Complementation of independent subsets and subspaces of generalized vector spaces

by

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Introduction. The axioms for generalized vector spaces which will be used in this paper are those which were introduced by G. B. Preston and the author and which were expanded upon in their joint paper [2]. In the above paper one may find proofs of the analogs of many of the theorems of classical vector space theory which are related to independence, as well as the equivalence of this system, in the proper case (cf. below), with that of R. Rado defined by I-functions [7]. The reader is also referred to the work of A. Kertész [5] in which he gives many interesting applications of a system of axioms clearly equivalent to those to be used here. All necessary definitions will be given in this work; however, it will be necessary to cite various results proved in [2].

The principle result in this work is that for any basis of an arbitrary generalized vector space there is a function which to any subspace of the space assigns a disjoint subset of this basis, which together with the subspace generates the whole space, in such a manner that this function is anti-monotonic. From this theorem follow the theorem of B. Banaschewski [1] for the general case and the theorem of K. Johnson [4] for the special case of modules over a division ring. The derivation in this paper of the result of Johnson depends on a property of dependence which does not hold for the general case; an example will be given to clarify this difference.

Definitions and axioms. A relation R between the subsets of a fixed set V is called a *dependence relation* if the following axioms hold (if subsets X and Y are in relation R. i.e., if XRY, we shall write $X \prec Y$ which is to be read 'X is dependent on Y' or simply 'X depends on Y') [1]:

L1. *If* $X \subset Y$ *then* $X \prec Y$.

L2. *If* $X_t \prec Y$ *for all* t *in some index set* T *then* $\sum X_t \prec Y$.

L3. *If* $X \prec Y$ *and* $Y \prec Z$ *then* $X \prec Z$.

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[1] $\emptyset$ denotes the empty set. $A \cdot B$ and $A+B$ denote the intersection and union, respectively, of two sets A and B.

L4. *If y and x are points of V and $y \prec X$ and $y \not\prec (X-x)$ then $x \prec \big((X-x)+y\big)$ (here for a one element set $\{x\}$ we use simply x).*

A subset X is *independent* if for each of its elements x, $x \not\prec (X-x)$; otherwise it is said to be *dependent*.

A dependence relation which further satisfies the following property is called a *proper dependence relation*:

L5. *For every subset X of V the property of independence is a property of finite character.*

The importance of L5 is further clarified by the following theorem (cf. [2], Lemma 6):

A dependence relation $\prec$ on the subsets of V is a proper dependence relation if and only if the following property holds:

(P) *For any element y of V and any subset X, $y \prec X$ if and only if $y \prec F$ where F is a finite subset of X.*

In this work only proper dependence relations will be considered. A set is called a *basis* if it is a maximal (with respect to inclusion) independent subset of V.

A *generalized vector space* consists of a couple $\langle V, \prec\rangle$ where $\prec$ is a proper dependence relation on V. When no confusion is likely to arise the set V will be used to denote the couple.

A subset A of a generalized vector space V is called a *subspace* of V if $x \in A$ implies $x \prec A$. It is not difficult to show that the intersection of an arbitrary collection of subspaces is again a subspace; thus, if B is any subset of V we define the subspace generated by B to be the intersection of all subspaces which contain B. This subspace will be denoted by $\overline{B}$. It can be shown that $\overline{B} = \{x\colon x \prec B\}$. The family of all subspaces of V will be denoted by $\mathcal{S}(V)$. If A is any subset of V then the family of all independent subsets of A will be denoted by $I(A)$.

Main results. The proof of the main theorem requires two lemmas.

LEMMA I. *If X and Y are subspaces such that $X\cdot Y = \overline{\emptyset}$ and z is any point of V which is not in $\overline{X+Y}$ then $X\cdot(\overline{Y+z}) = \overline{\emptyset}$.*

Proof. We suppose that x is in $X\cdot(\overline{Y+z})$. It follows that $x \prec X$ and $x \prec Y+z$. If x is in Y then x is in $X\cdot Y = \overline{\emptyset}$.

