Axioms of multiple choice

by

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Throughout this paper let $x \precsim y$ stand for "There exists a one-one mapping of x into y"; let m, n be variables ranging over the positive integers and let i, j be variables ranging over the non-negative integers (1).

Consider the following statement.

$Z(n)$: *On every set x of non-void sets there exists a function f such that for every $y \in x$ $f(y) \subseteq y$ and $0 \neq f(y) \precsim n$.*

$Z(1)$ is obviously equivalent to the axiom of choice. It will be proved that, for every n, $Z(n)$ is equivalent to the axiom of choice, and so is even $(\exists n)Z(n)$. It will also be shown that this answers affirmatively the question raised by Chang [1] whether the maximum principles ν_n mentioned there are equivalent to the axiom of choice.

On the other hand, it will be shown that the statement

$Z(\infty)$: *On every set x of non-void sets there exists a function f such that for every $y \in x$ $0 \neq f(y) \subseteq y$ and $f(y)$ is finite*

(which obviously follows from the axiom of choice) does not imply the axiom of choice (in a suitable system of set theory). Moreover, it is easily seen that $Z(\infty)$ in conjunction with

$C(\infty)$: *On every set x all of whose members are finite but not void there exists a function g such that $g(y) \in y$ for every $y \in x$*

is equivalent to the axiom of choice; however, it will be shown that for

$C(n)$: *On every set x all of whose members have exactly n members there exists a function g such that $g(y) \in y$ for every $y \in x$*

the conjunction $Z(\infty) \wedge (\forall n) C(n)$ does not imply the axiom of choice. An immediate corollary of this fact is that $Z(\infty) \wedge (\forall n) C(n)$ does not imply $C(\infty)$, and, *a fortiori*, $(\forall n) C(n)$ does not imply $C(\infty)$. The methods

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(1) An ordinal (and, in pariticular, a non-negative integer) in taken to be the set of all smaller ordinals.

used in the metamathematical part of this paper originate with Fraenkel [2] and were developed by Mostowski ([6], [7]).

We write

$W(n)$: *For every set z there exists an ordinal α and a function h on α such that $h(\beta) \precsim n$ for every $\beta < \alpha$ and $\bigcup_{\beta<\alpha} h(\beta) = z$.*

LEMMA 1. $(\forall n)\big(Z(n) \Longleftrightarrow W(n)\big)$.

Proof. Assume $Z(n)$. Let z be any set and let $P(z)$ be its power set. By $Z(n)$ there is a function f on $P(z)-\{0\}$ such that for every $y \in P(z)-\{0\}$ $f(y) \subseteq y$ and $0 \neq f(y) \precsim n$. We define h_β by transfinite recursion as follows. $h_\beta = f(z - \bigcup_{\gamma<\beta} h_\gamma)$ if $z - \bigcup_{\gamma<\beta} h_\gamma \neq 0$ and $h_\beta = 0$ if $z - \bigcup_{\gamma<\beta} h_\gamma = 0$. If for every β $h_\beta \neq 0$ then, since, as one easily sees, for $\beta \neq \gamma$ $h_\beta \cap h_\gamma = 0$ and hence $h_\beta \neq h_\gamma$, we have a one-one mapping of the class of all ordinals into $P(z)$, which is a contradiction. Therefore there is an ordinal α such that $h_\alpha = 0$ and hence $z - \bigcup_{\gamma<\alpha} h_\gamma = 0$, $z = \bigcup_{\gamma<\alpha} h_\gamma$. The function $h = \{\langle \beta, h_\beta\rangle \mid \beta < \alpha\}$ on α obviously satisfies the requirements of $W(n)$.

