morel, *On the arithmetic of order types*, Trans. Amer. Math. Soc. 92 (1959), pp. 48-71.


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*Reçu par la Rédaction le 7. 9. 1961*