différentes, les suites $\{E(10^i \cdot x_1)\}$ et $\{E(10^i \cdot x_2)\}$ sont incomparables. Il existe donc (axiome du choix) un ensemble T de $2^{\aleph_0}$ nombres réels distincts x pour lesquels les suites $\{E(10^i \cdot x)\}$ sont incomparables deux à deux.

Il est à noter que, dans le travail précité de Waraszkiewicz, c'est seulement la *puissance* $2^{\aleph_0}$ de l'ensemble des x et y pour lesquels les suites (3) sont incomparables et non pas sa *structure continue* qui intervient dans la construction de la famille $\mathcal{S}$ de $2^{\aleph_0}$ spirales S_x incomparables deux à deux (dans le sens défini au début). Ce fait permet de maintenir le résultat de Waraszkiewicz par la simple correction consistant à réduire la famille $\mathcal{S}$ à celle des spirales S_x où x ne parcourt que l'ensemble T de valeurs donné par le théorème 2.

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Suspension of transgression

by

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1. Introduction. In a fibre bundle, there exist relations between the homotopy invariants of the fibre, the total space, and the base. It is interesting to investigate how such relations depend on the fibration. For homology much is already known. Here (see also [7]) we study the suspension of the transgression homomorphism in the homotopy exact sequence of the fibration. Information about the transgression is obtained which adds something to our scanty knowledge of the relation between the homotopy groups of the fibre, the total space, and the base.

Consider a fibre bundle with fibre Y and projection $f\colon B \to X$, where B is the total space and X is the base. Let SY denote the suspension of Y. We study the composition

$$\pi_r(X) \xrightarrow{\Delta} \pi_{r-1}(Y) \xrightarrow{E} \pi_r(SY),$$

where Δ is the transgression and E is the suspension operator. The Hopf fibration of S^7 over S^4 provides an example where $E\Delta$ is not induced by a map $X \to SY$.

To construct SY we take C^0 and C^1, two copies of the cone on Y, and identify their bases, so that

$$Y = C^0 \cap C^1, \quad SY = C^0 \cup C^1.$$

Since fY is a point, we regard the cone on Y as a subspace of the mapping cylinder of $f\colon B \to X$. Let Γ^0 and Γ^1 be copies of this mapping cylinder, with their bases identified, and let ΣB denote their union, so that

$$(SY;\; C^0, C^1) \subset (\Sigma B; \Gamma^0, \Gamma^1).$$

We extend f to a fibration $g\colon \Sigma B \to X$ by defining $g(Sz) = fz$, where $Sz \subset \Sigma B$ denotes the suspension of $z \in B$. Thus if $x \in X$ then $g^{-1}x$ is the suspension of $f^{-1}x$, and an admissible map $SY \to g^{-1}x$ is the suspension of an admissible map $Y \to f^{-1}x$. Similarly Γ^i $(i = 0, 1)$ is represented as a fibre bundle over X with fibre C^i and projection $g|\Gamma^i$. The mapping cylinder of f contains X as a subspace, and so there exist cross-sections $h^i\colon X \to \Sigma B$ such that $h^i X \subset \Gamma^i$. Let $x_0 \in X, y_0 \in Y$ be basepoints, with

$x_0 = fy_0$. We use y_0 as basepoint for SY and ΣB, also for C^i and Γ^i. By translation of the basepoint along Sy_0 we identify [1] homotopy groups based at y_0 with those based elsewhere on Sy_0.

We choose the suspension operator E to be the one such that

$$\xi E \delta = \eta \,, \tag{1.1}$$

as shown in the following diagram, where δ denotes the boundary operator and ξ, η are injections.

$$\begin{array}{ccc} \pi_r(C^0, Y) & \xrightarrow{\eta} & \pi_r(SY, C^1) \\ \delta \downarrow & & \uparrow \xi \\ \pi_{r-1}(Y) & \xrightarrow[E]{} & \pi_r(SY) \end{array}$$

If C^0 and C^1 are interchanged then E is changed into $-E$.