Assume $W(n)$; let x be a set of non-void sets and let $U(x)$ be its union set. By $W(n)$ there is an ordinal α and a function h on α such that $h(\beta) \precsim n$ for every $\beta < \alpha$ and $\bigcup_{\beta<\alpha} h(\beta) = U(x)$. If $y \in x$ then $y = y \cap U(x) = \bigcup_{\beta<\alpha}\big(y \cap h(\beta)\big)$; since $y \neq 0$, there exists a β such that $y \cap h(\beta) \neq 0$. For $y \in x$ put $f(y) = y \cap h(\beta_y)$ where β_y is the least β such that $y \cap h(\beta) \neq 0$. The function f thus defined obviously satisfies the requirements of $Z(n)$.

THEOREM 2. $(\exists n) Z(n) \Longleftrightarrow Z(1)$.

Proof. Obviously $Z(1) \rightarrow (\exists n) Z(n)$. Assume $(\exists n) Z(n)$ and let m be the least n such that $Z(n)$. We want to prove $m = 1$. Assume $m > 1$, then $\sim Z(m-1)$. Let z be any set; by $W(m)$, which follows from $Z(m)$ by Lemma 1, there is an ordinal α and a function h such that $h(\beta) \precsim m$ for $\beta < \alpha$ and $\bigcup_{\beta<\alpha} h(\beta) = z$. Let $x = \{h(\beta) \times h(\gamma) \mid \beta, \gamma < \alpha\}$ (where $a \times b = \{\langle u, v\rangle \mid u \in a \wedge v \in b\}$). By $Z(m)$ there is a function f on $x - \{0\}$ such that $f(y) \subseteq y$ and $0 \neq f(y) \precsim m$ for $y \in x - \{0\}$. If

(1) *there exists a β such that $h(\beta) \neq 0$ and for all $\gamma < \alpha$ $h(\gamma) = 0$ or $\mathfrak{D}\Big(f\big(h(\beta) \times h(\gamma)\big)\Big)$ has exactly m members*

(where $\mathfrak{D}(u) = \{v \mid (\exists w)(\langle v, w\rangle \in u)\}$), then for each such β and any $\gamma < \alpha$ for which $h(\gamma) \neq 0$ since $h(\beta) \precsim m$ and $\mathfrak{D}\Big(f\big(h(\beta) \times h(\gamma)\big)\Big) \subseteq h(\beta)$, we have $\mathfrak{D}\Big(f\big(h(\beta) \times h(\gamma)\big)\Big) = h(\beta)$ and hence $h(\beta)$ has exactly m members; since also $f\big(h(\beta) \times h(\gamma)\big) \precsim m$, $f\big(h(\beta) \times h(\gamma)\big)$ is a function mapping $h(\beta)$ into $h(\gamma)$. Fixing one β as in (1) and a member t of $h(\beta)$ we define, for $\gamma < \alpha$, $h_1(\gamma) = f\big(h(\beta) \times h(\gamma)\big)(t)$ if $h(\gamma) \neq 0$ and $h_1(\gamma) = 0$ if $h(\gamma) = 0$; $h_2(\gamma) = h(\gamma) - h_1(\gamma)$. We have, for $\gamma < \alpha$, $h_1(\gamma) \precsim 1 \precsim m-1$, $h_2(\gamma) \precsim m-1$ (since if $h(\gamma) = 0$ then $h_2(\gamma) = 0$ and if $0 \neq h(\gamma) \precsim m$ then $h_1(\gamma)$ is a singleton and $h_2(\gamma) = h(\gamma) - h_1(\gamma) \precsim m-1$) and $\bigcup_{\gamma<\alpha}\big(h_1(\gamma) \cup h_2(\gamma)\big) = \bigcup_{\gamma<\alpha} h(\gamma) = z$. If, on the other hand,

