# Reversibility in absolute-valued algebras

by

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Let $A$ be an algebra, not necessarily associative, over the real field $R$, which is a normed linear space under a norm $|\ |$ satisfying, in addition to the usual requirements, the condition $|xy| = |x||y|$ for every $x, y$ of $A$. Such an algebra is called *absolute-valued*. A complete description of all absolute-valued division algebras was given by F. B. Wright ([4]), who proved that every such algebra is isotopic to one of the following: the real field $R$, the complex field $C$, the quaternion algebra $Q$ or the Cayley-Dickson algebra $D$. A. A. Albert ([1]) had previously established this result under the restriction that the algebra be algebraic in the sense of every element generating a finite-dimensional subalgebra. F. B. Wright and the present author have shown that an absolute-valued algebra with a unit element is isomorphic to one of the classical algebras $R$, $C$, $Q$, and $D$ ([3]). Infinite-dimensional absolute-valued algebras with an involution were studied in [2].

An element $x$ from $A$ is said to be *reversible* and to have $y$ as a *reverse* if

$$x+y-xy = x+y-yx = 0\,. \tag{1}$$

If an algebra has a unit element $e$, then $y$ is the reverse of $x$ if and only if $e-y$ is the inverse of $e-x$. In equation (1) there is no reference to the unit element and the relation between $x$ and $y$ can hold in any algebra regardless of the existence of a unit element. Therefore the concept of reverse is capable of replacing the inverse in algebras without unit element.

We say that an algebra satisfies the *reversibility condition* if all its elements except of a countable set are reversible. In the present note we shall give a complete description of all absolute-valued algebras satisfying the reversibility condition.

It is obvious that all the classical algebras $R$, $C$, $Q$ and $D$ satisfy the reversibility condition. Moreover, all their elements except the unit element are reversible and have exactly one reverse. We note that in arbitrary absolute-valued algebras the uniqueness of the reverse of elements $x$ with the norm $|x| \neq 1$ can easily be established. In fact, if $y_1$ and $y_2$ are the reverses of $x$, then $x+y_1-xy_1 = x+y_2-xy_2$, which implies the equality $|y_1-y_2| = |x||y_1-y_2|$. Hence, taking into account the inequality $|x| \neq 1$, we get the equality $y_1 = y_2$. In the sequel, for $|x| \neq 1$, the unique reverse of $x$ will be denoted by $x^-$.

THEOREM. *An absolute-valued algebra satisfying the reversibility condition is isomorphic to one of the following: the real field, the complex field, the quaternion algebra or the Cayley-Dickson algebra.*

Before proving the Theorem we shall prove some lemmas. In what follows $A$ will denote an absolute-valued algebra satisfying the reversibility condition. Further, $[x_1, x_2, \ldots, x_n]$ will denote the linear set spanned by the elements $x_1, x_2, \ldots, x_n$ from $A$. For each $x$ in $A$, we shall denote by $A(x)$ the subalgebra generated by $x$.

We shall use the following lemma, proved in [3] (p. 862).

LEMMA 1. *If the elements $x_1, x_2, \ldots, x_n$ from $A$ commute with one another, then $[x_1, x_2, \ldots, x_n]$ is an inner product space.*

For each $x$ in $A$, let us introduce the notation

(2) $$x^{[1} = x, \quad x^{[n+1} = x(x^{[n}); \quad x^{1]} = x, \quad x^{n+1]} = (x^{n]})x \quad (n = 1, 2, \ldots).$$

LEMMA 2. *The reverse of a reversible element $x$, with $|x| < 1$, can be expanded into the series*

$$x^- = -\sum_{n=1}^{\infty} x^{[n} = -\sum_{n=1}^{\infty} x^{n]}.$$

Proof. Using the notation

$$u_1 = x(x^-), \quad u_{n+1} = xu_n; \quad v_1 = (x^-)x, \quad v_{n+1} = v_n x \quad (n = 1, 2, \ldots),$$

we start from equation (1) and, by means of successive multiplication by $x$, obtain the equations

$$x^{[n+1} + u_n - u_{n+1} = 0, \quad x^{n+1]} + v_n - v_{n+1} = 0 \quad (n = 1, 2, \ldots).$$

Hence and by (1) we conclude that the reverse $x^-$ can be expanded in the following manner:

(3) $$x^- = -\sum_{n=1}^{N} x^{[n} + u_{N+1} = -\sum_{n=1}^{N} x^{n]} + v_{N+1} \quad (N = 1, 2, \ldots).$$

Since $|u_{N+1}| = |v_{N+1}| = |x^-||x|^N$ $(N = 1, 2, \ldots)$, the sequences $u_N$ and $v_N$ converge to 0 when $N$ tends to infinity. Passing to the limit in (3) we get the assertion of the lemma.

