On the order of the points of the n-dimensional Euclidean space

by

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1. In the research into foundations of mathematics the concept of *order* may be defined in various ways. For our purpose it is useful to recall one of the most simple and spontaneous definitions [1].

A set S of points or, more generally, elements, is said to be *ordered* if for the points of S there exists a relationship of *preceding* $<$ (as opposed to a relationship of *following* $>$) with the following characteristics:

(a) *For any two points* A *and* B, *either* $A < B$, *or* $B < A$.

(b) *For no point* A, $A < A$.

(c) *For any three points* A, B, C, *if* $A < B$ *and* $B < C$, *then* $A < C$.

The relationship of preceding can be extended to the subsets of S. The formulas $P < \alpha$ and $\alpha < \beta$ (P being a point, α and β point sets) will mean that P precedes every point of α, and, respectively, every point of α precedes every point of β.

The purpose of this paper is the search for the most general *normal order* of n-dimensional Euclidean space E_n; that is for the most general order of E_n, which satisfies, together with (a), (b), (c), the following condition:

(d) *Given any two points* A *and* B, *if* $A < B$, *any point* C *within the straight segment* AB (*any point* C *between* A *and* B) *satisfies* $A < C < B$.

2. In our research we assume all those axioms which define the *Euclidean affine geometry* of E_n $(n \geqslant 1)$, i.e.:

the generalized (for a generical n) *Euclidean axioms of incidence* (in German *Verknüpfung*),

the *Euclidean axiom of parallelism*,

the *axioms of the Euclidean order* on the straight line,

the so-called *axiom of Pasch*,

Dedekind's axiom of continuity.

(1) The case in question is the *total order*, according to N. Bourbaki's terminology (see *Les structures fondamentales de l'Analyse*, Paris 1939, Actualités Scientif. et Industr. N° 846, p. 34).

3. Lemma I. *Any normal order of E_n is also normal in every subset of E_n.*

No proof is needed.

Lemma II. *On a straight line any normal order must coincide with one of the two opposed natural Euclidean orders.*

This is almost evident.

In fact, let the order of the line be a normal order. Supposing that $A < B$: by hypothesis each point C between A and B is such that $A < C < B$. Furthermore each point D of the Euclidean extension $\overrightarrow{AB}$ of the segment AB (i.e. such that B lies between A and D) is such that $A < B < D$, because: we cannot have $D < A$; otherwise B, which lies between A and D, would also precede A, against the hypothesis $A < B$, and, on the other hand, if $A < D$, the point B, which lies between A and D, must precede D. Thus it can also be shown that each point E of the extension $\overrightarrow{BA}$ of the segment AB is such that $E < A < B$. Therefore, if the mutual order of two points of the line is known, its order is completely characterized and coincides with the corresponding Euclidean order.

From Lemmas I and II easily follows:

Lemma III. *Any normal order of E_n subordinates on a straight line one of the two opposed Euclidean orders.*

Definitions. Let an order of E_n and a pencil $\mathfrak{P}$ of parallel $(n-1)$-dimensional hyperplanes of E_n be given; we say that $\mathfrak{P}$ *is ordered according to one of the two opposed Euclidean orders*, if:

a) for any two hyperplanes α and β of $\mathfrak{P}$, either $\alpha < \beta$ or $\beta < \alpha$,

b) any line of E_n intersecting the hyperplanes of $\mathfrak{P}$ is ordered according to one of the two Euclidean orders.

Besides we say that E_n (for $n > 1$) is ordered *by parallel $(n-1)$-dimensional hyperplanes* if there exists a pencil $\mathfrak{P}$ of parallel $(n-1)$-dimensional hyperplanes with an Euclidean ordering.

Lemma IV. *If for a normal order of E_n $(n > 1)$ a $(n-1)$-dimensional hyperplane α precedes another $(n-1)$-dimensional hyperplane β, parallel to it, the (improper) pencil which comprises α and β has the one Euclidean order for which $\alpha < \beta$; then E_n is ordered by $(n-1)$-dimensional hyperplanes parallel to α and β.*

In order to prove the lemma, let us consider any two hyperplanes γ and δ, parallel to α and β, and any two lines r, r', not parallel to them. If we call A, B, C, D the points of intersection of α, β, γ, δ with r, and A', B', C', D' the points of intersection of α, β, γ, δ with r', we observe that $A < B$ and $A' < B'$; then also either $C < D$ and $C' < D'$, or $D < C$ and $D' < C'$ (Lemma III); that is either $\gamma < \delta$ or $\delta < \gamma$. Therefore Lemma III allows us to assert Lemma IV.

