Point-like decompositions of E^3

by

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1. Introduction. An upper semicontinuous decomposition G of E^n (Euclidean n space) is *point-like* if for each element g of G, $E^n - g$ is topologically equivalent to the complement of a point. We call such an upper semicontinuous decomposition a *point-like decomposition*.

If G is a point-like decomposition of E^2 (or E^1), the resulting decomposition space is topologically E^2 (or E^1) [17]. However, there are examples of point-like decompositions of E^3 whose decomposition spaces are topologically different from E^3. We give another such example in this paper and suggest a decomposition which may be an example.

Suppose that G is a point-like decomposition of E^3 and H is the collection of nondegenerate elements of G. It is known that G yields E^3 if it satisfies any one of the the following conditions:

1) Each element of G is a starlike body (Theorem 2 of [4]).

2) Each element of H lies in a horizontal plane [11].

3) There is a countable number of planes such that each element of H is an interval normal to one of these planes [14].

There are examples ([3], [12]) which show that the union of the following two conditions is not enough to ensure that G yields E^3.

4) Each element of H is a tame arc.

5) The sum of the elements of H is a G_δ set.

There are no published examples which resolve the question as to whether or not either of the following conditions implies that G yields E^3.

6) Each element of H is a straight line interval.

7) H is countable.

In this paper we give an example to show that Condition 7 does not imply that G yields E^3 and another example to cast doubt on the potency of Condition 6.

It is known (Theorems 3, 1 of [4]) that Condition 7 together with any one of Conditions 4, 5, 6 is enough to insure that G yields E^3. Open are

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the effects of imposing such conditions as that H is a family of planar disks or that H is a countable family of arcs (not necessarily tame) or even continuous curves.

2. An indecomposable plane continuum. Each nondegenerate element of the point-like decomposition of E^3 that we describe in Section 3 is a copy of the following plane continuum X which is described as the intersection of a countable number of annuli $A_1, A_2, A_3, \ldots$

The annulus A_1 shown in Figure 1 is bounded by two concentric circles. The annulus A_2 plus its bounded complementary domain lies in A_1 and wraps around A_1 twice with the ends of the "wraps" near each other on the left side of A_1 as shown in Figure 1. The annulus A_3 (only part of which is shown) wraps around A_2 in a similar fashion. Similarly, A_4 wraps around A_3, A_5 around A_4, ... The continuum $X = A_1 \cdot A_2 \cdot A_3 \cdot \ldots$ is the sum of a collection of semicircles.

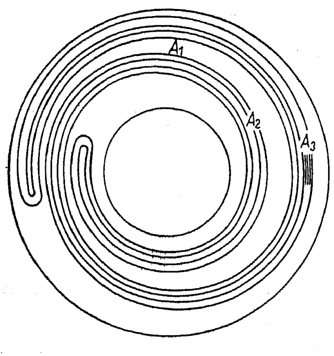

Fig. 1

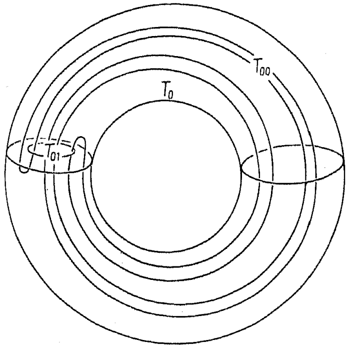

Fig. 2

Note that X is also the intersection of a sequence of topological disks. If X lies in a plane in E^3, it is the intersection of a sequence of tame cubes. Hence, it is point-like in E^3. Also, X may be regarded as the intersection of a sequence of solid tori in E^3. It is in this light that it is viewed in the next section.

3. A decomposition of E^3. In this section we describe a point-like decomposition G of E^3 such that G has only a countable number of nondegenerate elements, each of the nondegenerate elements is a copy of the indecomposable continuum X described in Section 2 and the decomposition space is topologically different from E^3. The proof that the decomposition space is different from E^3 is found in Section 5. The nondegenerate elements of G are described as the intersection of a sequence of tori.

Suppose that T_0 is a round solid torus as shown in Figure 2. Inside T_0 are two solid tori T_{00}, T_{01} linked as shown. These are solid even though they are shown as linear. Their sum essentially goes around T_0 twice. The diameter of T_{00} is comparatively large but the diameter of T_{01} is less than one half that of T_0. The center axis of T_{00} lies in the same plane as the axis of T_0 but the axis of T_{01} lies in a plane perpendicular to this one.

Just as T_{00} and T_{01} are built in T_0, so is a T_{0i0} and a T_{0i1} built in each T_{0i}. Similarly we define four T_{0ij0}'s, four T_{0ij1}'s, eight T_{0ijk0}'s, ... Each component of $Y = T_0 \cdot \sum T_{0i} \cdot \sum T_{0ji} \cdot \sum T_{0ijk} \cdot \ldots$ is an element of G. Also, each point of $E^3 - Y$ is an element of G.

Note that Y has a Cantor set of components. If $a_1 a_2 a_3 \ldots$ is an infinite decimal where each a_i is 0 or 1, we may let this decimal correspond to the component $T_{0a_1} \cdot T_{0a_1a_2} \cdot T_{0a_1a_2a_3} \cdot \ldots$ If the decimal has infinitely many 1's, the diameter of the component is 0. Hence, G has only a countable number of nondegenerate elements. We could define G so that each of these nondegenerate components either lies in the xy-plane or the yz-plane and any two of them are congruent under a similarity homeomorphism.

4. The intersection of disks with tori. In this section we consider the intersections of certain disks with the tori used in Section 3 to define Y. Although these tori were drawn as curved, we would get the same set Y if we had used polyhedral tori. We suppose in this section that each of T_0, T_{00}, T_{01} has a polyhedral boundary (is the sum of a finite number of triangular disks). We use $\mathrm{Bd}\, T$ to denote the boundary of solid torus T and $\mathrm{Int}\, T$ to denote $T - \mathrm{Bd}\, T$. For a disk D we use $\mathrm{Bd}\, D$ to denote its edge and $\mathrm{Int}\, D$ to denote $D - \mathrm{Bd}\, D$.

We shall consider the intersections of various polyhedral disks D with T_0, T_{00}, T_{01}. Each of the disks will be in general position with respect to $\mathrm{Bd}\, T$ ($T = T_0, T_{00}, T_{01}$) in that if p is a point of $D \cdot \mathrm{Bd}\, T$, there is a neighborhood U of p and a homeomorphism h of U onto E^3 such that $h(U \cdot \mathrm{Bd}\, T)$ is the $z = 0$ plane and $h(U \cdot D)$ is either the $x = 0$ plane or the $x = 0$, $z \geqslant 0$ halfplane. This ensures that each component of $D \cdot \mathrm{Bd}\, T$ is a polygonal simple closed curve.