COROLLARY. *For every element $x$ from $A$ the equality*

(4) $$x^{[n} = x^{n]} \quad (n = 1, 2, \ldots)$$

*holds.*



Indeed, for non-denumerably many real numbers $\lambda$ the elements $\lambda x$ are reversible and

$$(\lambda x)^- = -\sum_{n=1}^{\infty} \lambda^n x^{[n} = -\sum_{n=1}^{\infty} \lambda^n x^{n]},$$

provided $|\lambda x| < 1$. Hence noting that the coefficients of $\lambda^n$ are the same in both expressions we get formula (4). In the sequel by $x^n$ we shall denote the common value of $x^{[n}$ and $x^{n]}$. Definition (2) yields the formula

(5) $$x \cdot x^n = x^n x = x^{n+1} \quad (n = 1, 2, \ldots).$$

LEMMA 3. *For every pair $x_1, x_2$ of elements from $A$ we have the equality*

(6) $$x_2^2 x_1 + (x_1 x_2 + x_2 x_1) x_2 = x_1 x_2^2 + x_2(x_1 x_2 + x_2 x_1).$$

Proof. For every pair $\alpha, \beta$ of real numbers we have the equalities

$$(\alpha x_1 + \beta x_2)^2 (\alpha x_1 + \beta x_2) = \alpha^3 x_1^3 + \alpha\beta^2 x_2^2 x_1 + \alpha^2\beta (x_1 x_2 + x_2 x_1) x_1 + \\ + \alpha^2 \beta x_1^2 x_2 + \beta^3 x_1^3 + \alpha\beta^2 (x_1 x_2 + x_2 x_1) x_2,$$

$$(\alpha x_1 \beta x_2)(\alpha x_1 + \beta x_2)^2 = \alpha^3 x_1^3 + \alpha\beta^2 x_1 x_2^2 + \alpha^2 \beta x_1 (x_1 x_2 + x_2 x_1) + \\ + \alpha^2 \beta x_2 x_1^2 + \beta^3 x_1^3 + \alpha\beta^2 x_2 (x_1 x_2 + x_2 x_1),$$

which, by virtue of (5), imply

$$\alpha\beta^2 \left(x_2^2 x_1 + (x_1 x_2 + x_2 x_1) x_2\right) + \alpha^2 \beta \left(x_1^2 x_2 + (x_1 x_2 + x_2 x_1) x_1\right) \\ = \alpha\beta^2 \left(x_1 x_2^2 + x_2 (x_1 x_2 + x_2 x_1)\right) + \alpha^2 \beta \left(x_2 x_1^2 + x_1 (x_1 x_2 + x_2 x_1)\right).$$

Comparing the coefficients of $\alpha\beta^2$ on both sides of this equation we get formula (6).

LEMMA 4. *For every element $x$ from $A$ we have the relations*

$$(x^2)^2 \in [x^2, x^3]$$

*and*

(7) $$x^2 x^3 = x^3 x^2.$$

Proof. Substituting in formula (6) $x_1 = x^3$, $x_2 = x$ we get, in virtue of (5), equality (7).