(2) *for every β, $h(\beta) = 0$ or there exists a γ such that $h(\gamma) \neq 0$ and $\mathfrak{D}\Big(f\big(h(\beta) \times h(\gamma)\big)\Big) \precsim m-1$*

then for every $\beta < \alpha$ such that $h(\beta) \neq 0$ let γ_β be the least such γ. Put now, for $\beta < \alpha$, $h_1(\beta) = \mathfrak{D}\Big(f\big(h(\beta) \times h(\gamma_\beta)\big)\Big)$ if $h(\beta) \neq 0$ and $h_1(\beta) = 0$ if $h(\beta) = 0$; $h_2(\beta) = h(\beta) - h_1(\beta)$. If $h(\beta) \neq 0$ then, by definition of $h_1(\beta)$, $0 \neq h_1(\beta) \precsim m-1$ and, since $h_2(\beta) = h(\beta) - h_1(\beta)$, also $h_2(\beta) \precsim m-1$; if $h(\beta) = 0$ then $h_1(\beta) = h_2(\beta) = 0 \precsim m-1$. $\bigcup_{\beta<\alpha}\big(h_1(\beta) \cup h_2(\beta)\big) = \bigcup_{\beta<\alpha} h(\beta) = z$. In both cases (1) and (2) we define a function h^* on $\alpha+\alpha$ by $h^*(\mu) = h_1(\mu)$, $h^*(\alpha+\mu) = h_2(\mu)$ for $\mu < \alpha$. By what was shown above about h_1 and h_2 we have $h^*(\mu) \precsim m-1$ for $\mu < \alpha+\alpha$ and $\bigcup_{\mu<\alpha+\alpha} h^*(\mu) = z$. Thus we proved $W(m-1)$, which implies, by Lemma 1, $Z(m-1)$, contradicting our assumption.

Following Chang [1] we say, for $n \geqslant 2$, that a set x is *n-disjointed* if any distinct n members of x have an empty intersection. A subset y of x is said to be a maximal n-disjointed subset of x if y is n-disjointed and is not properly included in any n-disjointed subset of x. We consider the statement

ν_n: *Every set x has a maximal n-disjointed subset.*

As mentioned in Chang [1] the axiom of choice implies $(\forall n \geqslant 2)\nu_n$. The theorem $\nu_2 \rightarrow Z(1)$ of Vaught [9] (and its proof) is generalized as follows.

LEMMA 3. $(\forall n \geqslant 2)\big(\nu_n \rightarrow Z(n-1)\big)$.

Proof. Let x be a set of non-void sets. We have to prove that there exists a function f on x such that for every $y \in x$ $f(y) \subseteq y$ and $0 \neq f(y) \precsim n-1$. As is well known, we can assume without loss of generality that the set x is disjointed (i.e., 2-disjointed), since otherwise we can replace x by $\{\{\langle y, z\rangle \mid z \in y\} \mid y \in x\}$ etc. Let $u = \big\{\{y, \{t\}\} \mid y \in x \wedge t \in y\big\}$ and let v be a maximal n-disjointed subset of u. We put, for $y \in x$, $f(y) = \big\{t \mid \{y, \{t\}\} \in v\big\}$. As one easily sees, for $y \in x$, $f(y) \subseteq y$ since $v \subseteq u$, $f(y) \precsim n-1$ since v is n-disjointed and $f(y) \neq 0$ since x is disjointed, $y \neq 0$ and v is maximal.

By Lemma 3, Theorem 2 and what was mentioned in Chang [1] we have

THEOREM 4. $(\exists n \geqslant 2)\nu_n$ *is equivalent to the axiom of choice.*

For the metamathematical part of the paper we need the following Lemma 5. In Lemma 5, as well as in the refutation of (10) and the proof of (12) within the proof of Theorem 6, Mostowski [7] is followed closely. Let $p_0, p_1, \ldots$ be the primes in their natural order. If H is a group and L a subgroup of H we denote with $\mathrm{Ind}(H/L)$ the number of (left) cosets of H over L.

