$n = 1, 2, \ldots$ and $\lim \varrho(p, v_n) = 0$, by (9) and (10). Therefore $\lim \varrho(p, U_n) = 0$ and an infinite subsequence $U_{n_1}, U_{n_2}, \ldots$ (where $n_1 < n_2 < \ldots$) of mutually distinct sets can be chosen since no continuum U_n contains p, according to (1). Then the points

$$s \leqslant u_{n_1} < u_{n_2} < \ldots$$

constitute a converging sequence with $\lim u_{ni} = u \in B$, whence $u \in \overline{A \cap B}$, by (1), and $u \neq q$, by (4). Thus $s < u \in S$ follows, contrary to (6).

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A remark on duality

by

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We shall consider triples (X, U, A) in which

(i) *X is a connected compact Hausdorff space,*
(ii) *U is a connected open subset of X,*
(iii) $A = X - U$.

If U is dense in X then we say that (X, U, A) is a *compactification* of U. If further A is zero-dimensional then we say that (X, U, A) is a *light* compactification of U. For each triple (X, U, A) we may construct a light compactification $(X, U, A)^*$ of U by regarding each connected component of A as a single point. Thus $(X, U, A)^* = \big(X', U, c(A)\big)$ where $c(A)$ is the *component space* of A.

For every locally compact Hausdorff space U we have the Čech compactification $\big(\beta(U), U, \delta(U)\big)$ which gives rise to the *standard* light compactification $\big(\beta(U), U, \delta(U)\big)^* = \big(\beta'(U), U, \delta'(U)\big)$ of U. This one is characterized by the property that for each light compactification (X, U, A) of U there exists a unique map $\big(\beta'(U), U, \delta'(U)\big) \to (X, U, A)$ which is the identity on U.

The purpose of this note is to show that, in a sense that will be specified below, *among all the light compactifications (X, U, A) of U, the standard one can be characterized by the fact that X has the lowest possible connectivity in dimension* 1. Thus if X is 1-acyclic then (X, U, A) is necessarily the standard light compactification. These considerations imply that in any triple (X, U, A), if X is 1-acyclic then $c(A)$ and $\delta'(U)$ are homeomorphic. In particular, this holds if $X = S^n$ is the n-sphere, $n > 1$. In this case the result has been established recently by M. K. Fort, Jr. [1] solving a question raised by Kuratowski. Much earlier the case $n = 2$ was considered by L. E. J. Brouwer (see e.g. [2], p. 386).

The considerations are based on cohomology in dimensions 0 and 1. We summarize briefly the relevant facts.

(1) For each compact pair (X, A) we have an exact sequence

$$0 \to H^0(X, A) \to H^0(X) \to H^0(A) \to H^1(X, A) \to H^1(X) \to H^1(A).$$

(2) If $f: (Y, B) \to (X, A)$ then f determines a homomorphism of the sequence of (X, A) into that of (Y, B).

(3) If $f: (Y, B) \to (X, A)$ is such that f maps $Y - B$ homeomorphically onto $X - A$ then $H^1(f): H^1(X, A) \to H^1(Y, B)$ is an isomorphism.

(4) If X is connected then $H^0(X) = 0$.

(5) If $f: Y \to X$, $Y \neq 0$ and $H^0(f): H^0(X) \to H^0(Y)$ is an isomorphism then f is a homeomorphism.

(6) If X is zero-dimensional then $H^1(X) = 0$.

(7) If $f: Y \to X$ is monotone and surjective then $H^1(f): H^1(X) \to H^1(Y)$ is a monomorphism.

The conditions above are satisfied by the augmented Čech cohomology groups (based on finite open coverings) with any non-zero coefficient group.