Since, by (5), $xx^2 = x^2 x$, we infer, in view of Lemma 1, that $[x, x^2]$ is an inner product space. If $x$ and $x^2$ are linearly dependent, then, of course, $A(x) = [x] = [x^2]$, which implies the assertion of the Lemma. Now let us suppose that $x$ and $x^2$ are linearly independent. Let $z$ be an element from $[x, x^2]$ different from 0 and orthogonal to $x$. Of course, element $z$ can be written in the form $\alpha x + \beta x^2$, where

(8) $$\beta \neq 0.$$

Taking into account the equality

$$z^2 = \alpha^2 x^2 + 2\alpha\beta x^3 + \beta^2 (x^2)^2 \tag{9}$$

and (7), we conclude that $x^2$ commutes with $z^2$. Consequently, by Lemma 1, $[x^2, z^2]$ is an inner product space. Further, since $x$ and $z$ are orthogonal and commute with one another, we have the formula

$$|x^2 - z^2| = |(x+z)(x-z)| = |x+z||x-z| = |x|^2 + |z|^2 = |x^2| + |z^2|.$$

Hence it follows that $x^2$ and $z^2$ are linearly dependent. Consequently, according to (8) and (9), the element $(x^2)^2$ is a linear combination of $x^2$ and $x^3$, which completes the proof.

LEMMA 5. *For every $x$ from $A$, $x^2$ commutes with $x^4$, $x^2x^3$ and $(x^3)^2$.*

Proof. Substituting in (6) $x_1 = x^4$, $x_2 = x$, we get, by virtue of (5), the equality

$$x^2 x^4 = x^4 x^2. \tag{10}$$

Further, substituting $x_1 = x^3$, $x_2 = x^2$ and taking into account formula (7), we have

$$(x^2)^2 x^3 + 2(x^2x^3)x^2 = x^3(x^2)^2 + 2x^2(x^2x^3). \tag{11}$$

By Lemma 4, $(x^2)^2$ is a linear combination of $x^2$ and $x^3$. Thus, by (7), we have the equality

$$x^3(x^2)^2 = (x^2)^2 x^3. \tag{12}$$

Hence and from (11) the formula

$$(x^2x^3)x^2 = x^2(x^2x^3) \tag{13}$$

follows.

Finally, substituting in (6) $x_1 = x^2$, $x_2 = x^3$ and applying (7) we obtain the equality

$$(x^3)^2 x^2 + 2(x^2x^3)x^3 = x^2(x^3)^2 + 2x^3(x^2x^3). \tag{14}$$

By simple computations for every pair $\alpha$, $\beta$ of real numbers we get the equalities

$$\begin{aligned}(\alpha x + \beta x^2)^2(\alpha x + \beta x^2)^3 = {} & \alpha^5 x^2 x^3 + \alpha^3\beta^2 x^2\big(x(x^2)^2\big) + 2\alpha^4\beta x^2 x^4 + \\ & + \alpha^4\beta(x^2)^3 + \alpha^2\beta^3(x^2)^4 + 2\alpha^3\beta^2 x^2(x^2x^3) + \alpha^3\beta^2(x^2)^2x^3 + \\ & + \alpha\beta^4(x^2)^2\big(x(x^2)^2\big) + 2\alpha^2\beta^3(x^2)^2x^4 + \alpha^2\beta^3\big((x^2)^2\big)^2 + \beta^5(x^2)^2(x^2)^3 + \\ & + 2\alpha\beta^4(x^2)^2(x^2x^3) + 2\alpha^4\beta(x^3)^2 + 2\alpha^2\beta^3x^3\big(x(x^2)^2\big) + 4\alpha^3\beta^2x^3x^4 + \\ & + 2\alpha^3\beta^2x^3(x^2)^2 + 2\alpha\beta^4x^3(x^2)^3 + 4\alpha^2\beta^3x^3(x^2x^3)\end{aligned} \tag{15}$$



and

$$\begin{aligned}(\alpha x + \beta x^2)^3(\alpha x + \beta x^2)^2 = {} & \alpha^5 x^3 x^2 + \alpha^3\beta^2\big(x(x^2)^2x^2\big) + 2\alpha^4\beta x^4 x^2 + \\ & + \alpha^4\beta(x^2)^3 + \alpha^2\beta^3(x^2)^4 + 2\alpha^3\beta^2(x^2x^3)x^2 + \alpha^3\beta^2x^3(x^2)^2 + \\ & + \alpha\beta^4\big(x(x^2)^2\big)(x^2)^2 + 2\alpha^2\beta^3x^4(x^2)^2 + \alpha^2\beta^3\big((x^2)^2\big)^2 + \beta^5(x^2)^3(x^2)^2 + \\ & + 2\alpha\beta^4(x^2x^3)(x^2)^2 + 2\alpha^4\beta(x^3)^2 + 2\alpha^2\beta^3\big(x(x^2)^2\big)x^3 + 4\alpha^3\beta^2x^4x^3 + \\ & + 2\alpha^3\beta^2(x^2)^2x^3 + 2\alpha\beta^4(x^2)^3x^3 + 4\alpha^2\beta^3(x^2x^3)x^3.\end{aligned} \tag{16}$$