4. Theorem. *For $n > 1$, any normal order of E_n is an order by parallel $(n-1)$-dimensional hyperplanes.*

The following pages of the present paper include the inductive proof of the above-mentioned theorem.

(I) *Any normal order of the Euclidean plane E_2 is an order by parallel lines* (²).

Let the order of the plane be a normal order.

If we consider on the plane any line r and a point P outside of it, then one of the following two cases may occur:

A) $P < r$, or $r < P$,

B) on r there are points which precede P as well as those which follow P.

Case A. Let us suppose for example that $P < r$. Then it can be shown that the line s, passing through P and parallel to r, precedes r. Indeed, if it were not so, there should be a point C of r which precedes a point Q of s. Since $P < C$, we should have $P < Q$. Therefore, taken a point T on the extension $\overrightarrow{CQ}$, we have $P < T$ (Fig. 1) (³). On the other hand, on the extension $\overrightarrow{TP}$ there exists a point D of r. According to the above, $D < P$, while by hypothesis $P < D$. This contradiction proves that $Q < r$. Therefore $s < r$.

In the same way, if $r < P$, it may be shown that $r < s$.

Lemma IV allows us to state that in case A the plane is ordered by lines parallel to r.

Case B. On r there are points which precede P as well as points which follow P. Therefore we obtain a partition of all points of r in two Dedekind's sets, the first of which contains only points which precede P, while the second contains only points which follow P. To these two sets corresponds one (and only one) separation point M, which is either the last point of r preceding P, or the first point of r following P.

Let us consider any point P' outside the lines PM and r; we observe first of all that there is a point M' of r, which is the last point to precede or the first point to follow P'. Otherwise we would return to the case studied before, which is incompatible with this one.

We intend to show that the line PM is parallel to the line $P'M'$.

Let us erroneously suppose that the lines PM and $P'M'$ have a point S in common, and let us consider two cases:

(²) One particular order by parallel lines is the well-known *order of the complex plane*, examined by O. Stolz and J. A. Gmeiner (*Theoretische Arithmetik*, Part II, Leipzig 1902, p. 280).

(³) In the following figures we will replace the symbol $<$ by $\rightarrow$.

Case B1. $M = M' = S$. In this case, if P and P' were on the two opposed sides of r, the segment PP' should meet r at a point Q distinct from $S = M = M'$ (since P' is outside the line PM). We should have $P < Q < P'$ or $P' < Q < P$, according to whether P precedes P' or *vice versa*. But, on the other hand, M and M' coincide; so all points of r preceding P (if any, other than S) should also precede P' and *vice versa*. This means that P and P' should be on the same side of r.

However, it is obvious that they cannot both precede or both follow S. In fact, if we take a point T and a point T', the first on the extension $\overrightarrow{PS}$ and the other on $\overrightarrow{P'S}$, it is evident that one of the two segments PT'

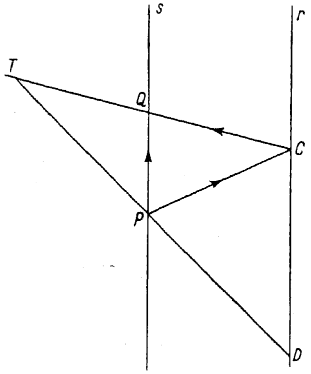

Fig. 1

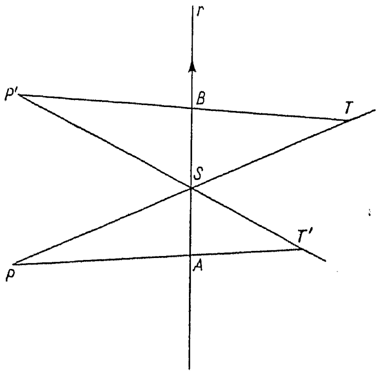

Fig. 2

and $P'T$ meets r at a point A which precedes S, while the other meets r at a point B which follows S (Fig. 2). Now, if $P < S$ and $P' < S$, we should not only obtain $P < T$ and $P' < T'$, but also $P < S < T'$ and $P' < S < T$, and therefore either $P < A$, or $P' < A$, which is obviously absurd. If, on the other hand, $S < P$ and $S < P'$, we should have $T' < P$ and $T < P'$, and consequently $B < P$ or $B < P'$, which is equally absurd.

