A property of accessible points

by

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It is seen in the first of the following examples (fig. 1) that a square can, after removing its centre p, be decomposed into disjoint continua with diameters greater than 1. By replacing the square by a locally connected continuum C contained in the plane $\mathcal{E}^2$, it is possible (fig. 2) to

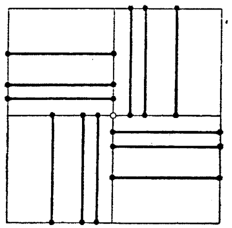

Fig. 1

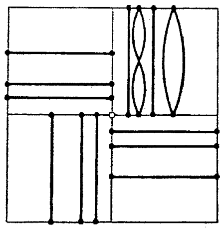

Fig. 2

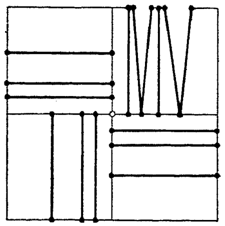

Fig. 3

put the point p into the boundary of C. Finally, losing the local connectedness of C, we can find such a decomposition (fig. 3) with p belonging to the boundary of a component of $\mathcal{E}^2 - C$.

However, in all the above examples the point p is not accessible (*) from the set $\mathcal{E}^2 - C$. This leads to the following

THEOREM. *Let p be a limit point of a set $A \subset \mathcal{E}^2$ such that A is locally compact at p. If the point p is accessible from the set $\mathcal{E}^2 - A$ and $\mathbf{C}$ is a collection of mutually disjoint continua filling up the set $A - \{p\}$, i.e.*

$$A - \{p\} = \bigcup_{C \in \mathbf{C}} C ,$$

then $\mathbf{C}$ contains arbitrarily small elements, i.e.

$$\inf_{C \in \mathbf{C}} \delta(C) = 0 .$$

(*) A point p is said to be *accessible* from a set X if a continuum C exists such that $p \in C \subset X \cup \{p\}$ and $C \neq \{p\}$ (C. Kuratowski, *Topologie II*, Warszawa 1961, p. 115). For X which are connected open subsets of locally connected continua this is equivalent to the existence of an arc C satisfying the same conditions (ibidem, p. 194).

Proof. Taking a disk $D' \subset \mathcal{E}^2$ and a continuum $C \subset \mathcal{E}^2$ such that p is the centre of D', $A \cap D'$ is a compact set, $A \cap C = \{p\}$ and $C \neq \{p\}$ (which exist by the local compactness of A at p and the accessibility of p from $\mathcal{E}^2 - A$, respectively), we see that the component C' of $C \cap D'$, containing p, either coincides with C if $C \cap D' = C$, or intersects the boundary of $C \cap D'$ in C if $C \cap D' \neq C$ (ibidem, p. 112). Hence $C' \neq \{p\}$ in any case. Then the closure of a component J of $C' - \{p\}$ intersects the boundary of $C' - \{p\}$ in C', i.e. $p \in \bar{J} \neq \{p\}$ (ibidem). Since $p \notin J \subset C' \subset D'$ and $A \cap J \subset A \cap C = \{p\}$, we have $J \subset D' - A$. Let us denote by R the component of $D' - A$, containing J. The set $A \cap D'$ being compact, R is a connected open set in D' and $\bar{R} - R \subset A$. It follows that $\bar{J} - R \subset A \cap \bar{J} \subset A \cap C = \{p\}$, which gives $\bar{J} \subset R \cup \{p\}$. The point p is thus accessible from R and we infer that an arc L' exists such that $p \in L' \subset R \cup \{p\}$.