By (7) the right-hand sides of the last equalities are identical. Thus, in view of (7), (10), (12) and (13), we have the equality

$$\begin{aligned}& \alpha\beta^4(x^2)^2\big(x(x^2)^2\big) + 2\alpha\beta^4(x^2)^2(x^2x^3) + 2\alpha\beta^4x^3(x^2)^3 + 2\alpha^2\beta^3(x^2)^2x^4 + \\ & + 2\alpha^2\beta^3x^3\big(x(x^2)^2\big) + 4\alpha^2\beta^3x^3(x^2x^3) + \alpha^3\beta^2x^2\big(x(x^2)^2\big) + 4\alpha^3\beta^2x^3x^4 \\ & = \alpha\beta^4\big(x(x^2)^2\big)(x^2)^2 + 2\alpha\beta^4(x^2x^3)(x^2)^2 + 2\alpha\beta^4(x^2)^3x^3 + 2\alpha^2\beta^3x^4(x^2)^2 + \\ & + 2\alpha^2\beta^3\big(x(x^2)^2\big)x^3 + 4\alpha^2\beta^3(x^2x^3)x^3 + \alpha^3\beta^2\big(x(x^2)^2\big)x^2 + 4\alpha^3\beta^2x^4x^3.\end{aligned} \tag{17}$$

Finally, comparing the coefficients of $\alpha^2\beta^3$ and $\alpha^3\beta^2$ on both sides of this equality, we get the following equalities:

$$(x^2)^2x^4 + x^3\big(x(x^2)^2\big) + 2x^3(x^2x^3) = x^4(x^2)^2 + \big(x(x^2)^2\big)x^3 + 2(x^2x^3)x^3, \tag{18}$$

$$x^2\big(x(x^2)^2\big) + 4x^3x^4 = \big(x(x^2)^2\big)x^2 + 4x^4x^3. \tag{19}$$

By Lemma 4, $(x^2)^2$ is a linear combination of $x^2$ and $x^3$: $(x^2)^2 = \lambda x^2 + \mu x^3$. Substituting this expression in (18) and (19) and applying (7) and (10), we obtain the equalities

$$\mu x^3x^4 + x^3(x^2x^3) = \mu x^4x^3 + (x^2x^3)x^3, \qquad x^3x^4 = x^4x^3, \tag{20}$$

which imply $x^3(x^2x^3) = (x^2x^3)x^3$. Combining this equality and (14), we obtain the relation $x^2(x^3)^2 = (x^3)^2x^2$, which together with (10) and (13) completes the proof of the lemma.

LEMMA 6. *For every element $x$ from $A$ we have the relation*

$$x^3 \in [x, x^2].$$

Proof. Contrary to our statement, let us suppose that $x^3 \notin [x, x^2]$. Then, of course, the elements $x$ and $x^2$ are linearly independent and, consequently, the space $[x, x^2, x^3]$ is three-dimensional. Since, by (5) and (7), the elements $x$, $x^2$ and $x^3$ commute with one another, we infer, in view of Lemma 1, that $[x, x^2, x^3]$ is an inner product space. Let $x$, $y$, $z$ be an orthonormal basis of $[x, x^2, x^3]$. Since $x$, $y$, $z$ commute with one another, we have the equalities

$$|x^2 - y^2| = |(x+y)(x-y)| = |x+y||x-y| = 2 = |x^2| + |y^2|, \tag{21}$$

$$|x^2 - z^2| = |(x+z)(x-z)| = |x+z||x-z| = 2 = |x^2| + |z^2| \tag{22}$$

and

$$|y^2-z^2| = |(y+z)(y-z)| = |y+z||y-z| = 2\,. \tag{23}$$

We note that $y^2$ and $z^2$ are linear combinations of $x^2$, $(x^2)^2$, $x^3$, $x^4$, $x^2x^3$, $x^3x^2$, $(x^3)^2$. Thus, by (7) and Lemmas 4 and 5, $x^2$ commutes with $y^2$ and $z^2$. Consequently, by Lemma 1, $[x^2, y^2]$ and $[x^2, z^2]$ are inner product spaces. Thus from (21) and (22) we get the equalities $y^2 = -x^2$ and $z^2 = -x^2$, which imply $y^2 = z^2$. But this contradicts formula (23). The lemma is thus proved.