Since the fibration $g\colon \Sigma B \to X$ admits a cross-section, its homotopy exact sequence splits and the injection

$$\sigma\colon \pi_r(SY) \to \pi_r(\Sigma B)$$

is a monomorphism. We shall prove

THEOREM (1.2). *The composition*

$$\sigma E \Delta\colon \pi_r(X) \to \pi_r(\Sigma B)$$

is equal to the difference $h^0_* - h^1_*$ *between the homomorphisms induced by the cross-sections* $h^0, h^1\colon X \to \Sigma B$.

COROLLARY (1.3). *Suppose that there exists a retraction* $e\colon \Sigma B \to SY$. *Then*

$$k^0_* - k^1_* = E\Delta\colon \pi_r(X) \to \pi_r(SY) \,,$$

where k^i_* *is induced by* $k^i = eh^i\colon X \to SY$.

The retraction condition is studied in [8]. It is fulfilled in the following example. Take $X = S^n$, the unit sphere in Euclidean $(n+1)$-space. Let T denote the antipodal transformation, whose degree is $(-1)^{n+1}$. Take B to be the $(n-1)$-sphere bundle of unit tangent vectors to S^n. Let $d(x, y)$ denote the geodesic distance between points $x, y \in S^n$. If $d(x, y) = \pi/2$ we identify (x, y) with the unit tangent at x in the direction of y. Thus B is identified with the subspace of $S^n \times S^n$ where $d(x, y) = \pi/2$. Similarly Γ^0 is identified with the subspace where $d(x, y) \leqslant \pi/2$, and Γ^1 with the subspace where $d(x, y) \geqslant \pi/2$, in such a way that $g(x, y) = x$ and

$$h^0(x) = (x, x) \,, \qquad h^1(x) = (x, Tx) \,.$$

[1] This overcomes the difficulty caused by using unreduced suspension and enables us to compare the homomorphisms in (1.2) below.

We identify y with (x_0, y) so that $SY = S^n$. A retraction of ΣB onto SY is given by $e(x, y) = y$. We obtain $k^0 = 1$ and $k^1 = T$. By (1.3), therefore,

$$1 - T_* = E\Delta \,, \tag{1.4}$$

where T_* denotes the automorphism of $\pi_r(S^n)$ induced by T.

In general SY is not a retract of ΣB and so (1.3) does not apply. To illustrate the use of (1.2) we take $X = S^n$. Let $\iota_n \in \pi_n(S^n)$ denote the class of the identity map. Consider the element $\theta = \Delta\iota_n \in \pi_{n-1}(Y)$, which is the obstruction to the existence of a cross-section. Let $\xi \in \pi_n(\Sigma B)$ denote the class of the cross-section h^1. Then $\xi + \sigma(\lambda)$ is the class of h^0, by (1.2), where $\lambda = E\theta \in \pi_n(SY)$. Notice that $h^0 \simeq h^1$ if and only if $\lambda = 0$. If $\alpha \in \pi_r(S^n)$ then

$$\sigma E\Delta(\alpha) = \big(\xi + \sigma(\lambda)\big) \circ \alpha - \xi \circ \alpha \,, \tag{1.5}$$

by (1.2), and the main theorem in § 6 of [4] can be used to express the right hand side of this relation in terms of multiple Whitehead products and generalized Hopf invariants. To be specific we define a homomorphism