Suppose on the contrary that

$$\inf_{C \in \boldsymbol{C}} \delta(C) = \varepsilon > 0$$

and let $D \subset D'$ be a disk with the centre p and a diameter $\delta(D) < \min\{\varepsilon, \delta(L')\}$. Then no element of $\boldsymbol{C}$ is contained in D and the arc L' intersects the boundary B of D. Moreover, since each $C \in \boldsymbol{C}$ is a continuum, every component of $C \cap D$ intersects B provided that $C \cap D \neq 0$ (ibidem). Denoting by $\boldsymbol{K}$ the collection of all components K of sets $C \cap D \neq 0$, where $C \in \boldsymbol{C}$, we thus have

$$(1) \qquad A \cap D - \{p\} = \bigcup_{K \in \boldsymbol{K}} K,$$

$$(2) \qquad B \cap K \neq 0 \quad \text{for} \quad K \in \boldsymbol{K}$$

and $\boldsymbol{K}$ consists of mutually disjoint continua.

Let, further, $L \subset L'$ be an arc with end points p and q such that $B \cap L = \{q\}$. Hence $L \subset D$ and $p \in A \cap L \subset A \cap L' \subset A \cap (R \cup \{p\}) = \{p\}$, i.e.

$$(3) \qquad A \cap L = \{p\},$$

which gives

$$(4) \qquad q \notin A \cap B = \overline{A \cap B},$$

the set $A \cap B = B \cap A \cap D'$ being compact.

Let us establish an order $<$ of the circumference B, beginning at the point q. Thus B is ordered by $<$ similarly to a right side open segment of the real line and $q \leqslant b$ for every $b \in B$. Moreover

(5) if $C_i \subset D$ are continua such that $b_i, c_i \in B \cap C_i$ for $i = 1, 2$ and $b_1 < b_2 < c_1 < c_2$, then $C_1 \cap C_2 \neq 0$.

Indeed, if the continua C_1, C_2 were disjoint, arcs $L_1, L_2 \subset D$ would exist having the same properties, and thus the union $B \cup L_1$ would contain a θ-curve, which is impossible (ibidem, p. 359).

Since p is a limit point of A, points $p_i \in A \cap D - \{p\}$ exist converging to p. By (1), we have $p_i \in K_i \in \boldsymbol{K}$ for $i = 1, 2, \ldots$, where no continuum K_i contains p. The distance $\varrho(p, K_i)$ tends thus to zero, as $i \to \infty$, and the sequence $K_1, K_2, \ldots$ contains infinitely many distinct sets. We may assume that all the continua $K_1, K_2, \ldots$ are distinct, whence they are mutually disjoint. Let $b_i \in B \cap K_i$ for $i = 1, 2, \ldots$, according to (2). Thus $b_i \neq b_j$ for $i \neq j$ and the sequence $b_1, b_2, \ldots$ contains an infinite subsequence of points coverging to a point $b \in B$. On the other hand, among these points an infinite monotone sequence may be chosen. Let us assume that all the points $b_1, b_2, \ldots$ constitute such a sequence and that $b_1 < b_2 < \ldots$ for convenience.

Therefore the set S of points $b \in B$ such that there exist distinct continua $K_i \in \boldsymbol{K}$ and points $b_i \in B \cap K_i$ $(i = 1, 2, \ldots)$, satisfying the conditions: $\lim \varrho(p, K_i) = 0$, $b_1 < b_2 < \ldots$ and $\lim b_i = b$, is not empty.

It follows from (1) that $S \subset \overline{A \cap B}$ and so, by (4), the point q does not belong to $\bar{S}$. This implies the existence of the least upper bound s of S. Hence

$$(6) \qquad b \leqslant s \quad \text{for} \quad b \in S$$

and it instantly follows from the definition of the set S that s belongs to S. Thus there are distinct continua $T_i \in \boldsymbol{K}$ and points $s_i \in B \cap T_i$ $(i = 1, 2, \ldots)$, satisfying

$$(7) \qquad \lim_{i \to \infty} \varrho(p, T_i) = 0,$$

$s_1 < s_2 < \ldots$ and $\lim s_i = s$. Hence

$$(8) \qquad s_i < s \quad \text{for} \quad i = 1, 2, \ldots$$

because $q \neq s$.