We now suppose that x is not in Y. Since Y is a subspace, this implies that $x \not\prec Y$. We thus obtain

$$x \prec Y+z \quad \text{and} \quad x \not\prec [(Y+z)-z];$$

hence from L4 it follows that

$$z \prec \big([(Y+z)-z]+x\big) = Y+x\,.$$

However

$$z \prec Y+x \prec X+Y$$

hence

$$z \in \overline{X+Y}$$

which is in contradiction to the hypothesis that z is not in $\overline{X+Y}$. Thus the only case which can arise is $x \in Y$. It follows that $X\cdot(\overline{Y+z}) \subset \overline{\emptyset}$. Since every subspace contains the subspace generated by the empty set, equality follows.

LEMMA II. *If X is a subspace of V and $\{Y_t\}$ is any tower of subsets of V such that for every t, $X\cdot\overline{Y_t} = \overline{\emptyset}$ then*

$$X\cdot\big(\overline{\textstyle\sum Y_t}\big) = \overline{\emptyset}\,.$$

Proof. Suppose that there is a point x in $X\cdot\big(\overline{\sum Y_t}\big)$. Then since the dependence relation is proper, $x \prec F$ where F is a finite subset of $\sum Y_t$. Since the Y_t form a tower, there is an index t_0 such that F is contained in Y_{t_0}. Thus $x \prec Y_{t_0}$ from which it follows that x is in $X\cdot\overline{Y_{t_0}} = \overline{\emptyset}$. It is clear that $X\cdot\big(\overline{\sum Y_t}\big) = \overline{\emptyset}$.

We are now in a position to prove the following theorem.

THEOREM I. *Let V be any vector space and B any basis of V. Then there is a function f which assigns to each subspace of V a subset of B which generates a disjoint subspace (except for $\overline{\emptyset}$) but which together with the given subspace generates all of V, further f does this in a monotonicly decreasing manner; symbolically,*

1. $X\cdot\overline{f(X)} = \overline{\emptyset}$.
2. $\overline{X+f(X)} = V$.
3. $X \subset Y$ *implies* $f(X) \supset f(Y)$.

Proof. The proof proceeds by means of a transfinite construction of the function f. We arrange B in a well ordered sequence

$$b_0, b_1, \ldots, b_t, \ldots, \qquad t \leqslant T\,,$$

where T is an ordinal number.

Let X be an arbitrary subspace of V; we now construct $f(X)$. We begin by defining P_0 as follows:

$$P_0 = \begin{cases} \emptyset & \text{if} \quad b_0 \text{ is in } X\,, \\ b_0 & \text{if} \quad b_0 \text{ is not in } X\,. \end{cases}$$

We now make the inductive hypothesis that for some ordinal $t \leqslant T$ we have previously defined the sequence

$$P_0, P_1, \ldots, P_s, \ldots, \qquad s < t\,,$$

which satisfies the conditions:

i. for all $s < t$, $X\cdot\overline{P_s} = \overline{\emptyset}$,

ii. for all $r < s < t$, $P_r \subset P_s \subset \{b_q\}_{q\leqslant s}$.

Let $Q_t = \sum_{s<t} P_s$. We define P_t by

$$P_t = \begin{cases} Q_t & \text{if } \quad b_t \text{ is in } \overline{X+Q_t}\,, \\ Q_t + b_t & \text{if } \quad b_t \text{ is not in } \overline{X+Q_t}\,. \end{cases}$$

From Lemma II we see that $\overline{X}\cdot\overline{Q_t} = \overline{\emptyset}$. From Lemma I we see that if b_t is not in $\overline{X+Q_t}$ then $\overline{X}\cdot\overline{Q_t+b_t} = \overline{\emptyset}$. Thus in either case $\overline{X}\cdot\overline{P_t} = \overline{\emptyset}$. Condition i of the inductive hypothesis is thus satisfied. Condition ii is also satisfied since for all $s<t$

$$P_s \subset Q_t \subset P_t \subset Q_t + b_t \subset (\{b_r\}_{r<t} + b_t)\,.$$

The induction is thus complete.

We define f by

$$f(X) = P_T\,.$$

It is an immediate consequence of the first condition of the inductive hypothesis that P satisfies condition 1 of the theorem.