Therefore we ought to have $P < S < P'$, or else $P' < S < P$. For obvious reasons of symmetry, it is sufficient to show the absurdity of the first of the two cases, in order to exclude both of them.

Since on the line PP' there are points which precede S (like P) as well as points which follow S (like P'), there must be (on the segment PP') a last point N preceding S, or a first point N following S (Fig. 3). It is not difficult to prove that, if T is a point on the line SN and R and R' are any two points lying on the two opposed sides of SN, R on the same side as the extension $\overrightarrow{P'P}$, and R' on the same side as the extension $\overrightarrow{PP'}$, then $R < T < R'$. In fact $S < R'$, because: if the line $R'S$ meets the line

PP' at a point V' of the segment NP' or of the extension $\overrightarrow{PP'}$, the point R' cannot lie on the extension $\overrightarrow{V'S}$ and therefore, in view of $S < V'$, we have $S < R'$; if, on the other hand, the line $R'S$ meets the line PP' at a point V of the segment NP or of the extension $\overrightarrow{P'P}$, the point R' must lie on the extension $\overrightarrow{VS}$, and therefore, in view of $V < S$, we again have $S < R'$; and if finally the line $R'S$ is parallel to the line PP', we can still find a segment $R'_1R'_2$ on the side of $\overrightarrow{PP'}$, which comprises R' and is not parallel to the line PP', and therefore state that S precedes R', as well as R'_1 and R'_2. Thus also $R < S$. From this our assertion follows easily.

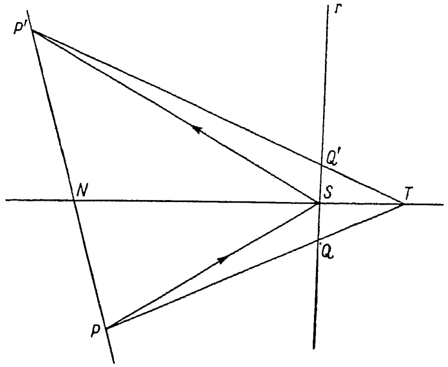

Fig. 3

If we now suppose N to be distinct from P, and we consider a point T on the extension $\overrightarrow{NS}$ (Fig. 3), the segment PT meets r at a point Q, which, on one hand, should follow P, since $P < T$, and, on the other hand, should precede P, since $Q < S$ (S is not preceded on r by any point which follows P). Therefore we have $N = P$. In the same way it may be shown that $N = P'$. We deduce $P = P'$, that is, a contradiction.

Therefore we have to discard Case B1, in which M and M' coincide in S.

Case B2. *M and M' are different.* Let us suppose that $M < M'$, because, if on the contrary we should have $M' < M$, it would be sufficient to change the function of the two couples (P, M) and (P', M').

If there were a point S common to the lines PM and $P'M'$, one of the following three cases would occur:

B2a. $M < S$, $M' < S$,
B2b. $S < M$, $S < M'$,
B2c. $M < S$, $S < M'$.

But we shall now prove that all of them are impossible.

Case B 2a. $M < S$ *and* $M' < S$. In this case there could not be $P < M < S$. Indeed, if it were so, for a given point T of the extension $\overrightarrow{M'S}$ (Fig. 4), the segment PT would meet r at a point Q lying on the extension $\overrightarrow{M'M}$. Now, since $P < S < T$, we should have $P < Q$. But, since $Q < M$, the definition of M calls for $Q < P$. Therefore the absurdity has been proved.