PROPERTY P. *The disk D has Property* P *with respect to T_0 if D is a polyhedral disk in general position with respect to T_0, T_{00}, T_{01} and* $\mathrm{Bd}\, D$ *is a simple closed curve on* $\mathrm{Bd}\, T_0$ *that circles* $\mathrm{Bd}\, T_0$ *meridianally.*

Note that Property P implies that some component of $D \cdot T_0$ is a punctured disk such that the outer boundary component (with respect to $\mathrm{Bd}\, D$) of the punctured disk circles $\mathrm{Bd}\, T_0$ meridianally and the other boundary components (if any) bound disks on $\mathrm{Bd}\, T_0$. To get such a component one could consider the various simple closed curves on $D \cdot \mathrm{Bd}\, T_0$ that circle $\mathrm{Bd}\, T_0$ meridianally, consider one of these which bounds a disk D' of D which contains no other one of the simple closed curves, and finally take the component of $D'\cdot T_0$ which contains $\mathrm{Bd}\,D'$.

THEOREM 1. *If D is a polyhedral disk in general position with respect to $\mathrm{Bd}\,T$ and J is a component of $D\cdot\mathrm{Bd}\,T$, then J either bounds a disk on $\mathrm{Bd}\,T$ or J circles $\mathrm{Bd}\,T$ once longitudinally and no times meridianally, or J circles $\mathrm{Bd}\,T$ once meridianally and no times longitudinally.*

Proof. Suppose that J does not bound a disk on $\mathrm{Bd}\,T$. Let J' be an inner such simple closed curve on D with respect to this property in the sense that J' is a simple closed curve on $D\cdot\mathrm{Bd}\,T$ which does not bound a disk on $\mathrm{Bd}\,T$ but the subdisk D' of D bounded by J' contains no other such simple closed curve.

The closure of the component of $D'-\mathrm{Bd}\,T$ containing J' is a punctured disk such that the boundary of each hole bounds a disk on $\mathrm{Bd}\,T$. If the holes in this punctured disk are filled by disks near $\mathrm{Bd}\,T$ and then the resulting disk is pushed slightly away from $\mathrm{Bd}\,T$ near these fill-ins, there results a disk D'' which does not intersect $\mathrm{Bd}\,T$ except in its boundary $\mathrm{Bd}\,D''=\mathrm{Bd}\,D'=J'$. If $\mathrm{Int}\,D''\subset T$, J' circles $\mathrm{Bd}\,T$ meridianally and if $\mathrm{Int}\,D''\cdot T=0$, J' circles $\mathrm{Bd}\,T$ longitudinally. Any other simple closed curve (such as J) on $\mathrm{Bd}\,T-J'$ which does not bound a disk on $\mathrm{Bd}\,T$ would circle $\mathrm{Bd}\,T$ in the same fashion. This is because $\mathrm{Bd}\,T-J'$ is an open annulus and any simple closed curve on an annulus which does not bound a disk on the annulus circles the annulus exactly once.

THEOREM 2. *The fundamental group of $E^3-(T_{00}+T_{01})$ is a free group on two generators. A simple closed curve that circles $\mathrm{Bd}\,T_0$ meridianally cannot be shrunk to a point in $E^3-(T_{00}+T_{01})$.*

Proof. If we were only going to prove the first half of the theorem we would merely note that there is a homeomorphism of E^3 onto itself that takes T_{00} and T_{01} onto two nonlinking circular solid tori. It is a bit complicated to see where a meridianal simple closed curve on $\mathrm{Bd}\,T_0$ goes under this homeomorphism so we formally compute the fundamental group of $E^3-(T_{00}+T_{01})$.

The fundamental group of $E^3-(T_{00}+T_{01})$ can be read from Figure 3. Where the eye is regarded as the starting point, the letters on the arrows represent a path starts at the eye and goes under T_{00} or T_{01} in the direction of the arrow and returns to the starting point, and relations result at perspective crossing points of T_{00} and T_{01}. If one admissible polygonal path is homotopic to another, it can be deformed in a series of jumps so that no jump involves going past more than one perspective crossing. Hence the word corresponding to the first path can be changed by a finite number of the relations into the word representing the other path.

As read from Figure 3 the fundamental group of $E^3-(T_{00}+T_{01})$ is given by generators a, b, c, d, x, y, z, w and relations $xd=ax$, $yd=dx$, $bz=yb$, $cz=wc$, $wc=cx$, $xb=ax$, $dx=xc$, $xc=bx$. The relations merely imply that $a=xbx^{-1}$, $c=x^{-1}bx$, $d=b$, $y=bxb^{-1}$, $z=x$, $w=x^{-1}bxb^{-1}x$. Hence the group is generated by b and x with no relations.

As may be seen by examining the simple closed curve J on the right, a simple closed curve circling $\mathrm{Bd}\,T_0$ meridianally may correspond to the element $cb^{-1}da^{-1}=x^{-1}bx^2b^{-1}x^{-1}$. Since this element is non-trivial in the free group, J cannot be shrunk to a point in $E^3-(T_{00}+T_{01})$.

THEOREM 3. *If D is a disk with Property P with respect to T_0, then either $D\cdot\mathrm{Bd}\,T_{00}$ contains a simple closed curve that circles $\mathrm{Bd}\,T_{00}$ meridianally or $D\cdot\mathrm{Bd}\,T_{01}$ contains a simple closed curve that circles $\mathrm{Bd}\,T_{01}$ meridianally.*

Proof. First we show that there is no loss of generality in supposing that no simple closed curve on $D\cdot\mathrm{Bd}\,T_{00}$ bounds a disk on $\mathrm{Bd}\,T_{00}$ and no simple closed curve on $D\cdot\mathrm{Bd}\,T_{01}$ bounds a disk on $\mathrm{Bd}\,T_{01}$. We do this by eliminating unwanted simple closed curves by a method used in the proof of Theorem 1. This elimination is accomplished by considering the punctured disk which is the closure of the component which contains $\mathrm{Bd}\,D$ of D minus the sum of the simple closed curves of $D\cdot(\mathrm{Bd}\,T_{00}+\mathrm{Bd}\,T_{01})$ which bound disks on $\mathrm{Bd}\,T_{00}+\mathrm{Bd}\,T_{01}$, then one-at-a-time (starting with a hole bounded by an inner most simple closed curve on $\mathrm{Bd}\,T_{00}+$ $+\mathrm{Bd}\,T_{01}$) fill in the holes of this punctered disk with disks on $\mathrm{Bd}\,T_{00}+$ $+\mathrm{Bd}\,T_{01}$, and push these fill-ins to one side of $\mathrm{Bd}\,T_{00}+\mathrm{Bd}\,T_{01}$. Hence we suppose that if J is a component of $D\cdot(\mathrm{Bd}\,T_{00}+\mathrm{Bd}\,T_{01})$, then J does not bound a disk on $\mathrm{Bd}\,T_{00}+\mathrm{Bd}\,T_{01}$.

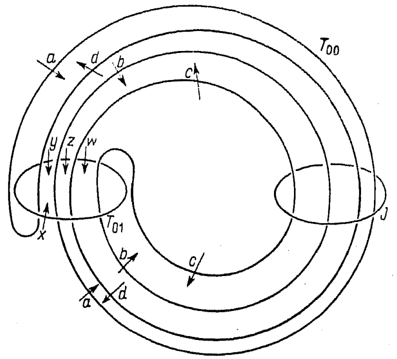

Fig. 3

If a component J of $D\cdot\mathrm{Bd}\,T_{00}$ is a simple closed curve that circles $\mathrm{Bd}\,T_{00}$ longitudinally, then to one side of J there is a simple closed curve K on $\mathrm{Bd}\,T_{00}$ that circles $\mathrm{Bd}\,T_{00}$ longitudinally and which misses D. There is a homeomorphism h of E^3 onto itself that shrinks T_{00} to so near K that h is the identity on $T_{01}+\mathrm{Bd}\,T_0$ and $D\cdot h(T_{00})=0$. Hence we suppose that D misses T_{00}.