It follows from the second inductive hypothesis and the definition of P_t that for all $t \leqslant T$

$$b_t \prec \overline{X+P_t} \subset \overline{X+P_T} = \overline{X+f(X)}\,.$$

From the axioms L1, L2, and L3 we obtain

$$B \prec \overline{X+f(X)}\,.$$

Since B is a basis, we get

$$V = \overline{B} \subset \overline{X+f(X)} \subset V\,.$$

Thus f satisfies the second condition of the theorem.

We now prove that f satisfies the third condition of the theorem. We suppose that $X \subset Y$ and that X and Y are subspaces of V. We will derive a contradiction from the assumption that

$$C = f(Y) - f(X) \neq \emptyset\,.$$

R_t and S_t will be used to denote the sets in the construction of $f(Y)$ which correspond to the sets P_t and Q_t, respectively, in the construction of $f(X)$.

Since C is a subset of a well-ordered set B, there is a least ordinal s for which b_s is in C.

From the choice of s it follows that

$$P_t \supset R_t\,, \qquad t < s\,,$$

and further that b_s is not in P_s. But b_s not in P_s can occur if and only if b_s is in $\overline{X+Q_s}$.

However $X \subset Y \subset \overline{Y+S_s}$, and for all $t<s$

$$b_t \prec \overline{Y+S_t} \prec \overline{Y+S_s}\,.$$

Since $Q_s \subset \{b_t\}_{t<s}$, it follows that $Q_s \prec \overline{Y+S_s}$ and hence

$$\overline{X+Q_s} \prec \overline{Y+S_s}\,.$$

Thus b_s is in $\overline{Y+S_s}$.

Therefore by the definition of R_s, b_s is not in R_s, and hence b_s is not in $f(Y)$. It follows that b_s is not in $C = f(Y) - f(X)$. This is contrary to the choice of s; the desired contradiction having been established the theorem is proved.

A function f from the set of subspaces $S(V)$ of V to $S(V)$ is said to be *quasi-orthogonal* if for any two subspaces X and Y in $S(V)$, f satisfies

1. $\overline{X}\cdot\overline{f(X)} = \overline{\emptyset}$,
2. $\overline{X+f(X)} = V$,
3. $X \subset Y$ implies $f(X) \supset f(Y)$.

B. Banaschewski [1] proved that in the case of modules over a division ring there exist quasi-orthogonal functions; the proof for generalized vector space may be found in [2]. It follows here as an immediate corollary to Theorem I.

THEOREM II. *Let V be a generalized vector space, then there exists a quasi-orthogonal function from the set of subspaces of V to the set of subspaces of V.*

Proof. Let g be the function given in Theorem I. Define f by $f(X) = \overline{g(X)}$ for each X in $S(V)$. It is immediate that g is quasi-orthogonal.

K. Johnson [4] has proved a similar type theorem on the existence of such functions from the set of independent subsets of V, $I(V)$ to $I(V)$. In order to more fully understand the relationships of these theorems we must first investigate the relationship between bases of subspaces which have only $\overline{\emptyset}$ in common (²). The next two lemmas will yield information to this end.

LEMMA III. *Let X and Y be two disjoint independent subsets of a generalized vector space, then: if $X+Y$ is independent then $\overline{X}\cdot\overline{Y} = \overline{\emptyset}$.*

Proof. We suppose that $X+Y$ is independent and will derive a contradiction from the existence of an element z in $(\overline{X}\cdot\overline{Y}) - \overline{\emptyset}$. By the choice of z it follows, since our dependence relation is proper, that there is a minimal non-empty finite subset X' of X such that $z \prec X'$. Let x be any element of X'. Then

$$z \prec X' \quad \text{but} \quad z \not\prec X' - x\,,$$

(²) (Added in proof.) Johnson's paper has appeared in the interim and the results of his paper are somewhat stronger than here indicated.

hence, by axiom L4, $x \prec (X'-x)+z$. Further, since z is in $\overline{Y}$, we know $z \prec Y$. Putting these facts together we get

$$x \prec (X'-x)+z \prec (X'-x)+Y \subset (X-x)+Y = (X+Y)-x\,.$$

This last statement is in contradiction to the independence of $X+Y$. The lemma is thus proved.