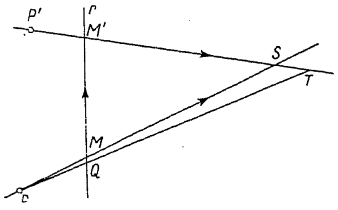

Fig. 4

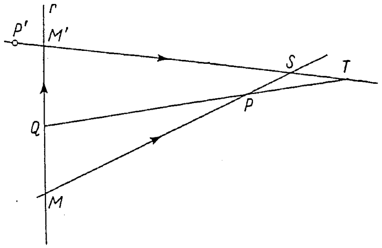

Fig. 5

Neither could we have $M < P < S$. In fact in this case, let us again consider a point T on the extension $\overrightarrow{M'S}$ (Fig 5). We see that the line PT meets r at a point Q of the segment MM', which lies on the extension $\overrightarrow{TP}$. Since $P < S < T$, we should have $Q < P$, which is again in contradiction to the definition of M, namely that $M < Q$, which would call for $P < Q$.

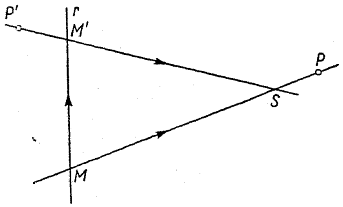

Fig. 6

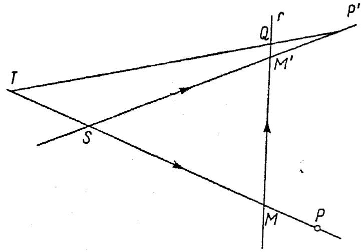

Fig. 7

And finally, there could not be $M < S \leqslant P$, since otherwise from $M' < S$ we could deduce $M' < P$, which is in contradiction to the definition of M, namely that $M < M'$ (Fig. 6), which would call for $P < M'$.

Therefore Case B2a is to be discarded.

Case B2b. $S < M$ *and* $S < M'$. In this case we could not have $S < M' < P'$. In fact, if we should choose instead a point T on the extension $\overrightarrow{MS}$ and call Q the point of intersection of the segment TP' with the line r, since $T < S < P'$, we should have $Q < P'$ (Fig. 7). But, since $M' < Q$, in accordance with the definition of M', we should have the contrary, that is $P' < Q$.

Neither could we have $S < P' < M'$. In fact in this case, if T is a point on the extension $\overrightarrow{MS}$, and if Q is the point of intersection of the segment MM' with the extension $\overrightarrow{TP'}$ (Fig. 8), since $T < S < P'$, we should have $P' < Q$, which is in contradiction to the definition of M', which calls for $Q < P'$.

And finally, we could not have $P' \leqslant S < M'$, because otherwise we should have $P' < M$, in contradiction to the definition of M', namely that $M < M'$, which calls for $M < P'$ (Fig. 9).

Therefore we can also discard Case B2b.

Case B2c. $M < S$ *and* $S < M'$. In this case we observe first of all that there are on r points (like M), which precede S, as well as points

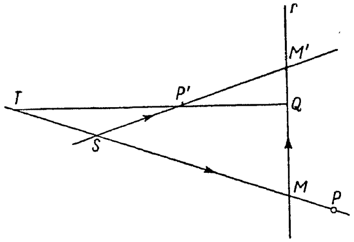

Fig. 8

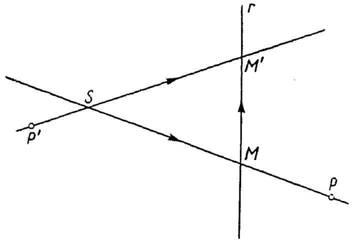

Fig. 9

(like M'), which follow S. Let the point N of the segment MM' be the last point of r to precede S, or the first one to follow S. Taking advantage of the proof made for a perfectly identical case (Case B1), we may state that of the two sides generated in the plane by the line NS, one, which contains the extension $\overrightarrow{M'M}$, consists of points preceding each point of the line NS, while the other one, which comprises the extension $\overrightarrow{MM'}$, consists of points following each point of the line NS.

Let us now suppose N different from M.

Then Case B2c can be divided into three subcases, which are all impossible.

We could not have $P < M < S$; otherwise, if T is a point on the extension $\overrightarrow{NS}$ and Q is the intersection of the segment PT with the extension $\overrightarrow{M'M}$ (Fig. 10), we ought to have $P < T$; then $P < Q$, in contradiction to the definition of M, namely that $Q < M$, which would require $Q < P$.

Accordingly we cannot have $M < P < S$, since otherwise, if T is again a point on the extension $\overrightarrow{NS}$ and Q is the intersection of r with the extension $\overrightarrow{TP}$ (Fig. 11), we should have $P < T$; then $Q < P$, in contradiction to the definition of M, namely that $M < Q$, which implies $P < Q$.