A system satisfying (1)-(7) can also be constructed without making appeal to homology theory. To this end consider the exact sequence

$$0 \to Z \xrightarrow{i} R \xrightarrow{\varphi} S \to 0$$

where R is the additive group of real numbers (with the usual topology), Z is the group of integers, i is the inclusion map, $S = R/Z$ is the circle-group and φ is the canonical factorization map. For each pair (X, A) we define $H^0(X, A)$ as the group of all continuous maps $f: X \to Z$ which are constant on A, divided by the subgroup of maps constant on X. The group $H^1(X, A)$ is defined as the group of all continuous maps $f: X \to S$ which are constant on A divided by the subgroup of all maps of the form φg where $g: X \to R$ is a continuous map constant on A. The homomorphism $H^0(A) \to H^1(X, A)$ is defined as follows: given $f: A \to Z$, consider an extension $f': X \to R$ and take the element of $H^1(X, A)$ given by $\varphi f'$. The verification of (1)-(7) is straightforward.

THEOREM 1. *Let $f: (Y, U, B) \to (X, U, A)$ be a mapping of two light compactifications of U, which is the identity on U. Then the map $H^1(X) \to H^1(Y)$ induced by f is an epimorphism. This epimorphism is an isomorphism if and only if f is a homeomorphism.*

Proof. If $B = 0$ then U is compact and $U = Y = X$. Thus we may assume $B \neq 0$. Since X and Y are connected, we have $H^0(X) = 0 = H^0(Y)$ by (4). Since A and B are zero-dimensional, it follows from (6) that $H^1(A) = 0 = H^1(B)$. Consequently, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & H^0(A) & \to & H^1(X, A) & \to & H^1(X) & \to 0 \\ & \downarrow \varphi_1 & & \downarrow \varphi_2 & & \downarrow \varphi_3 & \\ 0 \to & H^0(B) & \to & H^1(Y, B) & \to & H^1(Y) & \to 0 \end{array}$$

with exact rows. Further, φ_2 is an isomorphism by (3). It follows that φ_3 is an epimorphism, and that φ_3 is an isomorphism if and only if φ_1 is an isomorphism. By (5), φ_1 is an isomorphism if and only if the map $B \to A$ is a homeomorphism, i.e. if and only if f is a homeomorphism.

Let (X, U, A) be an arbitrary light compactification of U. Let

$$g: \big(\beta'(U), U, \delta'(U)\big) \to (X, U, A)$$

be the map of the standard light compactification of U in (X, U, A) which is the identity on U.

COROLLARY 2. *g induces an epimorphism $H^1(A) \to H^1\big(\delta'(U)\big)$. This epimorphism is an isomorphism if and only if g is a homeomorphism.*

COROLLARY 3. *If (X, U, A) is a light compactification of U such that $H^1(X) = 0$, then g is a homeomorphism and (X, U, A) is essentially the standard light compactification.*

COROLLARY 4. *If $(X_1, U_1, A_1), (X_2, U_2, A_2)$ are light compactifications such that $H^1(X_1) = 0 = H^1(X_2)$ then every homeomorphism $U_1 \leftrightarrow U_2$ admits an extension $X_1 \leftrightarrow X_2$. This extension is a homeomorphism $(X_1, U_1, A_1) \leftrightarrow (X_2, U_2, A_2)$.*

E. Michael and E. G. Skliarenko have recently announced [1] that this extension theorem remains valid without the assumption that U_1 and U_2 are open.

THEOREM 4. *Let (X, U, A) be a triple and let*

$$h: (X, U, A) \to (X, U, A)^* = \big(X', U, c(A)\big)$$

be the canonical map. If $H^1(X) = 0$ then $H^1(X') = 0$ and $\big(X', U, c(A)\big)$ is essentially the standard light compactification of U.

Indeed, $X \to X'$ being monotone, it follows from (7) that $H^1(X') \to H^1(X)$ is a monomorphism and thus $H^1(X') = 0$. Thus the conclusion follows from Corollary 3.

References

[1] M. K. Fort, Jr, *The complements of bounded, open connected subsets of Euclidean space*, Bull. Acad. Polon. Sci. 9 (1961), p. 457.

[2] K. Kuratowski, *Topologie II*, 3rd ed., Warszawa 1961.

[1] On the Prague Symposium on Topology (September 1961).

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