LEMMA 5. *Let I_{p_i} be the cyclic group of order p_i and $G_j = \prod_{i\geqslant j} I_{p_i}$ (Π stands for the weak direct product). For every n there exists a j such that for all subgroups H of G_j and L of H $L = H$ or $\mathrm{Ind}(H/L) > n$.*

Proof. For a given n take j to be the least such that $p_j > n$. Assume $1 < \mathrm{Ind}(H/L) = l \leqslant n$. Let $H = L + \varphi^{(1)}L + \ldots + \varphi^{(l-1)}L$ be the decomposition of H in cosets of L. Every $\varphi^{(i)}$ is a sequence $\langle \varphi_j^{(i)}, \varphi_{j+1}^{(i)}, \ldots\rangle$ where almost all $\varphi_k^{(i)}$ are the respective units 1_k of I_k. Suppose that $\varphi_k^{(i)} = 1_k$ for $k > q_i$ and let q be the greatest of the numbers $q_1, \ldots, q_{l-1}$. Let H^* be the subgroup of H containing all φ's such that $\varphi_{q+m} = 1_{q+m}$ for all m and let L^* be $H^* \cap L$. $\varphi^{(1)}, \ldots, \varphi^{(l-1)} \in H^*$. We shall show that the decomposition of H^* in cosets of L^* is $H^* = L^* + \varphi^{(1)}L^* + \ldots + \varphi^{(l-1)}L^*$. If $\varphi \in H^*$ then $\varphi \in H$ and hence $\varphi \in \varphi^{(i)}L$ for some $i < l$ ($\varphi^{(0)} = 1$), i.e., $\varphi = \varphi^{(i)}\psi$ where $\psi \in L$. But since $\varphi, \varphi^{(i)} \in H^*$, one has $\psi = \varphi^{(i)-1}\varphi \in H^*$, thus $\psi \in L^*$ and $\varphi \in \varphi^{(i)}L^*$. Hence $H^* = L^* + \varphi^{(1)}L^* + \ldots + \varphi^{(l-1)}L^*$. These cosets are disjoint because they are included in the respective L-cosets. Thus we have $\mathrm{Ind}(H^*/L^*) = l$. The projection of H^* into $\prod_{j\leqslant i\leqslant q} I_{p_i}$ is an isomorphism of H^* onto a subgroup H^{**} of $\prod_{j\leqslant i\leqslant q} I_{p_i}$. Let L^{**} be the projection of L^*. Thus $l = \mathrm{Ind}(H^*/L^*) = \mathrm{Ind}(H^{**}/L^{**})$ is a divisor of $\prod_{j\leqslant i\leqslant q} p_i$ and hence $l = 1$ or $l \geqslant p_j > n$, contradicting $1 < l \leqslant n$.

Let $\mathfrak{S}$ be the set theory given in Mostowski [6]. This is a set theory of the Bernays-Gödel type which permits the existence of urelements and which does not have the axiom of choice among its axioms.

THEOREM 6. *If $\mathfrak{S}$ (or the system A, B, C of Gödel [3]) is consistent then $Z(\infty)\wedge(\nabla n)C(n)\rightarrow Z(1)$ is unprovable in $\mathfrak{S}$* [1].

Proof. If $\mathfrak{S}$ is consistent then by the construction of Gödel [3] we know that the system A, B, C, D, E of [3] is consistent. The same conclusion is obtained also from the assumption that the system A, B, C of [3] is consistent (see the remark preceeding axiom D on p. 6 of [3] and also Shepherdson [8], pp. 164-165). From the consistency of the system A, B, C, D, E, taking the group $\mathfrak{G}$ in Mostowski [6] to be the one element group, we get the consistency of a system $\mathfrak{S}^*$ which is $\mathfrak{S}$ with the axiom of choice and the axiom "There exists a denumerable set of urelements" added. To show that $Z(\infty)\wedge(\nabla n)C(n)\rightarrow Z(1)$ is unprovable in $\mathfrak{S}$ we shall prove that the system $\mathfrak{S}+\{Z(\infty), (\nabla n)C(n), \sim Z(1)\}$ is consistent by giving an interpretation of the latter system in $\mathfrak{S}^*$. In order to do it we shall now proceed within $\mathfrak{S}^*$.