By (3) and the compactness of the set $A \cap D = D \cap A \cap D'$, a number $\varepsilon_1 > 0$ exists such that if the Hausdorff distance $d(L, X)$ between L and any compact subset X of D is less than ε_1, then $d(\{p\}, A \cap X) < 1/2$ (putting $d(Y, 0) = 0$ for every Y). Since the arc L is contained in the disk D and meets its boundary B only at the point q, there exists a homeomorphism h of D onto itself such that $h(p) = p$, h is the identity mapping on B and maps the straight line segment $\overline{pq}$ with end points p and q onto L (ibidem, p. 381). It follows from (1) and (7) that there are, in every neighbourhood of p, points

$$t \in T = \bigcup_{i=1}^{\infty} T_i,$$

satisfying $t \in D - \{p\}$, and—from (4) that there are, in every neighbourhood of q, points $b_1 \in B - \{q\}$, satisfying $a < b_1$ for each $a \in A \cap B$. Therefore we may find an arc in $D - \overline{pq}$ so that it is mapped by h onto an arc L_1' with end points t and b_1, and satisfying the inequality $d(L, L_1') < \varepsilon_1$. Hence $L_1' \subset D-L$ and $t \in L_1' \cap T$. We have $\overline{T} \subset \overline{A \cap D} = A \cap D \subset A$, according to (1) and the compactness of $A \cap D$. Thus every point $a \in \overline{T} \cap B$ belongs to $A \cap B$, whence $a < b_1$. Therefore b_1 does not belong to $\overline{T}$ and so there is an arc $L_1 \subset L_1'$ with end points v_1 and b_1 such that $L_1 \cap \overline{T} = \{v_1\}$. We get $v_1 \in A \cap L_1' \subset A \cap D - \{p\}$, whence

$$\varrho(p, v_1) \leqslant d(\{p\}, A \cap L_1') < 1/2$$

and, by (1), a continuum $U_1 \in \boldsymbol{K}$ exists such that $v_1 \in U_1$. Let us take an arbitrary point $u_1 \in B \cap U_1$, which exists according to (2).

If we had $u_1 < s$, a positive integer j would exist such that $u_1 < s_i$ and $T_i \neq U_1$ for $i > j$. Thus we should have $T_i \cap U_1 = 0$ for $i > j$. However, by (7), there would be points $t_i \in T_i$ such that $\lim \varrho(p, t_i) = 0$, whence $\operatorname{Lim} \overline{pt_i} = \{p\}$. Since $u_1 \in U_1 \cap B \subset A \cap B$, by (1), (4) would imply $q < u_1$. On the other hand, $s_i < b_1$ because we should also have $s_i \in A \cap B$ for $i > j$. Therefore $q < u_1 < s_i < b_1$ and applying (5) to the continua

$$C_1 = L \cup \overline{pt_i} \cup T_i\,, \qquad C_2 = L_1 \cup U_1\,,$$

we should obtain $C_1 \cap C_2 \neq 0$. But since $L_1 \subset L_1' \subset D-L$ and $U_1 \cap L \subset (A - \{p\}) \cap L = 0$ by (1) and (3), there would be $L \cap C_2 = 0$. Furthermore, $T_i \cap L_1 = T_i \cap \overline{T} \cap L_1 = T_i \cap \{v_1\} \subset T_i \cap U_1 = 0$, i.e. $T_i \cap C_2 = 0$ for $i > j$. Hence $\overline{pt_i} \cap C_2 \neq 0$ for $i > j$, which would give $\{p\} \cap C_2 = (\operatorname{Lim} \overline{pt_i}) \cap C_2 \neq 0$, i.e. $p \in C_2$, contrary to (1) and the inclusion $L_1 \subset D-L$ (because $U_1 \in \boldsymbol{K}$ and $p \in L$). We thus have $s \leqslant u_1$.