As a direct consequence of the last lemma and Lemma 4 we get the following

COROLLARY. *For every element $x$ from $A$ the equality $A(x) = [x, x^2]$ holds.*

LEMMA 7. *If $y$ is a reverse of $x$, then*

$$x^2y = yx^2, \tag{24}$$

$$y^2x = xy^2, \tag{25}$$

$$x(x^2y) = (x^2y)x, \tag{26}$$

$$x^2(x^2y) = (x^2y)x^2. \tag{27}$$

**Proof.** Equality (1) can be written in the form

$$xy = yx = x+y\,. \tag{28}$$

Substituting in formula (6) $x_1 = x$, $x_2 = y$ and taking into account the last formula, we get equality (24). Furthermore, by symmetry, we get formula (25). From (24) and (28) it follows that $y$ commutes with all elements of $[x, x^2]$ and, consequently, by the Corollary to Lemma 6, it commutes with all elements of $A(x)$. Thus using (28) and substituting in formula (23) $x_1 = y$, $x_2 = x+\lambda x^2$, we obtain for every real number $\lambda$ the equality

$$(yx^2)x + \lambda(yx^2)x^2 = x(yx^2) + \lambda x^2(yx^2)\,,$$

which implies (26) and (27).

LEMMA 8. *If $x$ is a reversible element of $A$ and if the subalgebra $A(x)$ is of dimension two, then all reverses of $x$ belong to $A(x)$.*

**Proof.** Let $y$ be an arbitrary reverse of $x$. Contrary to our statement, let us suppose that $y \notin A(x)$. By (5), (24) and (28), the elements $x$, $x^2$, $y$ commute with one another and, consequently, by Lemma 1, $[x, x^2, y]$ is an inner product space of dimension three. Let $u$ be an element from $[x, x^2, y]$ with the unit norm orthogonal to both $x$ and $x^2$. Since, by the Corollary to Lemma 6, $x^3$ and $(x^2)^2$ belong to $[x, x^2]$, the element $u^2$ is a linear combination of $x$, $x^2$, $x^2y$, $y$, $y^2$ and, consequently, by formulas (25), (26), (27) and (28), it commutes with both $x$ and $x^2$. Thus, by Lemma 1, $[u^2, x, x^2]$ is an inner product space. Further, using a representation theorem for commutative absolute-valued algebras proved in [3] (p. 865), we infer that the algebra $A(x)$, being of dimension two, is isomorphic either to the complex field $C$ or to the algebra $C^*$ of all complex numbers with the usual addition and scalar multiplication, where the product of $x_1$ and $x_2$ is equal to $\bar{x}_1 \cdot \bar{x}_2$. Since both these algebras contain an idempotent which is non-trivial (i.e. different from 0), there exists an idempotent $a$ belonging to $[x, x^2]$, with $|a| = 1$. Taking into account the orthogonality of $u$ and $a$, we have

$$|u^2-a| = |u^2-a^2| = |(u+a)(u-a)| = |u+a||u-a| = 2\,.$$

Since both $u^2$ and $a$ are elements of the inner product space $[u^2, x, x^2]$, the last equality implies $u^2 = -a$. Further, the isomorphism between $A(x)$ and $C$ or $C^*$ implies the existence of such an element $b$ of $A(x)$ that $b^2 = -a$. Thus $u^2 = b^2$. Hence, by the commutativity of $u$ with all the elements of $A(x)$, we have either $u = b$ or $u = -b$. Consequently, $u$ belongs to $[x, x^2]$. But the element $u$ is orthogonal to both $x$ and $x^2$ and is different from 0, which gives a contradiction. The lemma is thus proved.