LEMMA IV. *If X and Y are disjoint independent subsets of a vector space over a division ring D then:*

$$X+Y \text{ is independent if and only if } \overline{X}\cdot\overline{Y} = \{0\}\,.$$

Proof. Half of this lemma is contained in the above lemma. It is straight forward to construct an element in $\overline{X}\cdot\overline{Y}$ under the hypothesized circumstances if one further assumes that $X+Y$ is not independent, and is therefore left to the reader.

Of course the question arises concerning the possibility of proving Lemma IV for the case of generalized vector spaces. The following example yields a negative reply.

Let $V = E_2$ the Euclidean plane. We define the relation $X \prec Y$ to hold if

1. Y is a one element set and $X \subset Y$;

2. Y consists of two or more colinear points and X is a subset of the line passing through them;

3. Y contains three or more non-colinear points and X is arbitrary.

Let X be the set consisting of the two points, (0,0) and (0,1), and let Y consist of (1,0) and (1,1); thus $\overline{X}$ is the line $x=0$ and $\overline{Y}$ is the line $x=1$. Clearly $\overline{X}\cdot\overline{Y} = \overline{\emptyset} = \emptyset$. However the set $X+Y$ is not independent since, for example, the point (1,1) is dependent on the other three points of $X+Y$.

This example provides yet another proof of the fact that there are dependence relations which can not be represented as vector spaces over some division ring (cf. Lazarson [6] and Ingleton [3]).

By using Theorem I and Lemma IV one can now easily obtain the following special case of Johnson's Theorem.

THEOREM III. *Let B be any basis of V, a vector space over division ring, then there exists a function f from the independent subsets of V, $I(V)$, to the subsets of B, $I(B)$, such that for any sets X and Y in $I(V)$ the following hold:*

1. $X\cdot f(X) = \emptyset$,
2. $X+f(X)$ *is a basis,*
3. $X\subset Y$ *implies* $f(X)\supset f(Y)$.

Proof. Let g be the function given by Theorem I. Define f by $f(X) = g(\overline{X})$ for each X in $I(V)$. Verification of 1-3 present no difficulty.

For the proof of this theorem in full generality we refer the reader to Johnson's work.

Remarks. Because of the wide range of applicability of the axioms of abstract linear dependence relations (e.g. groups, rings, fields, differential algebra, etc.) it is to be expected that the above theorems may have many interesting consequences in other particular cases. As examples of what may be done along this line, by application of the single theorem that the cardinal of any two bases are the same, one should consult Kertész [5]. The author hopes to go into this aspect of the theory at a later date.

It will be noticed that in the proofs of Theorem I, and consequently in the proofs of Theorems II and III, the axiom of choice is used. As always in such cases one would like to know if the use of this axiom is essential, and if so are these theorems equivalent with the axiom of choice. It can be shown that all of these theorems are independent of the usual axioms of set theory (in any of the ordinary axiomatization, e.g., the Bernays-Gödel system). It can further be shown that various restrictions of these theorems, either singly or in combination are equivalent to the axiom of choice. A more detailed discussion of the relationships between the axiom of choice and various restrictions of these and other theorems on generalized vector spaces, the proofs of which require the axiom of choice, will be the topic of a forthcoming paper of the author.

References *

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* (Added in proof.) For different axiomatis formulations of abstract dependence the reader may wish to consult G. Birkhoff, *Lattice theory*, New York 1948; O. Haupt, G. Nöbeling and C. Pauc, *Über Abhängigkeitsräume*, Journ. f. d. reine u. ang. Math. 181 (1940), pp. 193-217; J. Schmidt, *Mehrstufige Austauschstrukturen*, Zeit. Math. Log. u. Grund. d. Math. 2 (1956), pp. 233-249; A. P. Robertson and J. D. Weston, *A general basis theorem*, Proc. Edinburgh Math. Soc. 11 (1954), pp. 139-141.

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