If some continuum T_k contained the point v_1, we should have $T_k = U_1$ and could assume $s_k = u_1$, since $s_k \in B \cap T_k$ and the point u_1 had been chosen arbitrarily in the set $B \cap U_1$. This, however, is impossible, since $s_k < s$ according to (8). Hence v_1 belongs to none of T_i $(i = 1, 2, \ldots)$, i.e. the union T of T_i does not contain v_1 and thus $v_1 \in (\overline{T} - T) \cap U_1$.

Now, supposing that for some $n = 1, 2, \ldots$ a continuum $U_n \in \boldsymbol{K}$ and points u_n, v_n are given such that

$$s \leqslant u_n \in B \cap U_n\,, \qquad v_n \in (\overline{T} - T) \cap U_n\,, \tag{9}$$

$$\varrho(p, v_n) < 1/2^n\,, \tag{10}$$

we shall define U_{n+1}, u_{n+1} and v_{n+1}.

In fact, we have $U_n \subset D-L$, according to (1) and (3). Thus $\varrho(L, U_n) > 0$. By the same reasoning as previously (for $n+1$ instead of 1), a number $\varepsilon_{n+1} > 0$ exists such that if $d(L, X) < \varepsilon_{n+1}$, then $d(\{p\}, A \cap X) < 1/2^{n+1}$ for an arbitrary compact subset X of D. Similarly, we find an arc $L_{n+1}' \subset D-L$ which intersects T, satisfies

$$d(L, L_{n+1}') < \min\{\varepsilon_{n+1}, \varrho(L, U_n)\}$$

and has b_{n+1} as an end point, where $b_{n+1} \in B - \{q\}$ and $a < b_{n+1}$ for each $a \in A \cap B$. Then also $L_{n+1}' \subset D - U_n$, $b_{n+1} \notin \overline{T}$ and an arc $L_{n+1} \subset L_{n+1}'$ exists with end points v_{n+1} and b_{n+1} such that $L_{n+1} \cap \overline{T} = \{v_{n+1}\}$. Hence

$$\varrho(p, v_{n+1}) \leqslant d(\{p\}, A \cap L_{n+1}') < 1/2^{n+1}\,,$$

i.e. (10) holds for $n+1$ instead of n, and there is a continuum $U_{n+1} \in \boldsymbol{K}$ containing v_{n+1}. Thus a point $u_{n+1} \in B \cap U_{n+1}$ exists, according to (2). The proof that $s \leqslant u_{n+1}$ is quite similar to the preceding one, namely that $s \leqslant u_1$; it is sufficient to put everywhere $n+1$ instead of 1. Similarly, the set T does not contain the point v_{n+1}. We thus get (9) for $n+1$ instead of n.

Moreover, we have $u_n < u_{n+1}$. To show this, suppose on the contrary that $u_{n+1} \leqslant u_n$. Since

$$v_{n+1} \in L_{n+1} \cap U_{n+1} \subset L_{n+1}' \cap U_{n+1} \subset (D - U_n) \cap U_{n+1}\,,$$

the elements U_n, U_{n+1} of $\boldsymbol{K}$ are distinct, i.e. disjoint, and thus $u_n \neq u_{n+1}$, by (9). Hence $u_{n+1} < u_n$. It follows by the inclusion $L_{n+1} \subset D - U_n$ that the subcontinua U_n and $L_{n+1} \cup U_{n+1}$ of disk D are disjoint. Let G denote the component of $D - (L_{n+1} \cup U_{n+1})$, containing U_n. Thus G is an open set in D (ibidem, p. 163) and contains the point u_n.

If a point s_i $(i = 1, 2, \ldots)$ belonged to G, it would be joined to u_n with an arc $I \subset G$ (ibidem, p. 182). But since $u_n \in A \cap B$, we should have $u_n < b_{n+1}$. On the other hand, (8) and (9) would give $s_i < s \leqslant u_{n+1}$, whence $s_i < u_{n+1} < u_n < b_{n+1}$. Applying (5) to the continua

$$C_1 = I\,, \qquad C_2 = L_{n+1} \cup U_{n+1}\,,$$

we should obtain $C_1 \cap C_2 \neq 0$, which is impossible since $I \subset G \subset D - (L_{n+1} \cup U_{n+1})$. Thus no point s_i belongs to G $(i = 1, 2, \ldots)$.