In a similar fashion, it can be shown that unless Theorem 3 is true we may assume that D misses T_{01} also. However, this would contradict Theorem 2 which says that $\mathrm{Bd}\,D$ cannot be shrunk to a point in E^3- $-(T_{00}+T_{01})$.

THEOREM 4. *If D is a disk with Property* P *with respect to T_0, then D contains two mutually exclusive subdisks such that each of the subdisks has Property* P *with respect to T_{00} or each of the subdisks has Property* P *with respect to T_{01}.*

Proof. As in the proofs of Theorems 1 and 3 we adjust D so that the adjusted disk contains no simple closed curve that bounds a disk on either $\mathrm{Bd}\,T_{00}$ or $\mathrm{Bd}\,T_{01}$. If we can find two disks satisfying the conclusion of Theorem 4 on the adjusted disk, we can find two on D. For convenience in notation we suppose that the adjusted disk is D. It follows from Theorem 3 that D contains a disk D' that has Property P with respect to either T_{00} or T_{01}. We suppose that D' has Property P with respect to T_{00}. Furthermore, we suppose that D' is maximal in the sense that it is not a proper subdisk of any other subdisk of D that has Property P with respect to T_{00}. Note that although $\mathrm{Bd}\,D$ can be shrunk to a point in the complement of T_{00}, $\mathrm{Bd}\,D'$ cannot be shrunk to a point in the complement of the central curve of T_{00}.

Since $\mathrm{Bd}\,D'$ is not homotopic to $\mathrm{Bd}\,D$ in $(E^3-\mathrm{Int}\,T_{00})$, $D-D'$ contains a simple closed curve J such that J circles $\mathrm{Bd}\,T_{00}$ meridianally. To see that this is true consider the annulus A in D bounded by $\mathrm{Bd}\,D$ and $\mathrm{Bd}\,D'$. Some point of A must lie in $\mathrm{Int}\,T_{00}$ since $\mathrm{Bd}\,D'$ can be pulled to $\mathrm{Bd}\,D$ on A but not in $E^3-\mathrm{Int}\,T_{00}$. Since $\mathrm{Bd}\,D$ does not intersect T_{00}, $A\cdot\mathrm{Bd}\,T_{00}$ contains a component J other that $\mathrm{Bd}\,D'$. The disk in D bounded by J is another disk in D that has Property P with respect to T_{00}.

THEOREM 5. *If D_1, D_2 are two mutually exclusive disks each with Property* P *with respect to T_0, then D_i $(i=1,2)$ contains a subdisk D_i' such that either each of D_1', D_2' has Property* P *with respect to T_{00} or each of D_1', D_2' has Property* P *with respect to T_{01}.*

Proof. We shall show that D_i $(i=1,2)$ contains a simple closed curve J_i such that either J_1 and J_2 lie on $\mathrm{Bd}\,T_{00}$ and circle T_{00} meridianally or J_1 and J_2 lie on $\mathrm{Bd}\,T_{01}$ and circle T_{01} meridianally. Then D_i' will be the subdisk of D_i bounded by J_i.

Following an argument similar to that employed in the proofs of Theorems 1 and 3, we assume with no loss of generality that no simple closed curve on D_i bounds any disk on either $\mathrm{Bd}\,T_0$, $\mathrm{Bd}\,T_{00}$, $\mathrm{Bd}\,T_{01}$. Furthermore we suppose that $D_i\cdot\mathrm{Bd}\,T_0=\mathrm{Bd}\,D_i$. Then $\mathrm{Int}\,D_i\subset\mathrm{Int}\,T_0$.

If D_1 contains a simple closed curve that circles $\mathrm{Bd}\,T_{00}$ longitudinally, there is a simple closed curve K on $\mathrm{Bd}\,T_{00}$ that circles T_{00} longitudinally such that K misses each of D_1 and D_2. There is then a homeomorphism h of E^3 onto itself fixed on $\mathrm{Bd}\,T_0$ and T_{01} that pulls T_{00} so near K that $(D_1+D_2)\cdot h(T_{00})=0$. It then follows from Theorem 3 that each D_1, D_2 contains a simple closed curve that circles $\mathrm{Bd}\,T_{01}$ meridianally. Hence, we suppose that neither D_1 nor D_2 contains any simple closed curve that circles either $\mathrm{Bd}\,T_{00}$ or $\mathrm{Bd}\,T_{01}$ longitudinally.

Since it follows from Theorem 4 that each D_i intersects one of T_{00}, T_{01}, we assume with no loss of generality that D_1 intersects T_{00}. The condition of Theorem 5 is satisfied if D_2 intersects T_{00} so we suppose that D_2 misses T_{00} and intersects T_{01}. Also we suppose that D_1 misses T_{01}. We complete the proof of Theorem 5 by showing that it is impossible that $D_1\cdot T_{01}=D_2\cdot T_{00}=0$.

Consider the universal covering space of T_0. It is represented by Figure 4 where it appears that T_0 has been rolled out to make infinitely many copies of T_{00} and T_{01}. Let $T_{01}', T_{01}'', T_{01}'''$ be three adjacent copies of T_{01} and T_{00}' be the copy of T_{00} that links T_{01}' and T_{01}'' as shown. It follows from Theorem 3 of [3] that there is an arc A_1' from T_{01}' to T_{00}' and an arc A_2'

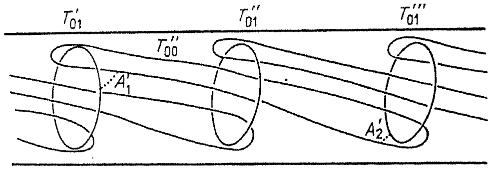

Fig. 4

from T_{00}' to T_{01}''' such that these arcs miss the various copies of D_1 and D_2. With no loss of generality we suppose that the projections of A_1' and A_2' into T_0 are mutually exclusive arcs A_1 and A_2.

Let B_1 be an arc in T_{00} from the last point of A_1 to the first point of A_2 and B_2 be an arc in T_{01} from the last point of A_2 to the first point of A_1. We suppose the B's intersect the A's at only the points indicated so that $A_1+B_1+A_2+B_2$ is a simple closed curve that circles T_0 twice. This simple closed curve would be the sum of two arcs, one A_2+B_2 missing D_1 and the other A_1+B_1 missing D_2 if $D_1\cdot T_{01}=0=D_2\cdot T_{00}$. This is impossible as pointed out in the following paragraph.

Since D_1 and D_2 chop T_0 into two 3-cells C_1, C_2, we may regard an arc going from D_1 to D_2 in C_i as going halfway around T_0. The arc A_1+B_1 would not go as much as once around T_0 since it misses D_2 and the arc A_2+B_2 would go as much as once around T_0 since it misses D_1. This contradicts the fact that the closed curve $A_1+B_1+A_2+B_2$ goes around T_0 exactly twice.