Let K be a denumerable set of urelements. We define $K_0 = K$, $K_\eta = K \cup \bigcup_{\mu<\eta} P(K_\mu)$, where $P(x)$ is the power set of x. If there exists an η such that $x \in K_\eta$ we say that x is a K-element. Let Φ be the group of all permutations of K. For $\varphi \in \Phi$ and a K-element x we define $\varphi(x)$ by recursion as follows.

(3) *For $x \in K$ $\varphi(x)$ is already defined; for $x \notin K$ $\varphi(x) = \{\varphi(y) \mid y \in x\}$* [2].

One can easily verify (see [6]) that

(4) $$(\varphi\psi)(x) = \varphi(\psi(x)), \quad 1(x) = x \quad \text{(where 1 is the identity)},$$

(5) $$\varphi(x \cap y) = \varphi(x) \cap \varphi(y).$$

Since K is denumerable, there is a one-one correspondence between the denumerable set $\{\langle p_i, q\rangle \mid i \in \omega \wedge q < p_i\}$ and K. Let $k_{i,q}$ be the member of K corresponding to $\langle p_i, q\rangle$, thus $K = \{k_{i,q} \mid i \in \omega \wedge q < p_i\}$. Let $K^{(i)} = \{k_{i,q} \mid q < p_i\}$; let χ_i be that member of Φ for which $\chi_i(k_{j,q}) = k_{j,q}$ for $j \neq i$, $\chi_i(k_{i,q}) = k_{i,q+1}$ for $q < p_i - 1$, and $\chi_i(k_{i,p_i-1}) = k_{i,0}$. Let Ψ be the subgroup of Φ generated by $\{\chi_i \mid i \in \omega\}$. Let the variables a, b, c range over the set of finite subsets of ω. A permutation $\varphi \in \Psi$ is said to be *b-identical* if $\varphi(k_{i,q}) = k_{i,q}$ for every $i \in b$ and $q < p_i$; the group of all b-identical members of Ψ will be denoted with Ψ^b. An element x is said to be *b-symmetric* if x is a K-element and $\varphi(x) = x$ for every $\varphi \in \Psi^b$. We define M_η by recursion as follows.

(6) $$\begin{cases} M_0 = K \text{ and for } \eta > 0 \\ x \in M_\eta \Longleftrightarrow (\nabla y \in x)(\exists \xi < \eta)(y \in M_\xi) \wedge (\exists b)\ (x \text{ is } b\text{-symmetric}). \end{cases}$$

If there exists an η such that $x \in M_\eta$ we say that x is an M-element. By (6) x is an M-element if and only if every member of x is an M-element and for some b x is b-symmetric. A class X is called an *M-class* if $(\nabla y \in X)$ *(y is an M-element)* $\wedge(\exists b)(\nabla\varphi \in \Psi^b)(\nabla y \in X)(\varphi(y) \in X)$. One can easily verify that

(7) $$M_0 = K \cup \{0\},$$

(8) $$x \in M_\eta \wedge \varphi \in \Psi \rightarrow \varphi(x) \in M_\eta,$$

(9) $$x, y \in M_\eta \rightarrow \{x, y\} \in M_{\eta+1} \wedge \langle x, y\rangle \in M_{\eta+2}.$$

[1] By the method of Mendelson [5] one can prove Theorem 6 with the conclusion replaced by "$Z(\infty)\wedge(\nabla n)C(n)\rightarrow Z(1)$ is unprovable in the system A, B, C of Gödel [3]".

[2] Actually, a new term with two variables, φ and x, is defined here and notation different from the functional notation $\varphi(x)$ should be used, but since no confusion can arise we use the same notation.