LEMMA 9. *For every element $x$ from $A$ different from 0, the subalgebra $A(x)$ is isomorphic to either the real field or the complex field.*

**Proof.** By the representation theorem for commutative absolute-valued algebras every subalgebra generated by one element different from 0 is isomorphic to one of the following: the real field, the complex field or the algebra $C^*$ ([3], p. 865). Contrary to our statement, let us suppose that there exists an element $x_0$ in $A$ such that $A(x_0)$ is isomorphic to $C^*$. We can then find a pair $e_0$, $i_0$ of elements of $A(x_0)$ such that $e_0^2 = e_0$, $e_0i_0 = i_0e_0 = -i_0$, $i_0^2 = -e_0$ and $A(x_0) = [e_0, i_0]$. Let us consider a non-denumerable family of elements from $A(x_0)$ of the form $\lambda e_0 + (1-\lambda^2)^{1/2}i_0$, where $|\lambda| \neq \frac{1}{2}$ and $|\lambda| < 1$. Since the algebra $A$ satisfies the reversibility condition, there exists a number $\lambda_0$ such that

$$|\lambda_0| < 1\,, \qquad |\lambda_0| \neq \tfrac{1}{2} \tag{29}$$

and the element $y_0 = \lambda_0 e_0 + (1-\lambda_0^2)^{1/2} i_0$ is reversible. By simple computations we get the equalities

$$e_0 = -(1-4\lambda_0^2)^{-1}(y_0^2 + 2\lambda_0 y_0)\,,$$
$$i_0 = (1-4\lambda_0^2)^{-1}(1-\lambda_0^2)^{-1/2}\big(\lambda_0 y_0^2 + (1-2\lambda_0^2)y_0\big)\,,$$

which show that the subalgebra $A(y_0)$ is of dimension two. Thus, by Lemma 8, $A(y_0)$ contains all reverses of $y_0$. Representing a reverse of the element $y_0$ in the form $\alpha e_0 + \beta i_0$ we deduce from (1) the following equations for $\alpha$ and $\beta$:

$$(1-\lambda_0)\alpha + (1-\lambda_0^2)^{1/2}\beta = -\lambda_0\,,$$
$$(1-\lambda_0^2)^{1/2}\alpha + (1+\lambda_0)\beta = -(1-\lambda_0^2)^{1/2}\,.$$

However, it is very easy to verify that these equations have no solution whenever $\lambda_0$ satisfies inequalities (29), which implies a contradiction. The lemma is thus proved.

LEMMA 10. *For every pair $e_1, e_2$ of linearly independent idempotents in $A$ there exists an element $v \in A$ such that $A(v)$ is of dimension two, $e_1 \in A(v)$ and $A(v) \subset [e_1, e_2, (e_1-e_2)^2]$.*

Proof. First we shall prove that there exists a sequence $\lambda_1, \lambda_2, \ldots$ of real numbers tending to 0 such that all the subalgebras $A(e_1+\lambda_n e_2)$ are of dimension two. Contrary to this, let us suppose that there exists a positive number $\omega$ such that the subalgebras $A(e_1+\lambda e_2)$ are of dimension less than two whenever $|\lambda| < \omega$. Thus for $|\lambda| < \omega$ we have the equalities

$$(e_1+\lambda e_2)^2 = \alpha_1(e_1+\lambda e_2), \quad (e_1-\lambda e_2)^2 = \alpha_2(e_1-\lambda e_2), \tag{30}$$

where $\alpha_1$ and $\alpha_2$ are real numbers depending on $\lambda$. Hence and from the equality

$$(e_1+\lambda e_2)^2+(e_1-\lambda e_2)^2 = 2e_1+2\lambda^2 e_2$$

we get the equality

$$2e_1+2\lambda^2 e_2 = \alpha_1(e_1+\lambda e_2)+\alpha_2(e_1-\lambda e_2).$$

Thus, by the linear independence of $e_1$ and $e_2$,

$$2-\alpha_1-\alpha_2 = 0, \quad 2\lambda^2-\lambda\alpha_1+\lambda\alpha_2 = 0$$

and, consequently, $\alpha_1 = 1+\lambda$, $\alpha_2 = 1-\lambda$. Now equality (30) can be rewritten in the form