THEOREM 6. *Suppose that S is a polyhedral 2-sphere in* $\mathrm{Int}\,T_0$ *such that S does not lie in either T_{00} or T_{01}, S is in general position with respect to* $\mathrm{Bd}\,T_{00}+\mathrm{Bd}\,T_{01}$, *$S$ contains no simple closed curve that bounds a disk in either* $\mathrm{Bd}\,T_{00}$ *or* $\mathrm{Bd}\,T_{01}$, *and the bounded component of E^3-S intersects T_{00}.*

Then S contains a disk D such that either D has Property P *with respect to T_{00} or D has Property* P *with respect to T_{01}.*

Proof. If S intersects $\mathrm{Bd}\,T_{00}$, $S\cdot\mathrm{Bd}\,T_{00}$ contains a simple closed curve J such that either J circles $\mathrm{Bd}\,T_{00}$ meridianally (in which case the condition of Theorem 6 is easily established) or circles $\mathrm{Bd}\,T_{00}$ longitudinally. In this latter case there is a simple closed curve K on $\mathrm{Bd}\,T_{00}$ such that K circles $\mathrm{Bd}\,T_{00}$ longitudinally and K lies in the bounded component of E^3-S. There is a homeomorphism h of E^3 onto itself that is fixed on $T_{01}+\mathrm{Bd}\,T_0$ which takes T_{00} so near K that $h(T_{00})$ lies in the bounded component of E^3-S. Hence we suppose that T_{00} lies in the bounded component of E^3-S.

In a similar fashion it is shown that unless Theorem 6 is true, we can assume with no loss of generality that T_{01} lies in the bounded component of E^3-S. However, this would violate Theorem 2 since a simple closed curve circling $\mathrm{Bd}\,T_0$ meridianally can be shrunk to a point in the unbounded component of E^3-S.

5. The decomposition space Γ. We use Γ to denote the decomposition space whose points are the elements of G. There are several ways of showing that the decomposition space Γ is different from E^3.

Perhaps the easiest way to show that Γ is topologically different from E^3 is to follow the methods used in [3]. Using this approach, one would show that if Γ is topologically equivalent to E^3, then there is a pseudo isotopy fixed outside T_0 which shrinks elements of G to points. It would then be shown that this is impossible since any homeomorphism h of E^3 onto itself that is fixed outside T_0 takes some T_{0i} onto a set of large size, some T_{0ij} in this T_{0i} would have an image of large size under h, some T_{0ijk} in it would have large size under h, ...

Although we could carry out the program suggested in the preceding paragraph, we do not suggest that it is easy. On the contrary, one can find from examining the proof in [3] that such an approach can be quite tedious.

We show that Γ is topologically different from E^3 by showing that there are points of Γ without small neighborhoods bounded by 2-spheres. Although this approach is more difficult than the one outlined above, it has the merits that it is different from that used in [3] and gives a handy property to distinguish the decomposition space from E^3. Let f be the map of E^3 onto Γ that takes each element of G onto a unique point of Γ.

THEOREM 7. *If g is a nondegenerate element of G, $f(T_0)$ contains no neighborhood of $f(g)$ whose boundary is a 2-sphere.*

Proof. This theorem is established by applying the results of Section 4. Assume that there is a neighborhood N of $f(g)$ in $f(T_0)$ such that $\mathrm{Bd}\,N$ is a 2-sphere. We show in five steps that this is impossible.

Step 1. *Putting boundaries in convenient positions.* In this paragraph we show that if there is an N, there is one such that $f^{-1}(\mathrm{Bd}\,N)$ is locally polyhedral except possibly at points of Y. Just because $\mathrm{Bd}\,N$ is a 2-sphere in Γ is no reason to suppose that $f^{-1}(\mathrm{Bd}\,N)$ is a 2-sphere in E^3. In fact it may even not be a 2-manifold. However $f^{-1}(\mathrm{Bd}\,N)$ is locally a 2-manifold away from Y and $f^{-1}(\mathrm{Bd}\,N)-Y$ is a 2-manifold in E^3-Y. Note that $f^{-1}(\mathrm{Bd}\,N)$ separates g from $\mathrm{Bd}\,T_0$ in E^3. There is a positive number ε such that if h is a map of $f^{-1}(\mathrm{Bd}\,N)$ into E^3 that moves no point more than ε, then $hf^{-1}(\mathrm{Bd}\,N)$ separates g from $\mathrm{Bd}\,T_0$ (Theorem VI, 10 of [13]). Let $F(x)$ be the real function which is the minimum of ε and one half the distance from x to Y. By regarding $f^{-1}(\mathrm{Bd}\,N)-Y$ as a 2-manifold in E^3-Y it follows from Theorem 7 of [6] that there is a homeomorphism h of $f^{-1}(\mathrm{Bd}\,N)$ into E^3 such that h does not move x by more than $F(x)$ and $hf^{-1}(\mathrm{Bd}\,N)$ is locally polyhedral at $h(x)$ if $F(x)>0$. Note that $E^3-hf^{-1}(\mathrm{Bd}\,N)$ is the sum of two components U_1, U_2 containing g, $\mathrm{Bd}\,T_0$ respectively. Then $f(U_1)$ is an open set bounded by the 2-sphere $fhf^{-1}(\mathrm{Bd}\,N)$. Replacing N by $f(U_1)$ we get an open set in $f(T_0)$ containing $f(g)$ such that $\mathrm{Bd}\,f(U_1)$ is a 2-sphere whose inverse under f is locally polyhedral except possibly on Y. For convenience in notion we suppose that $N=f(U_1)$ and $f^{-1}(\mathrm{Bd}\,N)$ is locally polyhedral except possibly at Y.

As mentioned at the beginning of Section 4 there is no loss of generality in supposing that the boundaries of the T's are polyhedral. Hence, we suppose that each $\mathrm{Bd}\,T$ is polyhedral and in general position with respect to $f^{-1}(\mathrm{Bd}\,N)$.

Step 2. $\mathrm{Bd}\,N$ *contains a disk D such that $f^{-1}(\mathrm{Bd}\,D)$ circles some* $\mathrm{Bd}\,T$ *meridianally.* Since $f(g)$ lies in N, there is a $T_{0a_1a_2\ldots a_n}$ such that $f(T_{0a_1a_2\ldots a_n})$ lies in N. We show that for some $T_{0b_1b_2\ldots b_m}$ $(m\leqslant n)$, $f^{-1}(\mathrm{Bd}\,N)$ contains a simple closed curve that circles $\mathrm{Bd}\,T_{0b_1b_2\ldots b_m}$ meridianally. We do this by applying Theorem 6.