As shown, essentially, by Mostowski [6], if we interpret $\mathfrak{S}$ in $\mathfrak{S}^*$ by replacing the primitive notions '*element*', '*class*', 'ϵ' and '0' of $\mathfrak{S}$ by the notions 'M-*element*', 'M-*class*', 'ϵ' and '0' of $\mathfrak{S}^*$, respectively, all the axioms (and the theorems) of $\mathfrak{S}$ will go over to theorems of $\mathfrak{S}^*$. In order to prove the consistency of $\mathfrak{S}+\{Z(\infty), (\forall n)C(n), \sim Z(1)\}$ we have to show that $Z(\infty)$ and $(\forall n)C(n)$ go over to theorems of $\mathfrak{S}^*$, whereas $Z(1)$ goes over to a sentence refutable in $\mathfrak{S}^*$. Using the standard absoluteness results, which are proved in Mostowski [6], we see that $Z(1)$ goes over to a sentence equivalent to

(10) *For every* M-*element* x *which is a set of non-void sets there is an* M-*element* f *which is a function such that for every* $y \in x$ $f(y) \in y$,

that $Z(\infty)$ goes over to a sentence equivalent to

(11) *For every* M-*element* x *which is a set of non-void sets there is an* M-*element* f *which is a function such that for every* $y \in x$ $0 \neq f(y) \subseteq y$ *and* $f(y)$ *is finite*

(here we use the fact that the notion of finiteness is absolute with respect to this interpretation—see Lemma 1 of [4]), and that $(\forall n)C(n)$ goes over to a sentence equivalent to

(12) *For every* n *and for every* M-*element* x *which is a set of sets which have exactly* n *members each there exists an* M-*element* f *which is a function such that for every* $y \in x$ $f(y) \in y$.

To refute (10) take for x the set $\{K^{(i)} \mid i \in \omega\}$ which, by (7), is easily seen to be in M_2. Let f be as in (10); then, since f is an M-element, there is a b such that f is b-symmetric. Let $j \notin b$; then, since χ_j is b-identical, $\chi_j(f)=f$. Since $f(K^{(j)}) \in K^{(j)}$, we have $f(K^{(j)}) = k_{j,q}$ for some $q < p_j$, i.e., $\langle K^{(j)}, k_{j,q}\rangle \in f$ and hence, by (3), $\langle K^{(j)}, \chi_j(k_{j,q})\rangle = \chi_j(\langle K^{(j)}, k_{j,q}\rangle) \in \chi_j(f) = f$. By definition of χ_j, $\chi_j(k_{j,q}) \neq k_{j,q}$ and hence $\langle K^{(j)}, k_{j,q}\rangle, \langle K^{(j)}, \chi_j(k_{j,q})\rangle \in f$ contradicts the assumption that f is a function.

Now we shall prove (11). Let u, v be K-elements; we define $u \sim_b v \Leftrightarrow (\exists \varphi \in \Psi^b)(\varphi(u) = v)$. One can easily see, by (4), that $\sim_b$ is reflexive, symmetric and transitive. For a K-element u let $E_b(u) = \{v \mid v \sim_b u\}$. Let

$$\Psi_c = \{\varphi \mid \varphi \in \Psi \wedge (\forall i)(\forall q)\big(i \notin c \wedge q < p_i \to \varphi(k_{i,q}) = k_{i,q}\big)\}.$$

Let $\psi \in \Psi^b$, then $\psi = \tau\sigma$ where τ and σ are defined by

$$i \in a \wedge q < p_i \to \sigma(k_{i,q}) = k_{i,q} \wedge \tau(k_{i,q}) = \psi(k_{i,q}),$$

$$i \notin a \wedge q < p_i \to \sigma(k_{i,q}) = \psi(k_{i,q}) \wedge \tau(k_{i,q}) = k_{i,q}.$$

Obviously $\sigma \in \Psi^{a \cup b}$ and $\tau \in \Psi_{a-b}$. Let u be an M-element, then u is a-symmetric for some a. For $\psi \in \Psi^b$ we have, by (4), $\psi(u) = (\tau\sigma)(u) = \tau(\sigma(u)) = \tau(u)$ (where τ and σ are as above) and hence $E_b(u) = \{\psi(u) \mid \psi \in \Psi^b\} = \{\tau(u) \mid \tau \in \Psi_{a-b}\}$. It is easily seen that Ψ_{a-b} has exactly $\prod_{i \in a-b} p_i$ members and hence $E_b(u)$ is finite. Since u is an M-element, then, by (8), all the members of $E_b(u)$ are M-elements. Let $\varphi \in \Psi^b$