$$e_1+\lambda^2 e_2+\lambda(e_1e_2+e_2e_1) = (1+\lambda)(e_1+\lambda e_2),$$

whence the formula $e_1e_2+e_2e_1 = e_1+e_2$ follows. Further, in view of the last equality, we obtain $(e_1-e_2)^2 = 0$. Since absolute-valued algebras contain no divisors of zero, we have $e_1 = e_2$, which contradicts the linear independence of $e_1$ and $e_2$. Thus there exists a sequence $\lambda_1, \lambda_2, \ldots$ tending to 0, for which $A(e_1+\lambda_n e_2)$ $(n = 1, 2, \ldots)$ are of dimension two, and, consequently, by Lemma 9, are isomorphic to the complex field. Hence we infer that there exist elements $v_n$ with unit norm, orthogonal to $v_n^2$, such that $A(e_1+\lambda_n e_2) = A(v_n)$ $(n = 1, 2, \ldots)$. By the Corollary to Lemma 6, all the elements $v_n$ and $v_n^2$ are contained in the unit sphere of the subspace $[e_1, e_2, (e_1-e_2)^2]$. Thus the sequence $\lambda_1, \lambda_2, \ldots$ contains a subsequence convergent to an element $v$ such that $v, v^2 \in [e_1, e_2, (e_1-e_2)^2]$ and $v$ is orthogonal to $v^2$. Since $e_1+\lambda_n e_n \in A(v_n)$ $(n = 1, 2, \ldots)$ and $e_1 = \lim_{n\to\infty}(e_1+\lambda_n e_2)$, we have $e_1 \in A(v)$. Further, from the orthogonality of $v$ and $v^2$ it follows that $A(v)$ is of dimension two, which completes the proof.

LEMMA 11. *If $e_1$ and $e_2$ are idempotents from $A$, then $(e_1-e_2)^2$ commutes with $e_1$ and $e_2$.*



Proof. By symmetry it suffices to prove that $(e_1-e_2)^2$ commutes with $e_1$. Substituting in formula (23) $x_1 = e_2$, $x_2 = e_1$ we get the equality

$$e_1e_2+(e_1e_2+e_2e_1)e_1 = e_2e_1+e_1(e_1e_2+e_2e_1).$$

In other words, $e_1$ commutes with $e_2-e_1e_2-e_2e_1$. Hence and from the equality $(e_1-e_2)^2 = e_1+e_2-e_1e_2-e_2e_1$ we get the assertion of the lemma.

Proof of the Theorem. To prove the Theorem it is sufficient to show that the algebra $A$ has a unit element (see [3], p. 863). By Lemma 9 every subalgebra $A(x)$ $(x \in A, x \neq 0)$ has a unit element. Consequently, it suffices to prove that the algebra $A$ contains exactly one nontrivial idempotent. Contrary to this, let us suppose that there exist two non-trivial idempotents, $e_1$ and $e_2$. Of course, $e_1$ does not belong to $A(e_2)$ and, consequently, $e_1$ and $e_2$ are linearly independent.

First let us consider commuting idempotents. By Lemma 1, $[e_1, e_2]$ is then an inner product space. Since $e_1$ and $e_2$ are linearly independent, the space $[e_1, e_2]$ is of dimension two. Therefore we can find in $[e_1, e_2]$ an element $c$ with unit norm and orthogonal to $e_1$. Writing $c = \alpha e_1+\beta e_2$, where $\alpha$ and $\beta$ are real numbers, we have the equality

$$c^2 = (\alpha^2+\alpha\beta)e_1+(\beta^2+\alpha\beta)e_2-\alpha\beta(e_1-e_2)^2. \tag{31}$$

Hence and from Lemma 11 we infer that $c^2$ commutes with $e_1$ and, consequently, by Lemma 1, $[e_1, c^2]$ is an inner product space. Using the orthogonality of $e_1$ and $c$ we have the equality

$$|c^2-e_1| = |(c+e_1)(c-e_1)| = |c+e_1|\,|c-e_1| = 2,$$

which implies $c^2 = -e_1$. Thus, by (31), $(\alpha^2+\alpha\beta+1)e_1+(\beta^2+\alpha\beta)e_2 = \alpha\beta(e_1-e_2)^2$. By the linear independence of $e_1$ and $e_2$ the right-hand side of the last equality is different from 0. Thus, $e_1e_2 \in [e_1, e_2]$. In other words $[e_1, e_2]$ is a commutative two-dimensional subalgebra of $A$. Being isomorphic either to the complex field or to the algebra $C^*$ (see [3], p. 865), it is generated by an element of $A$ and, consequently, by Lemma 9, it is isomorphic to the complex field. But the complex field does not contain two non-trivial idempotents, which gives a contradiction. Thus we have proved that the algebra $A$ does not contain any pair of commuting non-trivial idempotents.