To apply Theorem 6 we must replace $f^{-1}(\mathrm{Bd}\,N)$ by a polyhedral 2-sphere. (We recall that $f^{-1}(\mathrm{Bd}\,N)$ need not even be a 2-sphere.) There is a punctured polyhedral disk A on $f^{-1}(\mathrm{Bd}\,N)$ such that $f^{-1}(\mathrm{Bd}\,N)-A$ lies on $\sum\mathrm{Int}\,T_{0c_1c_2\ldots c_{n+2}}$. The boundary components of A can be shrunk to points in $\sum\mathrm{Int}\,T_{0c_1c_2\ldots c_{n+1}}$. Hence, there results a singular 2-sphere. By the Sphere Theorem [18] there is a polyhedral 2-sphere S in $A+\sum\mathrm{Int}\,T_{0c_1c_2\ldots c_{n+1}}$ such that $T_{0a_1a_k\ldots a_n}$ lies in the bounded component of E^3-S.

We assume that S does not contain any simple closed curve that bounds a disk on any $\mathrm{Bd}\,T_{0c_1c_2\ldots c_i}$ $(i\leqslant n)$. That we can do this follows from arguments used in the proofs of Theorems 1 and 3. Let $T_{0d_1d_2\ldots d_j}$ be the smallest T containing S. We note that $j<n$. It follows from Theorem 6 that S contains a simple closed curve J such that J circles one of

$T_{0d_1d_2...d_j0}$, $T_{0d_1d_2...d_j1}$ meridianally. This simple closed curve would belong to A and hence to $f^{-1}(\mathrm{Bd}\,N)$.

With no loss of generality we assume that $J \subset \mathrm{Bd}\,T_{00}$.

Step 3. *The disk D of Step 2 contains two mutually exclusive subdisks D_1, E_1 such that $f^{-1}(\mathrm{Bd}\,D_1)$, $f^{-1}(\mathrm{Bd}\,E_1)$ circles the same one of* $\mathrm{Bd}\,T_{000}$, $\mathrm{Bd}\,T_{001}$ *meridianally.* Just as we applied the Sphere Theorem in Step 2, we apply Dehn's Lemma in this step to replace $f^{-1}(D)$ by a disk D' in $f^{-1}(D)+\sum T_{0ijk}$ that contains $f^{-1}(D)-\sum T_{0ij}$. It follows from Theorem 4 that D' contains two mutually exclusive disks D_1', E_1' with Property P with respect to the same one of T_{000}, T_{001}. If D_1, E_1 are the subdisks of D which have the same boundaries as $f(D_1')$, $f(E_1')$, they are the disks promised by Step 3.

Step 4. *Disks D_1, E_1 contain disks D_2, E_2 respectively such that there is a T_{00ij} such that $f^{-1}(\mathrm{Bd}\,D_2)$, $f^{-1}(\mathrm{Bd}\,E_2)$ each circles* $\mathrm{Bd}\,T_{00ij}$ *meridianally.* This follows from Theorem 5 applied to disks replacing $f^{-1}(D_1)$, $f^{-1}(E_1)$.

Step 5. *D_1 intersects E_1.* It follows from repetitions of Step 4 that there are sequences of disks $D_1, D_2, D_3, \ldots$ and $E_1, E_2, E_3, \ldots$ such that $D_1 \supset D_2 \supset D_3 \supset \ldots$, $E_1 \supset E_2 \supset E_3 \supset \ldots$, and $f^{-1}(\mathrm{Bd}\,D_i)$, $f^{-1}(\mathrm{Bd}\,E_i)$ circle meridianally the same $\mathrm{Bd}\,T_{00c_1c_2...c_i}$. This implies that D_1 intersects E_1.

The assumption that there is a neighborhood N of $f(g)$ in $f(T_0)$ whose boundary is a 2-sphere led to the contradiction between Steps 3 and 5.

6. A decomposition H of E^3 into points and segments.

In this section we describe a decomposition H of E^3 into points and straight segments. Although this decomposition looks enough like the decomposition described in [3] to suggest that the decomposition space may be topologically different from E^3, it looks enough like the decomposition in [7] to make one cautious in venturing such a conjecture.

A descriprion of H. Let Θ_1 and Θ_2 be two horizontal planes in E^3 with Θ_1 below Θ_2. Each nondegenerate element of H is a straight segment with one end in Θ_1 and the other in Θ_2. Each horizontal plane that intersects one of the segments will intersect the sum of the segments in a topological Cantor set.

The sum of the nondegenerate elements of H is the intersection of a decreasing sequence of open sets $U_1, U_2, \ldots$ Each U_i is a tubular neighborhood of a finite graph G_i. Figure 5 shows G_1. We describe it as follows.

Let $Q(1), Q(2), Q(3), Q(4)$ be four points in Θ_1 and $P(1)$, $P(2)$ be two points in Θ_2 such that the line through $P(1), P(2)$ is not parallel to any line through any two of the Q's. We use $[P(i), Q(j)]$ to denote the straight segment from $P(i)$ to $Q(j)$. Then G_1 is the sum of the eight $[P(i), Q(j)]$'s as shown in Figure 5. U_1 is a tubular neighborhood of G_1.

We obtain G_2 by replacing each major segment $[P(i), Q(j)]$ in G_1 by a copy of G_1. The four copies of G_1 in G_2 that are near $P(i)$ $(i = 1, 2)$ link each other and the two copies of G_1 in G_2 near $Q(j)$ $(j = 1, 2, 3, 4)$ cross in a manner to be described presently. Before describing the eight copies of G_1 in G_2 in detail, we remark that their sum lies in U_1. Just as we obtained G_2 by replacing each major segment in G_1 by a copy of G_1, we obtain G_3 by replacing each major segment of G_2 by a copy of G_1. Continuing in this fashion we get a sequence of finite graphs $G_1, G_2, G_3, G_4, \ldots$ and a sequence of tubular neighborhoods $U_1, U_2, U_3, U_4, \ldots$ of these graphs such that $G_{i+1} \subset U_{i+1} \subset \overline{U}_{i+1} \subset U_i$. The tubular neighborhoods get progressively so thin that the intersection of the U_i's is the sum of a Cantor set of segments each with one end on Θ_1 and the other on Θ_2. These segments are the nondegenerate elements of the decomposition H.

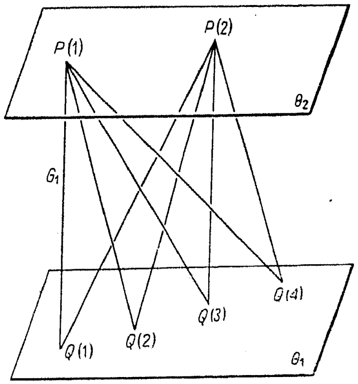

Fig. 5

The copy of G_1 in G_2 replacing $[P(i), Q(j)]$ is denoted by $G(i, j)$. It is the sum of eight straight segments $[P(i,j,m), Q(i,j,n)]$ $(m = 1, 2;$ $n = 1, 2, 3, 4)$ where the $P(i,j,m)$'s are points in Θ_2 near $P(i)$ and the $Q(i,j,n)$'s are points in Θ_1 near $Q(j)$. We shall describe these P's and Q's presently but first show how we want $G(i,j)$ and $G(i,k)$ to entwine near $P(i)$.