$$\varphi(E_b(u)) = \{\varphi(\psi(u)) \mid \psi \in \Psi^b\} = \{(\varphi\psi)(u) \mid \psi \in \Psi^b\}$$

and since Ψ^b is a group, the right-hand side is $\{\psi(u) \mid \psi \in \Psi^b\} = E_b(u)$. Thus $E_b(u)$ is b-symmetric and hence $E_b(u)$ is an M-element.

Given a set x as in (11), since x is an M-element, there is a b such that x is b-symmetric. Let $z = \{U(E_b(t)) \mid t \in x\}$. By the axiom of choice (in $\mathfrak{S}^*$) there exists a function g on z such that $g(s) \in s$ for each $s \in z$. Let

$$f = \Big\{\Big\langle y, E_b\Big(g\big(U(E_b(y))\big)\Big) \cap y\Big\rangle \mid y \in x\Big\}.$$

For $y \in x$, $g\big(U(E_b(y))\big) \in U(E_b(y))$, hence there is some $\varphi \in \Psi^b$ such that $g\big(U(E_b(y))\big) \in \varphi(y)$. By (8), $\varphi(y)$ is an M-element and hence $g\big(U(E_b(y))\big)$ is also an M-element and therefore $E_b\Big(g\big(U(E_b(y))\big)\Big)$ is an M-element. Since, as is easily seen, if u is a-symmetric and v is c-symmetric then, by (5), $u \cap v$ is $a \cup c$-symmetric, the intersection of two sets which are M-elements is an M-element. For $y \in x$, y is an M-element and therefore, by what was just said and by (9), it follows that all the members of f are M-elements. To show that f itself is an M-element we still have to show that f is b-symmetric. For this purpose it is enough to show that for every $\varphi \in \Psi^b$ $\varphi(f) \subseteq f$, since then, for $\varphi \in \Psi^b$, $\varphi^{-1} \in \Psi^b$ and

$$f = 1(f) = (\varphi^{-1}\varphi)(f) = \varphi^{-1}(\varphi(f)) \subseteq \varphi(f) \subseteq f$$

and hence $\varphi(f) = f$. By (3) and (5)

$$\varphi(f) = \Big\{\Big\langle \varphi(y), \varphi\Big(E_b\Big(g\big(U(E_b(y))\big)\Big)\Big) \cap \varphi(y)\Big\rangle \mid y \in x\Big\}$$

and since $E_b\Big(g\big(U(E_b(y))\big)\Big)$ is b-symmetric,

$$\varphi(f) = \Big\{\Big\langle \varphi(y), E_b\Big(g\big(U(E_b(y))\big)\Big) \cap \varphi(y)\Big\rangle \mid y \in x\Big\}.$$

For a fixed $y \in x$ put $\varphi(y) = z$. Since $\varphi \in \Psi^b$ and x is b-symmetric, we have $z \in x$. Since $z \sim_b y$, we have $E_b(z) = E_b(y)$. Thus

$$\Big\langle \varphi(y), E_b\Big(g\big(U(E_b(y))\big)\Big) \cap \varphi(y)\Big\rangle = \Big\langle z, E_b\Big(g\big(U(E_b(z))\big)\Big) \cap z\Big\rangle \in f$$

and hence $\varphi(f) \subseteq f$.