$$\varphi\colon \pi_r(SY) \to \pi_{r+n-1}(SY)$$

as follows. The image of the monomorphism σ is the kernel of the epimorphism

$$\varrho\colon \pi_r(\Sigma B) \to \pi_r(X)$$

induced by the fibre map g. By naturality of the Whitehead product

$$\varrho[\xi, \sigma(\beta)] = [\varrho(\xi), \varrho\sigma(\beta)] = 0 \,,$$

where $\beta \in \pi_r(SY)$, and so $[\xi, \sigma(\beta)] = \sigma(\gamma)$, where $\gamma \in \pi_{r+n-1}(SY)$. We define $\varphi(\beta) = \gamma$. Let $\lambda_1, \lambda_2, \ldots$ denote the sequence of elements in the homotopy groups of SY defined by $\lambda_1 = \lambda$ and $\lambda_m = \varphi(\lambda_{m-1})$ $(m \geqslant 2)$. The main theorem in § 6 of [4], applied to (1.5) above, shows that

$$E\Delta(\alpha) = \sum_m \lambda_m \circ \alpha_m + \sum_{l<m} [\lambda_l, \lambda_m] \circ \alpha_{l,m} + \sum_{k \geqslant l < m} \big[\lambda_k, [\lambda_l, \lambda_m]\big] \circ \alpha_{k,l,m} + \ldots \,, \tag{1.6}$$

where the remainder involves quadruple and higher Whitehead products, and the elements $\alpha_m, \alpha_{l,m}, \alpha_{k,l,m}, \ldots$ are generalized Hopf invariants of α. In particular

$$E\Delta(\alpha) = \sum_m \lambda_m \circ \alpha_m \,, \tag{1.7}$$

modulo the kernel of the suspension

$$\pi_r(SY) \to \pi_{r+1}(S^2 Y) \,.$$

In the notation of Hilton [4] we have $\alpha_1 = \alpha$, $\alpha_2 = H_0(\alpha)$, $\alpha_3 = H_1(\alpha)$, $\alpha_{1,2} = H_2(\alpha)$; if $r \leqslant 4n-4$ then all the other generalized Hopf invariants in (1.6) are zero for dimensional reasons.

There remains the problem of how to compute the "structural constants" $\lambda_1, \lambda_2, \ldots$ In § 3 below we solve this problem in the case of sphere-bundles over spheres, and thereby obtain an expression for $E\Delta$ which is convenient for applications.

2. Proof of (1.2). Consider first the commutative diagram shown below, where all the homomorphisms are injections.

$$\begin{array}{ccccc} \pi_r(C^1, Y) & \xrightarrow{\eta} & \pi_r(SY, C^0) & \xleftarrow{\xi} & \pi_r(SY) \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu} & & \downarrow{\scriptstyle\sigma} \\ \pi_r(\Gamma^1, B) & \xrightarrow[\eta']{} & \pi_r(\Sigma B, \Gamma^0) & \xleftarrow[\xi']{} & \pi_r(\Sigma B) \end{array}$$

Because g is an extension of f it follows that the injections

$$\pi_r(B, Y) \to \pi_r(\Gamma^1, C^1), \qquad \pi_r(\Gamma^0, C^0) \to \pi_r(\Sigma B, SY),$$

are isomorphisms. Therefore the triads

$$(\Gamma^1; B, C^1), \qquad (\Sigma B; \Gamma^0, SY)$$

have trivial homotopy groups, by the upper homotopy exact sequence, and hence λ and μ are isomorphisms, by the lower homotopy exact sequence. Also the image of the monomorphism σ coincides with the kernel of the epimorphism

$$\varrho\colon \pi_r(\Sigma B) \to \pi_r(X)$$

induced by g. Since $\xi'\sigma = \mu\xi$, which is an isomorphism, it follows that ξ' and ϱ yield a direct sum decomposition of $\pi_r(\Sigma B)$. Thus (1.2) will be proved by showing that

$$\text{(2.1)} \qquad \begin{array}{l} \text{a)} \\ \text{b)} \end{array} \left\{ \begin{array}{l} \varrho\sigma E\Delta = \varrho h^0_* - \varrho h^1_*\,, \\ \xi'\sigma E\Delta = \xi' h^0_* - \xi' h^1_*\,. \end{array} \right.$$

The first of these relations is obvious, since $\varrho\sigma = 0$ and $gh^0 = 1 = gh^1$.