Now let us assume that

$$e_1e_2 \neq e_2e_1. \tag{32}$$

By Lemma 10, the idempotent $e_1$ belongs to a two-dimensional subalgebra $A(v)$ contained in $[e_1, e_2, (e_1-e_2)^2]$. Of course, $e_1$ and $v$ are linearly independent and commute with one another. Writing the element $v$ in the form $v = \lambda e_1+\mu(e_1-e_2)^2+\nu e_2$ and taking into account Lemma 11, we see that $e_1$ commutes with $\nu e_2 = v-\lambda e_1-\mu(e_1-e_2)^2$, which, according to (32), implies the equality $\nu = 0$. Further, from the linear independence of $e_1$ and $v$ the inequality $\mu \neq 0$ follows. Thus $(e_1 - e_2)^2 \in A(v)$ and $e_1$, $(e_1-e_2)^2$ are linearly independent. Hence and from the isomorphism between $A(v)$ and the complex field it follows that the subalgebra $A\big((e_1-e_2)^2\big)$ is of dimension two. Thus $A\big((e_1-e_2)^2\big) = A(v)$ and, consequently, $e_1 \in A\big((e_1-e_2)^2\big)$. By symmetry, we also have the relation $e_2 \in A\big((e_1-e_2)^2\big)$, which shows that the subalgebra $A\big((e_1-e_2)^2\big)$ contains two non-trivial idempotents. But this contradicts the isomorphism between $A\big((e_1-e_2)^2\big)$ and the complex field. The Theorem is thus proved.

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# A characterization of abelian groups of automorphisms of a simply ordering relation *

by

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A *binary relation* $S$ is a set of ordered pairs $\langle x, y\rangle$ of elements $x$ and $y$; the *field* of $S$, denoted by $F(S)$, is the set of all elements $x$ for which there exists an element $y$ such that either $\langle x, y\rangle \in S$ or $\langle y, x\rangle \in S$. A binary relation $S$ is a *simply ordering relation* if for any elements $x, y, z \in F(S)$,

(i) $\langle x, x\rangle \in S$,

(ii) for $x \neq y$, either $\langle x, y\rangle \in S$ or $\langle y, x\rangle \in S$, but not both,

and

(iii) if $\langle x, y\rangle \in S$ and $\langle y, z\rangle \in S$, then $\langle x, z\rangle \in S$.

A set $X$ is said to be *simply ordered by a relation* $S$, if $S$ is a simply ordering relation and $X \subseteq F(S)$. Two binary relations $S$ and $T$ are *isomorphic*, in symbols $S \cong T$, if there exists a one-to-one mapping $f$ of $F(S)$ onto $F(T)$ such that for $x, y \in F(S)$, $\langle x, y\rangle \in S$ if and only if $\langle f(x), f(y)\rangle \in T$. The mapping $f$ is called an *isomorphism of $S$ onto $T$*. If the range of $f$ is a proper subset of $F(T)$ then $f$ is an *isomorphism of $S$ into $T$*; if $S$ and $T$ are the same relation, then the isomorphism onto is called an *automorphism* of $S$. Given a binary relation $S$, the set of automorphisms of $S$, denoted by $G(S)$, is a group under the usual operations of functional composition and inverse. In this paper we are interested in those groups $G(S)$ which are groups of automorphisms of a simply ordering relation $S$. We shall prove the following theorem. Let $G$ be an abelian group. A necessary and sufficient condition that $G$ be isomorphic to a group $G(S)$, for some simply ordering relation $S$, is that $G$ be isomorphic to a direct (cartesian) product $\prod_{i \in I} G_i$ of groups $G_i$ each of which is a subgroup of the additive group of real numbers. This result will be provided as a consequence to several lemmas.

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