We still have to show that, for $y \in x$, $0 \neq f(y) \subseteq y$ and $f(y)$ is finite. $f(y) \subseteq y$ follows directly from the definition of f. As was mentioned earlier, there is a $\varphi \in \Psi^b$ such that $g\big(U(E_b(y))\big) \in \varphi(y)$ hence, by (3) and (4), $\varphi^{-1}\Big(g\big(U(E_b(y))\big)\Big) \in y$. But, since $\varphi^{-1} \in \Psi^b$, $\varphi^{-1}\Big(g\big(U(E_b(y))\big)\Big) \in E_b\Big(g\big(U(E_b(y))\big)\Big)$, hence $\varphi^{-1}\Big(g\big(U(E_b(y))\big)\Big) \in f(y)$, i.e., $f(y) \neq 0$. $f(y)$ is finite since $g\big(U(E_b(y))\big)$ is an M-element and, as was remarked above, for every M-element u $E_b(u)$ is finite and hence $f(y) \subseteq E_b\Big(g\big(U(E_b(y))\big)\Big)$ is finite.

To prove (12) let n and x be as in (12). Since x is an M-element, x is b-symmetric for some b. For the given n let j be as in Lemma 5. Without loss of generality we can assume that $j \subseteq b$ (since x is, *a fortiori*, $b \cup j$-symmetric). Since x is b-symmetric, $u \in x \wedge u \sim_b v \rightarrow v \in x$ (by (3)) and hence x is the union of a set of $\sim_b$-equivalence classes. By the axiom of choice there is a subset z of x which contains exactly one representative of each equivalence class and hence $x = \bigcup_{y \in z} E_b(y)$. Since $z \subseteq x$, every member y of z has n members. By the axiom of choice there is a function g on z such that for every $y \in z$ $g(y) \in y$. Let $H_y = \{\varphi \mid \varphi \in \Psi^b \wedge \varphi(y) = y\}$ and let $L_y = \{\varphi \mid \varphi \in H_y \wedge \varphi\big(g(y)\big) = g(y)\}$. By (3), if $\tau \in H_y$ then $\tau\big(g(y)\big) \in y$. If τ and σ are respective members of distinct left cosets of H_y over L_y then $\tau\big(g(y)\big) \neq \sigma\big(g(y)\big)$ (since if $\tau\big(g(y)\big) = \sigma\big(g(y)\big)$ then, by (4), $g(y) = (\tau^{-1}\sigma)\big(g(y)\big)$ and $\tau^{-1}\sigma \in L_y$, which is impossible) and hence y has at least $\mathrm{Ind}(H_y/L_y)$ members, i.e., $\mathrm{Ind}(H_y/L_y) \leqslant n$. Since we assumed that $j \subseteq b$, we have $H_y \subseteq \Psi^b \subseteq \Psi^j$. Ψ^j is easily seen to be isomorphic to $\prod_{i \geqslant j} I_{p_i}$ and hence, by Lemma 5 and $\mathrm{Ind}(H_y/L_y) \leqslant n$, $L_y = H_y$. Put $f = \big\{\langle \varphi(y), \varphi\big(g(y)\big)\rangle \mid y \in z \wedge \varphi \in \Psi^b\big\}$. Since $g(y) \in y \in x$, y and $g(y)$ are M-elements and hence, by (8) and (9), every member of f is an M-element. Since Ψ^b is a group, f is b-symmetric by (3) and (4). The domain of f is $\{\varphi(y) \mid y \in z \wedge \varphi \in \Psi^b\}$ $= \bigcup_{y \in z} \{\varphi(y) \mid \varphi \in \Psi^b\} = \bigcup_{y \in z} E_b(y) = x$. To see that f is a function let $\varphi(y) = \varphi'(y')$ for $y, y' \in z$, $\varphi, \varphi' \in \Psi^b$, then $y = \varphi^{-1}\varphi'(y')$, i.e., $y \sim_b y'$ and since z contains exactly one representative from each $\sim_b$-equivalence class, we have $y = y'$. Thus $y = \varphi^{-1}\varphi'(y)$, hence $\varphi^{-1}\varphi' \in H_y$ and since $L_y = H_y$, $\varphi^{-1}\varphi'\big(g(y)\big) = g(y)$, hence $\varphi'\big(g(y)\big) = \varphi\big(g(y)\big)$ and f is a function. To complete the proof we note that, by $g(y) \in y$ and (3), we have $\varphi\big(g(y)\big) \in \varphi(y)$.

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