Entwining near $P(i)$. Figure 6 shows how certain segments of the $G(1,j)$'s look in perspective near $P(1)$. The general rule is that for $j < k$, $[P(i,j,s), Q(i,j,t)]$ passes between the eye and $[P(i,k,r), Q(i,k,u)]$ for $t \leqslant u$ and $[P(i,k,r), Q(i,k,u)]$ passes between the eye and $[P(i,j,s), Q(i,j,t)]$ for $u < t$. $P(i,j,1)$ and $P(i,j,2)$ are so close together (as are $P(i,k,1)$ and $P(i,k,2)$) that the crossings are the same irrespective of whether or not r (or s) is 1 or 2.

Entwining near $Q(j)$. Figure 7 shows how one broken arc in $G(1,j)$ entwines with four broken arcs in $G(2,j)$ near $Q(j)$. Each of the broken arcs in $G(1,j)$ entwines with these four arcs in $G(2,j)$ in this fashion but we did not show this in Figure 7 to avoid complication. Note that $[P(1,j,1), Q(1,j,m)]$ passes between the eye and $[P(2,j,1), Q(2,j,n)]$ $(n = 1, 2, 3, 4)$, $[P(2,j,1), Q(2,j,n)]$ passes between the eye and $[P(1,j,2), Q(1,j,m)]$, and $[P(i,j,2), Q(1,j,m)]$ passes between the eye and $[P(2,j,2), Q(2,j,n)]$.

The existence of the $G(i,j)$'s. It remains to be shown that we can really get $G(i,j)$'s that entwine as we have suggested. As a step toward doing this, we replace each $[P(i), Q(j)]$ by a triangular disk $T(i,j)$. The triangles are as shown in Figure 8 and are described as follows.

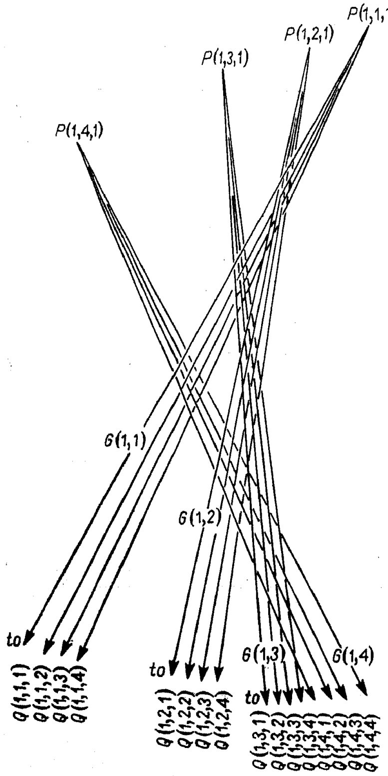

Fig. 6

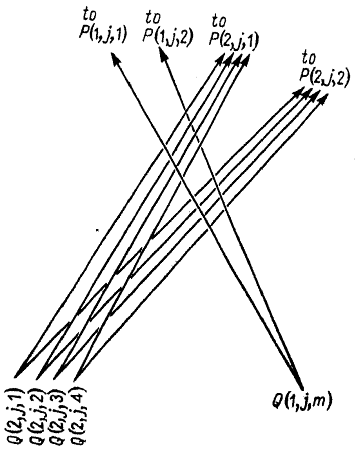

Fig. 7

Through $Q(j)$ consider a short segment $I(Q(j))$ parallel to $[P(1), P(2)]$. The base of $T(i,j)$ is on $I(Q(j))$. There is a line L parallel to $[P(1), P(2)]$ and a segment $I(P(i))$ on L near $P(i)$ such that $T(i,1)\cdot L = T(i,2)\cdot L = T(i,3)\cdot L = T(i,4)\cdot L = I(P(i))$. The lateral edges of $T(1,j)$ and $T(2,j)$ cross as shown. We suppose that $T(i,j)$ is close to $[P(i), Q(j)]$ and lies in U_1.

In describing the $Q(i,j,k)$'s it is convenient to think of L (and hence the base of $T(i,j)$) as running from left front to right back, with $I(P(2))$ to the right of $I(P(1))$. Let $Q(i,j,1)$ be the left most point of the base of $T(i,j)$, $Q(i,j,4)$ the right most point, $Q(i,j,2)$ a point that divides the base in the ratio 1 to 2 from left to right, and $Q(i,j,3)$ the point that divides it in the ratio 2 to 1.

The vertices $P(i,j,1)$ and $P(i,j,2)$ of $G(i,j)$ in Θ_2 are obtained by adjusting the apex of $T(i,j)$, but before describing this adjustment, let us comment on a restriction that if placed on these P's will ensure that $G(1,j)$ and $G(2,j)$ entwine near $Q(j)$ as shown in Figure 7. If $P(1,j,1), P(2,j,1)$ are on the same side of the plane of $T(1,j)+T(2,j)$

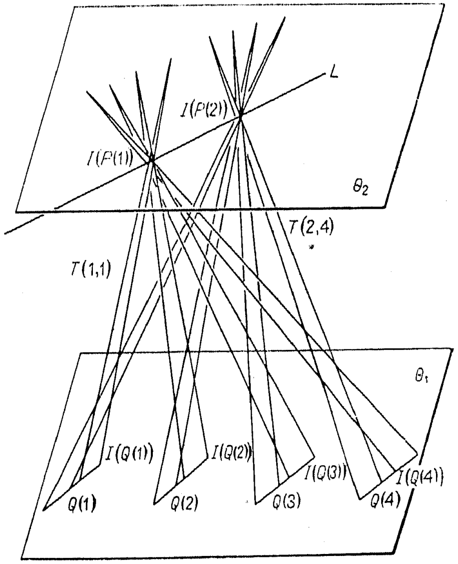

Fig. 8

as the eye with $P(2,j,1)$ only half as far from this plane as $P(1,j,1)$ and $P(1,j,2), P(2,j,2)$ are on the other side of this plane with $P(1,j,2)$ nearer than $P(2,j,2)$, the desired entwining is obtained.

As another step toward obtaining the $P(i,j,k)$'s we move the apex $A(i,j)$ of $T(i,j)$ varying amounts backward to the right. $B(i,j)$ is the point in Θ_2 obtained by moving $A(i,j)$ enough to the right that $[B(i,j), Q(i,j,1)]\cdot I(P(i))$ divides $I(P(i))$ in the ratio j to $12-j$ from right to left. If $P(i,j,1)$ and $P(i,j,2)$ are obtained from $B(i,j)$ by slight adjustments moving them out of the plane of $T(i,j)$ as suggested in the preceding paragraph, the $G(i,j)$'s entwine near $P(i)$ as suggested by Figure 6.

An intermediate step. Instead of considering eight components of U_2 in U_1, it is more convenient at times to consider two open sets in U_1 as shown in Figure 9 and then consider four components of U_2 in each of these open sets. In Figure 9 we do not emphasize straightness, but the figure is useful in showing the entwining near the Q's as suggested in Figure 7. This intermediate step is useful in Section 8 where we make suggestions as to why the decomposition space of H may be topologically different from E^3.