And finally, we cannot have $M < S \leqslant P$; otherwise, if Q is a generical point within the segment MN, we should have $Q < S$, according to the definition of N, and $P < Q$, according to the definition of M; this means that we should have simultaneously $Q < P$ and $P < Q$ (Fig. 12).

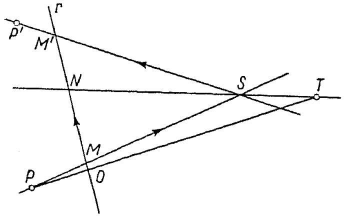

Fig. 10

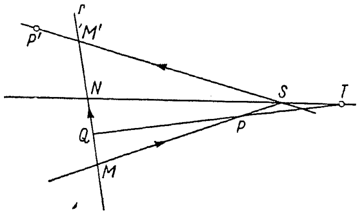

Fig. 11

Analogous contradictions are found if we suppose N different from M' and we study the (only) three subcases into which we can break down Case B2c:

$$S < M' < P', \quad S < P' < M', \quad P' \leqslant S < M'.$$

We deduce that not only $N = M$, but also $N = M'$; then $M = M'$ This contradiction proves that Case B2c must be discarded.

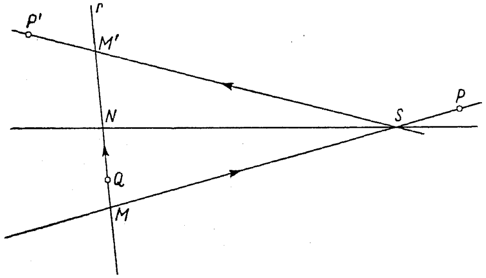

Fig. 12

Therefore, Case B2 also brings us to the conclusion that there cannot exist a point S common to the lines PM and $P'M'$. It follows that these lines are parallel.

Now let us consider any point P'' of the line PM; there certainly exists a last point M'' of r preceding P'' or a first point M'' following P''. Since the latter statement is obviously valid even if we change the point-couple (P, M) into (P', M') and the point-couple (P', M') into (P'', M''), we can conclude that the line $P''M''$ is parallel to $P'M'$. Therefore $P''M''$ coincides with PM and M'' coincides with M. We then deduce that all the points of the line PM, in particular P and P'', are preceded by the points of r which precede M, and are followed by those which follow M. Besides any other line $P'M'$ parallel to PM will have analogous properties with respect to the line r and to the point M'.

Thus we have proved that also in Case B the plane is ordered by parallel lines (Lemma IV). These are the lines parallel to the line PM.

In conclusion our assertion (I) is completely proved.

(II) *If for any m, such that $2 \leqslant m < n$, a normal order of the space E_m is an order by parallel $(m-1)$-dimensional hyperplanes, the same occurs for $m = n$.*

Proof. Let the order of the space E_n be a normal order. We will consider any $(n-1)$-dimensional hyperplane a and any point P outside of it, and we will first show that, if $P < a$, the $(n-1)$-dimensional hyperplane β, passing through P and parallel to a, precedes a.

Let us consider in fact any point $Q \neq P$ of β and any point C of a. The plane π, passing through the points P, Q, C, meets a in a line r passing through C and β in the line PQ parallel to r (Fig. 13 corresponding to case $n = 3$). But since each point of r is preceded by P, in view of Lemma I and of the proof of the case A of assertion (I), π shows an order by lines parallel to r, and the line PQ precedes the line r. Then $Q < C$; that is $\beta < a$.

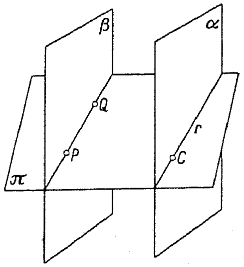

Fig. 13

In the same way we may prove that, if $a < P$, then $a < \beta$.

In these two cases E_n is ordered by $(n-1)$-dimensional hyperplanes parallel to a, as follows immediately from Lemma IV.

