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Added in proof. J. E. Kist has pointed out that, in the presence of completeness, the hypothesis that A be uniformly closed in Theorems 5.6 and 5.7 is redundant. (See, e.g. [31], p. 30.)

A note on 0-dimensional compact groups

by

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Ivanovskiĭ ([3]), Kuz'minov ([4]), and Hulanicki ([2]) have recently published proofs of the fact that a 0-dimensional infinite compact group is homeomorphic with the Cartesian product of a number of 2-element discrete spaces, thus answering a question raised by P. S. Aleksandrov. Since all three of these proofs are somewhat complicated, it appears worth while to present a simplified version of Hulanicki's elegant proof of the theorem. We prove slightly more, as follows.

THEOREM. *Let G be a 0-dimensional, infinite, compact topological group satisfying the T_0 separation axiom, and let $\mathfrak{m}$ be the least cardinal number of an open basis at the identity e of G. Then G, regarded only as a topological space, is homeomorphic with the space $\{a, b\}^{\mathfrak{m}}$, where $\{a, b\}$ is a discrete space and $a \neq b$.*

Proof. We give the proof in a number of steps.

(I) Let $\{U_\iota\}_{\iota\in I}$ be an open basis at e having cardinality $\mathfrak{m}$. By a well-known theorem of Pontryagin ([5], p. 140, Theorem 17), there is a normal compact open subroup V_ι of G such that $V_\iota \subset U_\iota$, for each $\iota \in I$. Now well order the family $\{V_\iota\}_{\iota\in I}$, and rewrite it as $\{V_1, V_2, V_3, \ldots, V_\alpha, \ldots\}$, where α runs through all ordinals less than (say) the first ordinal μ with cardinal $\mathfrak{m}$. (Note that $\mathfrak{m}$ must be infinite.) With no loss of generality, we may suppose that $V_1 = G$. For every ordinal β, $1 < \beta < \mu$, let $N_\beta = \bigcap_{\alpha<\beta} V_\alpha$, and let $N_1 = G$. It is clear that every N_β is a normal subgroup of G. Fix an ordinal $\beta < \mu$, and let X be any subset of G that is the intersection of sets of the form $a_1V_{\beta_1} \cup a_2V_{\beta_2} \cup \ldots \cup a_sV_{\beta_s}$, where $a_j \in G$ and $\beta_j < \beta$ $(j = 1, 2, \ldots, s)$. It is obvious that $N_\beta X = X$.

(II) We next define certain open and closed subsets of G. Let $\boldsymbol{B}$ be the set of ordinals $\beta < \mu$ for which $V_\beta \subsetneqq N_\beta V_\beta$. Both V_β and $N_\beta V_\beta$ are open and closed normal subgroups of G. The quotient groups $N_\beta V_\beta / V_\beta$ and $G/N_\beta V_\beta$ are compact and discrete and hence finite. Let $b_\beta^{(1)} N_\beta V_\beta, \ldots, b_\beta^{(m_\beta)} N_\beta V_\beta$

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be the distinct cosets of $N_\beta V_\beta$ in G, and let $a_\beta^{(1)}V_\beta, \ldots, a_\beta^{(k_\beta)}V_\beta$ be the distinct cosets of V_β in $N_\beta V_\beta$. For each integer j, $j = 1, 2, \ldots, k_\beta$, let

$$A_\beta^{(j)} = b_\beta^{(1)} a_\beta^{(j)} V_\beta \cup b_\beta^{(2)} a_\beta^{(j)} V_\beta \cup \ldots \cup b_\beta^{(m_\beta)} a_\beta^{(j)} V_\beta .$$

Thus each $A_\beta^{(j)}$ is a finite union of cosets of V_β, and each $A_\beta^{(j)}$ intersects every coset of $N_\beta V_\beta$ in a coset of V_β. It is obvious that $A_\beta^{(j)} \cap A_\beta^{(l)} = \emptyset$ if $j \neq l$ and that

$$\bigcup_{j=1}^{k_\beta} A_\beta^{(j)} = G .$$

We next note that if X is a nonvoid subset of G and $j = 1, 2, \ldots, k_\beta$, then $(N_\beta X) \cap A_\beta^{(j)} \neq \emptyset$. Since $\bigcup_{l=1}^{m_\beta} b_\beta^{(l)} N_\beta V_\beta = G$, there is some $b_\beta^{(l)}$ for which $(N_\beta X) \cap (b_\beta^{(l)} N_\beta V_\beta) \neq \emptyset$. Since $a_\beta^{(j)}$ is in $N_\beta V_\beta$, we have $b_\beta^{(l)} N_\beta V_\beta = b_\beta^{(l)} a_\beta^{(j)} N_\beta V_\beta = N_\beta b_\beta^{(l)} a_\beta^{(j)} V_\beta$. Thus there are elements $a_1, a_2 \in N_\beta$, $x \in X$, and $y \in V_\beta$ such that $a_1 x = a_2 b_\beta^{(l)} a_\beta^{(j)} y$. Since $b_\beta^{(l)} a_\beta^{(j)} \in A_\beta^{(j)}$, it follows at once that $(N_\beta X) \cap A_\beta^{(j)} \neq \emptyset$.