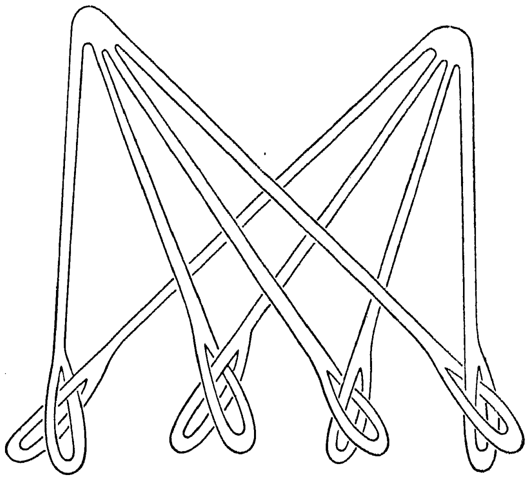

Fig. 9

7. Disks in Υ. We use Υ to denote the decomposition space whose points are the elements of H of the last section and f to be the map taking E^3 onto Υ. For each element y of Υ, $f^{-1}(y)$ is an element of H. We use C to denote the Cantor set in Υ which is the image of the sum of the nondegenerate elements of H. We note that $\Upsilon - C$ is a 3-manifold.

If X, Y are sets in a 3-manifold M, X is locally tame mod Y if for each point p of $X-Y$ there is a neighborhood N of p in M and a homeomorphism h of $\overline{N}$ onto a cube in E^3 such that $h(X\cdot\overline{N})$ is a polyhedron in the cube. Even if M is not a 3-manifold but $M-Y$ is one, we extend the notion to say that X is locally tame mod Y if for each point p of $X-Y$ there is such a neighborhood N of p. Hence we speak of locally tameness of sets in Υ mod C even though Υ may not be a manifold.

As a help toward learning the properties of Υ we consider how a disk in E^3 may be associated with a disk in Υ.

THEOREM 8. *Suppose that D is a disk in Υ such that D is locally tame* mod C *and* $C\cdot \mathrm{Bd}\,D = 0$. *Then for each integer i there is a tame disk D_i in E^3 such that $f(\mathrm{Bd}\,D_i) = \mathrm{Bd}\,D$ and $D_i \subset f^{-1}(D) + U_i$.*

Proof. As a result of Theorem 9 of [9], Theorem 9 of [8], or Theorem 1 of [16], we suppose with no loss of generality that $f^{-1}(D)$ is locally polyhedral mod $f^{-1}(C)$. Since the theorem is true for a particular integer i if is true for a larger one, we suppose that we are proving the theorem for an i so large that $f^{-1}(\mathrm{Bd}\,D)\cdot U_i = 0$.

Let X be a punctured disk in $D-C$ such that the boundary components of X are $\mathrm{Bd}\,D$ and simple closed curves in $f(U_{i+1})$. Then $f^{-1}(X)$ is a punctured disk in E^3. We suppose with no loss of generality that the boundary components of $f^{-1}(X)$ are $f^{-1}(\mathrm{Bd}\,D)$, $J_1, J_2, \ldots, J_n$ where J_j is a polygon in U_{i+1}. Since each simple closed curve in U_{i+1} can be shrunk to a point U_i, J_j bounds a polyhedral singular disk in U_i. Let Y be the sum of $f^{-1}(X)$ and singular disks, one corresponding to each J_j. Note that Y is a polyhedral singular disk all of whose singularities lie in U_i.

It follows from Dehn's lemma as proved by Papakyriokopoulos [18] that for each neighborhood N of the sum of the singularities of Y, there is a nonsingular polyhedral disk D_i in $Y+N$ with the same boundary as D. To ensure that D_i satisfies conditions of Theorem 8, we only need to take N to lie in U_i.

The next three theorems contain the questionable hypothesis that Υ is topologically E^3.

THEOREM 9. *Suppose that* $\mathrm{Bd}\,U_1$ *is polyhedral and $J_1, J_2, \ldots, J_n$ is a finite collection of mutually exclusive polygonal simple closed curves on* $\mathrm{Bd}\,U_1$ *each of which can be shrunk to a point in U_1. Then if Υ is topologically equivalent to E^3, there is an integer j and a finite collection of mutually exclusive polyhedral disks $D_1, D_2, \ldots, D_n$ such that* $\mathrm{Bd}\,D_i = J_i$, $\mathrm{Int}\,D_i \subset U_1$, *and no component of U_j intersects two D_i's.*

Proof. It follows from Theorem 9 of [8], Theorem 9 of [9] or Theorem 1 of [16] that we can suppose that f is piecewise linear except at C. We do this.

Since J_i can be shrunk to a point in U_1, it follows from Dehn's lemma [18] that J_i bounds a disk E_i which lies except for J_i in U_1. Then $f(J_i)$ bound the singular disk $f(E_i)$. Although $f(E_i)$ may not be polyhedral, there is a singular polyhedral disk X_i such that $\mathrm{Bd}\,X_i = f(J_i)$, $\mathrm{Int}\,X_i \subset f(U_1)$, and X_i has no singularities on $\mathrm{Bd}\,X_i$. It follows from Dehn's lemma as proved by Papakyriokopoulos [18] that $\mathrm{Bd}\,X_i$ bounds a polyhedral disk Y_i which lies except for its boundary in $f(U_1)$.

We wish to adjust the Y's so that the resulting disks will not intersect each other. Adjust Y_2 so that it is in general position with respect to Y_1. If Y_1, Y_2 intersect, consider a disk D in Y_1 such that $\mathrm{Int}\,D\cdot Y_2 = 0$ and

$\operatorname{Bd} D \subset Y_2$. Replace the disk in Y_2 bounded by $\operatorname{Bd} D$ by D and then shove the adjusted Y_2 to one side of Y_1 near D. This reduces the number of intersections of Y_1 and Y_2. A repetition of this procedure permits us to adjust Y_2 so that the adjustment misses Y_1. In a similar fashion we fix each of $Y_2, Y_3, \ldots, Y_n$ so that the resulting disks miss Y_1. Then $Y_3, Y_4, \ldots, Y_n$ are adjusted further so that the resulting disks will intersect neither Y_1 nor the new Y_2. In a finite number of such steps we can untangle the Y's from each other so we suppose with no loss of generality that they do not intersect. Note that $f^{-1}(Y_i)$ need not be a disk. Since Y_i is polyhedral in Υ and f is piecewise linear except on C, $f^{-1}(Y_i)$ is locally polyhedral except on C.

Let j be an integer so large that the closure of no component of $f(U_j)$ intersects two of the Y's. We suppose that $f(\operatorname{Bd} U_j)$ is in general position with respect to each Y.

It follows from Theorem 8 that there are tame disks F_i $(i = 1, 2, \ldots, n)$ such that $\operatorname{Bd}(F_i) = f^{-1}(\operatorname{Bd} Y_i) = J_i$, and F_i lies in $f^{-1}(Y_i) + U_j$. The closure of no component of U_j intersects two of the F's. It follows from Theorem 8 of [6] that a slight adjustment in the F's takes them into polyhedral disks that satisfy the conditions of Theorem 9.

We recall that an isotopy of E^3 onto itself is a one parameter family h_t $(0 \leqslant t \leqslant 1)$ of homeomorphisms of E^3 onto E^3. (If we permit h_1 to be a map rather than a homeomorphism we call the family a *pseudoisotopy* as mentioned in the beginning of Section 5.)