We prove (2.1b) as follows. Since $h^0 X \subset \Gamma^0$, we have at once that $\xi' h^0_* = 0$. Let $f_*\colon \pi_r(B, Y) \to \pi_r(X)$ denote the isomorphism induced by f. Then

$$\Delta f_* = \delta'\colon \pi_r(B, Y) \to \pi_{r-1}(Y)\,,$$

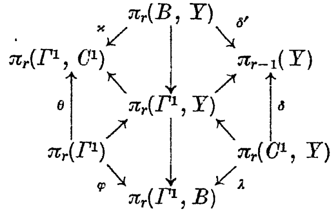
where δ' denotes the boundary operator, and so

$$\xi'\sigma E\Delta f_* = \mu\xi E\delta' = \mu\eta\delta^{-1}\delta'\,,$$

by (1.1). Also $\mu\eta = \eta'\lambda$, by naturality, and so

$$\text{(2.2)} \qquad \xi'\sigma E\Delta f_* = \eta'\lambda\delta^{-1}\delta'\,.$$

Consider next the commutative diagram shown below, where the homomorphisms into $\pi_{r-1}(Y)$ are boundary operators and the others are injections.



The sequences leading through the centre of the hexagon are exact. Also θ and δ are isomorphisms, since C^1 is contractible. Therefore

$$\lambda\delta^{-1}\delta' = -\varphi\theta^{-1}\varkappa\,,$$

by the hexagonal lemma ((1.15.1) of [3]). Hence and from (2.2) we obtain that

$$\text{(2.3)} \qquad \xi'\sigma E\Delta f_* = -\eta'\varphi\theta^{-1}\varkappa\,.$$

Finally we consider the diagram shown below, where ψ is the injection and $g_\#$ is the isomorphism induced by $g\,|\,\Gamma^1$.

$$\begin{array}{ccccc} \pi_r(\Gamma^1, C^1) & \xleftarrow{\theta} & \pi_r(\Gamma^1) & \xrightarrow{\varphi} & \pi_r(\Gamma^1, B) \\ \downarrow{\scriptstyle g_\#} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\eta'} \\ \pi_r(X) & \xrightarrow[h^1_*]{} & \pi_r(\Sigma B) & \xrightarrow[\xi']{} & \pi_r(\Sigma B, \Gamma^0) \end{array}$$

Commutativity on the right is obvious; on the left it holds because h^1 a cross-section whose image lies in Γ^1. Thus

$$\eta'\varphi = \xi'\psi = \xi' h^1_* g_\#\theta\,,$$

and so

$$\eta'\varphi\theta^{-1}\varkappa = \xi' h^1_* g_\#\varkappa = \xi' h^1_* f_*\,,$$

since $f = g\,|\,B$. Hence and from (2.3),

$$\xi'\sigma E\Delta f_* = -\xi' h^1_* f_*\,.$$

Since f_* is an isomorphism, this completes the proof of (2.1b), and consequently the proof of (1.2).

3. Sphere-bundles over spheres. Let B be an oriented $(q-1)$-sphere bundle over S^n, where $n, q \geqslant 2$. Then B determines an element $\chi \in \pi_{n-1}(R_q)$, as described in § 1 of [9], where R_q denotes the rotation group in euclidean q-space. Bundles which determine the same χ are equivalent as sphere-bundles, and so the transgression operator

$$\Delta\colon \pi_r(S^n) \to \pi_{r-1}(S^{q-1})$$

in the fibre homotopy sequence can properly be regarded as a function of χ. The action of R_q on S^{q-1}, given by rotation of the basepoint, carries χ into $\theta = \Delta\iota_n \in \pi_{n-1}(S^{q-1})$. We shall obtain an expression for

$$E\Delta\colon \pi_r(S^n) \to \pi_r(S^q)$$

in terms of the elements

$$\lambda = E\theta \in \pi_n(S^q)\,, \qquad \mu = -J\chi \in \pi_{n+q-1}(S^q)\,,$$

where $J\chi$ is obtained from χ by means of the Hopf construction (see § 3 of [10]).