It follows from (9) that $v_{n+1} \in U_{n+1} - T_i$ for $i = 1, 2, \ldots$ So T_i and U_{n+1} are distinct elements of $\boldsymbol{K}$, whence

$$T_i \cap L_{n+1} = T_i \cap \{v_{n+1}\} \subset T_i \cap U_{n+1} = 0\,,$$

i.e. $T_i \subset D - (L_{n+1} \cup U_{n+1})$. The continuum T_i containing the point s_i which does not belong to the component G must therefore be contained in $D - G$ for $i = 1, 2 \ldots$ We thus obtain the inclusion $T \subset D - G$ which, by (9), yields the contradiction

$$v_n \in \overline{T} \cap U_n \subset \overline{D - G} \cap U_n = (D - G) \cap U_n = 0$$

since $U_n \subset G$; and the inequality $u_n < u_{n+1}$ is shown.

In this way, we have got infinite sequences of continua $U_n \in \boldsymbol{K}$, points $u_n \in B \cap U_n$ and points $v_n \in U_n$ satisfying $s \leqslant u_n < u_{n+1}$ for

$n = 1, 2, \ldots$ and $\lim \varrho(p, v_n) = 0$, by (9) and (10). Therefore $\lim \varrho(p, U_n) = 0$ and an infinite subsequence $U_{n_1}, U_{n_2}, \ldots$ (where $n_1 < n_2 < \ldots$) of mutually distinct sets can be chosen since no continuum U_n contains p, according to (1). Then the points

$$s \leqslant u_{n_1} < u_{n_2} < \ldots$$

constitute a converging sequence with $\lim u_{ni} = u \in B$, whence $u \in \overline{A \cap B}$, by (1), and $u \neq q$, by (4). Thus $s < u \in S$ follows, contrary to (6).

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A remark on duality

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We shall consider triples (X, U, A) in which

(i) *X is a connected compact Hausdorff space,*
(ii) *U is a connected open subset of X,*
(iii) *$A = X - U$.*

If U is dense in X then we say that (X, U, A) is a *compactification* of U. If further A is zero-dimensional then we say that (X, U, A) is a *light* compactification of U. For each triple (X, U, A) we may construct a light compactification $(X, U, A)^*$ of U by regarding each connected component of A as a single point. Thus $(X, U, A)^* = (X', U, c(A))$ where $c(A)$ is the *component space* of A.

For every locally compact Hausdorff space U we have the Čech compactification $(\beta(U), U, \delta(U))$ which gives rise to the *standard* light compactification $(\beta(U), U, \delta(U))^* = (\beta'(U), U, \delta'(U))$ of U. This one is characterized by the property that for each light compactification (X, U, A) of U there exists a unique map $(\beta'(U), U, \delta'(U)) \to (X, U, A)$ which is the identity on U.

The purpose of this note is to show that, in a sense that will be specified below, *among all the light compactifications (X, U, A) of U, the standard one can be characterized by the fact that X has the lowest possible connectivity in dimension* 1. Thus if X is 1-acyclic then (X, U, A) is necessarily the standard light compactification. These considerations imply that in any triple (X, U, A), if X is 1-acyclic then $c(A)$ and $\delta'(U)$ are homeomorphic. In particular, this holds if $X = S^n$ is the n-sphere, $n > 1$. In this case the result has been established recently by M. K. Fort, Jr. [1] solving a question raised by Kuratowski. Much earlier the case $n = 2$ was considered by L. E. J. Brouwer (see e.g. [2], p. 386).

The considerations are based on cohomology in dimensions 0 and 1. We summarize briefly the relevant facts.

(1) For each compact pair (X, A) we have an exact sequence

$$0 \to H^0(X, A) \to H^0(X) \to H^0(A) \to H^1(X, A) \to H^1(X) \to H^1(A).$$