Let us suppose now that P neither precedes nor follows a. In view of Lemma I and of the hypothesis of the inductive proof, the hyperplane a is ordered by parallel $(n-2)$-dimensional hyperplanes. The improper pencil $\mathfrak{P}$ of these $(n-2)$-dimensional hyperplanes comprises certainly one (and only one) element μ which separates the hyperplanes which precede P from those which do not precede P (because Dedekind's axiom is valid for any improper pencil with a Euclidean ordering). It is not difficult to prove that the $(n-2)$-dimensional hyperplanes of $\mathfrak{P}$ which follow μ follow also P. Indeed, if there were a $(n-2)$-dimensional hyperplane μ' of $\mathfrak{P}$, following μ, which did not follow the point P, there should be another hyperplane μ'' of $\mathfrak{P}$, such that $\mu < \mu'' < \mu$, which could precede P; but that is absurd. We will prove that the space E_n is ordered by $(n-1)$-dimensional hyperplanes parallel to $P\mu$.

In fact let us consider a generical point P' outside the $(n-1)$-dimensional hyperplanes $P\mu$ and α. Consequently there must be on α an $(n-2)$-dimensional hyperplane μ' of $\mathfrak{P}$, such that every other hyperplane of $\mathfrak{P}$ precedes or follows P', according to whether it precedes or follows μ'. If this were not so, we should fall again into the case of an order by $(n-1)$-dimensional hyperplanes parallel to α, which is incompatible with this one.

First of all we will show that the $(n-1)$-dimensional hyperplanes $P\mu$ and $P'\mu'$ are parallel.

Let us consider a plane π, passing through P and P', but not parallel to the $(n-2)$-dimensional hyperplanes μ and μ'; it meets α in a line r and μ' respectively at two points N and N' of r (Fig. 14, corresponding to case $n=3$). Because of the particular relationship of μ to P and of μ' to P', we find that the point N is the last one of r to precede P, or the first one to follow P, and similarly that N' is the last point of r to precede P', or the first one to follow P'. Therefore the plane π is ordered by lines parallel to PN, as well as by lines parallel to $P'N'$ (in view of Lemma I and of the proof of Case B of assertion (I)); but this can happen only if the lines PN and $P'N'$ are parallel to each other. The two $(n-1)$-dimensional hyperplanes $P\mu$ and $P'\mu'$, containing two parallel $(n-2)$-dimensional hyperplanes (μ and μ') and two lines, not belonging to them, but parallel (PN and $P'N'$), are therefore parallel in their turn.

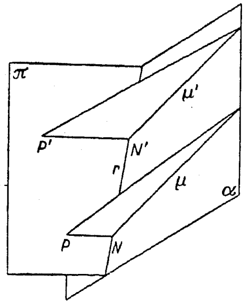

Fig. 14

Let us now take a generical point P'' of the $(n-1)$-dimensional hyperplane $P\mu$, and consider the corresponding $(n-2)$-dimensional hyperplane μ'', which, of all the $(n-2)$-dimensional hyperplanes of α parallel to μ, separates those which precede P'' from those which follow it. We observe that the $(n-1)$-dimensional hyperplane $P''\mu''$ is parallel to $P'\mu'$ and therefore must coincide with $P\mu$. We may conclude that μ'' coincides with μ.

From the latter assertion we deduce that all the points of the $(n-1)$-dimensional hyperplane $P\mu$ are preceded by the points of α which precede μ, and are followed by those which follow μ. The same can be said, referring to the corresponding $(n-2)$-dimensional hyperplane μ', of every other $(n-1)$-dimensional hyperplane $P'\mu'$ parallel to $P\mu$.

Therefore it easily follows from Lemma IV that the space E_n is ordered by $(n-1)$-dimensional hyperplanes parallel to $P\mu$. In conclusion our assertion (II) is completely proved.

5. The theorem of no. 4 and Lemmas I, II, III allow us to give an inductive characterisation for the most general normal order of E_n.

Any normal order of E_n,

for $n=1$, coincides with one of the two opposed Euclidean orders of the line E_1;

for $n>1$, coincides with an order by parallel $(n-1)$-dimensional hyperplanes, which subordinates on each of these hyperplanes a normal order (which will be an Euclidean order if $n=2$, and an order by parallel $(n-2)$-dimensional hyperplanes if $n>2$).

Since this inductive definition of the most general normal order is a constructive definition, it proves also the existence of normal orders for any n.

Reçu par la Rédaction le 15. 4. 1961