Now let $\beta_1, \beta_2, \ldots, \beta_s$ be any distinct elements of $\boldsymbol{B}$; we may suppose that $\beta_1 < \beta_2 < \ldots < \beta_s$. Then we have

$$A_{\beta_1}^{(j_1)} \cap A_{\beta_2}^{(j_2)} \cap \ldots \cap A_{\beta_s}^{(j_s)} \neq \emptyset , \tag{1}$$

if $1 \leqslant j_l \leqslant k_{\beta_l}$ $(l = 1, 2, \ldots, s)$. It follows from (I) that $A_{\beta_1}^{(j_1)} = N_{\beta_2} A_{\beta_1}^{(j_1)}$. By the preceding paragraph, we have $A_{\beta_1}^{(j_1)} \cap A_{\beta_2}^{(j_2)} = (N_{\beta_2} A_{\beta_1}^{(j_1)}) \cap A_{\beta_2}^{(j_2)} \neq \emptyset$. Applying (I) and the preceding paragraph again, together with finite induction, we verify (1).

To complete our construction, we need one more fact, *viz.*: if x, y are in G and $x \neq y$, then there is a $\beta \in \boldsymbol{B}$ and a j, $1 \leqslant j \leqslant k$, such that $A_\beta^{(j)}$ contains one and only one of the points x and y. Since $x^{-1}y \neq e$, there is an $\alpha < \mu$ such that $x^{-1}y \notin V_\alpha$. Let β be the least of all these α's. Then $x^{-1}y \in N_\beta$, so that $x^{-1}y \in N_\beta V_\beta$ and $x^{-1}y \notin V_\beta$. Hence we have $V_\beta \subsetneqq N_\beta V_\beta$, so that $\beta \in \boldsymbol{B}$. In the notation used above to define the sets $A_\beta^{(j)}$, we have $x, y \in b_\beta^{(t)} N_\beta V_\beta$ for some t, $1 \leqslant t \leqslant m_\beta$. Since $x^{-1}y \notin V_\beta$, we have $x \in b_\beta^{(t)} a_\beta^{(j)} V_\beta$ and $y \in b_\beta^{(t)} a_\beta^{(l)} V_\beta$, where $j \neq l$. Hence $x \in A_\beta^{(j)}$ and $y \in A_\beta^{(l)}$, so that $y \notin A_\beta^{(j)}$.

(III) To finish the proof, we form the Cartesian product $Y = \underset{\beta \in B}{\boldsymbol{P}} \{1, 2, \ldots, k_\beta\}$, each finite space $\{1, 2, \ldots, k_\beta\}$ being given the discrete topology. Define a mapping Φ of G into Y by: $\Phi(x) = (j_\beta)_{\beta \in \boldsymbol{B}}$, where j_β is the integer such that $x \in A_\beta^{(j)}$, for each $\beta \in \boldsymbol{B}$. Since all of the sets $A_\beta^{(j)}$ are open and closed, Φ is a continuous mapping. Since the sets $A_\beta^{(j)}$ separate points of G, Φ is one-to-one. Hence Φ is a homeomorphism, and $\Phi(G)$ is a compact and hence closed subspace of Y. In view of (1), $\Phi(G)$ is dense in Y, and thus $\Phi(G) = Y$.

The constructions in (I) and (II) above show that $\overline{\overline{\boldsymbol{B}}} \leqslant \mathfrak{m}$. Since G and Y are homeomorphic, we must have $\overline{\overline{\boldsymbol{B}}} = \mathfrak{m}$, inasmuch as $\overline{\overline{\boldsymbol{B}}}$ is the smallest cardinal number of an open basis at an arbitrary point of Y. To show that Y is homeomorphic to $\{a, b\}^{\mathfrak{m}}$, write $\boldsymbol{B}$ as the union of a family of pairwise disjoint countably infinite sets. For each countably infinite $\boldsymbol{B}_1 \subset \boldsymbol{B}$, $\underset{\beta \in \boldsymbol{B}_1}{\boldsymbol{P}} \{1, 2, \ldots, k_\beta\}$ is a 0-dimensional compact metric space without isolated points. By a classical theorem, this product is homeomorphic to $\{a, b\}^{\aleph_0}$ (see for example [1], p. 121, Satz VI′). This implies immediately that Y is homeomorphic to $\{a, b\}^{\mathfrak{m}}$.

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