Let $\pi_r'(S^q)$ denote the subgroup $\pi_r(S^q)$ consisting of elements which are obtainable by the Hopf construction. We compute the homomorphism

$$\varphi\colon \pi_r(S^q) \to \pi_{r+n-1}(S^q)\,,$$

on this subgroup by proving

THEOREM (3.1). *Write $\nu = -[\mu, \iota_q] \in \pi_{n+2q-2}(S^q)$. Let $\beta \in \pi_r'(S^q)$ have generalized Hopf invariant $\beta' = H_0(\beta) \in \pi_r(S^{2q-1})$. Then*

$$(-1)^{(n+1)(q+r)}\varphi(\beta) = \mu \circ E^{n-1}\beta + \nu \circ E^{n-1}\beta'\,.$$

According to § 7 of [9], B can be described as the neighbourhood bundle of ΣB determined by the cross-section h^1. Since ξ is the class of h^1, it follows from (3.7) of [9] that $[\xi, \eta] = \sigma(\mu)$, where $\eta = \sigma(\iota_q)$. By definition

$$\sigma\varphi(\beta) = [\xi, \sigma(\beta)] = [\xi, \eta \circ \beta]\,,$$

to which we apply the Barcus-Barratt formula (7.4) of [1]. Since $\beta \in \pi_r'(S^q)$, the higher generalized Hopf invariants $H_1, H_2, \ldots$ are trivial, by Lemma 4 of [5], and so we obtain (²)

$$\begin{aligned}(-1)^{(n+1)(q+r)}[\xi, \eta \circ \beta] &= [\xi, \eta] \circ E^{n-1}\beta - \big[[\xi, \mu], \eta\big] \circ E^{n-1}\beta' \\ &= \sigma(\mu) \circ E^{n-1}\beta + \sigma(\nu) \circ E^{n-1}\beta'\,.\end{aligned}$$

(²) Some sign changes are necessary because the conventions of [1] are different from those of [9].

Since σ is a monomorphism, this completes the proof of (3.1). Further use of the Barcus-Barratt formula enables us to compute $H_0(\nu)$ and hence obtain an expression for $H_0\varphi(\beta)$. Moreover, it follows from the remarks on the lower part of page 76 of [6] and from (3.1) that

$$\text{(3.2)} \qquad \varphi\pi_r'(S^q) \subset \pi_{r+n-1}'(S^q)\,,$$

and so by iteration of (3.1) we can compute the "structural constants" $\lambda_1, \lambda_2, \ldots$ in (1.6). Notice, furthermore, that the remainder in (1.6) is trivial, since by (6.10) of [4] the quadruple Whitehead products are zero in the homotopy groups of S^q. The triple Whitehead products are zero when q is odd.

Suppose, to give an example, that $E^{n-1}\mu = 0$. This condition is satisfied by the Stiefel manifolds of 2-frames (see § 1 of [9]) when represented as sphere-bundles over spheres in the usual way. Repeated application of (3.1) shows that $\lambda_m = 0$ for $m \geqslant 4$, that $[\lambda_l, \lambda_m] = 0$ for $m \geqslant 3$, and that $\big[\lambda_k, [\lambda_l, \lambda_m]\big] = 0$ for $m \geqslant 2$. Thus (1.6) reduces to the relation

$$\text{(3.3)} \qquad E\Delta(\alpha) = \lambda \circ \alpha + \lambda_2 \circ H_0(\alpha) + \lambda_3 \circ H_1(\alpha) + [\lambda_1, \lambda_2] \circ H_2(\alpha)\,,$$

under the condition $E^{n-1}\mu = 0$.

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