THEOREM 10. *If Υ is topologically E^3, then for each integer i and each positive number ε there is an integer j and an isotopy h_t $(0 < t \leqslant 1)$ of E^3 onto itself such that*

$h_0 = I$ (the identity),

$h_t = I$ outside U_i, and

(diameter of each image of a component of U_j under h_1) $< \varepsilon$.

Proof. Let N be a polyhedral tubular neighborhood of G_{i+1} so thin that $N \subset U_{i+1}$ and there is a finite number of mutually exclusive horizontal disks $E_1, E_2, \ldots, E_n$ such that $\operatorname{Bd} E_i \subset \operatorname{Bd} N$, $\operatorname{Int} E_i \subset N$, and the closure of each component of $N - \sum E_a$ is a tame 3-cell of diameter less than $\varepsilon/2$.

Let g_t $(0 \leqslant t \leqslant 1)$ be an isotopy of E^3 onto itself such that $g_0 = I$, $g_t = I$ outside U_i, g_1 is piecewise linear and takes U_{i+1} onto N.

It follows from Theorem 9 that there is an integer j and a finite collection of mutually exclusive polyhedral disks $D_1, D_2, \ldots, D_n$ such that for $a = 1, 2, \ldots, n$, $\operatorname{Bd} D_a = g_1^{-1}(\operatorname{Bd} E_a)$, $\operatorname{Int} D_a \subset U_{i+1}$, and no component of U_j intersects two D_a's.

We now show that there is an isotopy k_t $(0 \leqslant t \leqslant 1)$ of E^3 onto itself such that $k_0 = I$, $k_t = I$ outside U_{i+1}, and k_1 takes D_1 onto $g_1^{-1}(E_1)$. The first part of our isotopy is to free D_1 from $g_1^{-1}(E_2)$. Suppose D_1 and $g_1^{-1}(E_2)$ are in general position and that T is a 2-sphere in their sum which is the sum of a disk in D_1 and a disk in $g_1^{-1}(E_2)$. It follows from Alexander's Sphere Theorem ([1], [15]) that we can suppose with no loss of generality that T is the boundary of a tetrahedron $abcd$ whose base abc is the disk in $T \cdot g_1^{-1}(E_2)$. Since the three lateral faces of a tetrahedron can be pushed down onto the base of a tetrahedron by an isotopy that does not move any point that is far from the solid tetrahedron, we find that there is an isotopy of E^3 onto itself that is fixed outside U_{i+1} and which pushes D_1 onto $\big(D_1 - (abd + acd + bcd)\big) + abc$. More details of such an isotopy are found in the proof of Theorem 7.1 of [10]. The isotopy may be continued to push D_1 to one side of abc and thereby reduce the number of components of the intersection of the adjusted D_1 and $g_1^{-1}(E_2)$. A finite number of such reductions changes D_1 so that its image does not intersect $g_1^{-1}(E_2)$. Another sequence of isotopies alters D_1 until it does not intersect any $g_1^{-1}(E_i)$ $(i \neq 1)$. In constructing this sequence of isotopies, care is taken not to alter any simplification accomplished at a preceding stage. We suppose that $k_{1/2}(D_1)$ is a polyhedral disk which lies except for its boundary in the 3-cell C which is the closure of a component of $N - \big(g_1^{-1}(E_2) + g_1^{-1}(E_3) + \ldots + g_1^{-1}(E_n)\big)$. It follows from Alexander's Sphere Theorem that there is a homeomorphism of C onto itself that is fixed on $\operatorname{Bd} C$ which takes $k_{1/2}(D_1)$ onto $g_1^{-1}(E_1)$. It follows from Alexander's Deformation of an n-Cell Theorem [2] that k_t $(0 \leqslant t \leqslant \frac{1}{2})$ can be extended to an isotopy k_t $(0 \leqslant t \leqslant 1)$ such that $k_s = k_{1/2}$ $(\frac{1}{2} \leqslant s \leqslant 1)$ outside C and $k_1(D_1) = g_1^{-1}(E_1)$.

By repeating the procedure suggested in the preceding paragraph for each D_i, we find that there is an isotopy K_t $(0 \leqslant t \leqslant 1)$ of E^3 onto itself such that $K_0 = I$, $K_t = I$ outside U_{i+1}, and K_1 takes each D_a onto the corresponding $g_1^{-1}(E_a)$. The required isotopy h_t is $g_t K_t$.

Let V be a component of U_j. To see that diameter $h_1(V) < \varepsilon$, note that V lies in the sum of a D_a and two adjacent components of $U_{i+1} - \sum D_a$. However, each component of $U_{i+1} - \sum D_a$ goes under h_1 into a component of $N - \sum E_a$ and hence goes into a set of diameter less than $\varepsilon/2$. Hence V would go into a set of diameter less than ε.

THEOREM 11. *If Υ is topologically E^3, then for each integer i there is a pseudo isotopy h_t $(0 \leqslant t \leqslant 1)$ of E^3 onto itself such that*

$h_0 = I$,

$h_t = I$ outside U_i,

h_t is a homeomorphism for $0 \leqslant t < 1$, and

$h_1^{-1}(p)$ is an element of H for each point p of E^3.

Proof. This result follows from repeated applications of Theorem 10.

It follows from Theorem 10 that there is an integer i_1 and an isotopy h_t $(0 \leqslant t \leqslant \frac{1}{2})$ such that $h_0 = I$, $h_t = I$ outside U_i, and diameter $h_{1/2}(V) < \frac{1}{2}$ for each component V of U_{i_1}.

Let ε_1 be a positive number so small that diameter $h_{1/2}(X) < 1/3$ if X is a subset of U_{i_2} of diameter less than ε_1.

It follows from Theorem 10 that there is an integer i_2 and an isotopy k_t $(\frac{1}{2} \leqslant t \leqslant \frac{3}{4})$ of E^3 onto itself such that

$k_{1/2} = I$,

$k_t = I$ *outside* U_{i_1}, *and*

diameter $k_{3/4}(V) < \varepsilon_1$ *if* V *is a component of* U_{i_1}.

Then h_t $(\frac{1}{2} \leqslant t \leqslant \frac{3}{4})$ is $k_t h_{1/2}$. Note that diameter $h_{3/4}(V) < \frac{1}{3}$ if V is a component of U_{i_2}.

Continuing in this fashion we define h_t $(\frac{3}{4} \leqslant t \leqslant \frac{7}{8})$ so that $h_t = h_{3/4}$ outside U_{i_2} and there is an integer i_3, such that diameter $h_{7/8}(V) < \frac{1}{4}$ if V is a component of U_{i_3}. Also h_t $(\frac{7}{8} \leqslant t \leqslant \frac{15}{16})$ is such that $h_t = h_{7/8}$ outside U_{i_3} and there is an i_4 such that diameter $h_{15/16}(V) < \frac{1}{5}$ if V is a component of U_{i_4}. In similar fashion we define h_t $(\frac{15}{16} \leqslant t \leqslant \frac{31}{32})$, h_t $(\frac{31}{32} \leqslant t \leqslant \frac{63}{64})$, ... The limit of h_t as t approaches 1 is h_1.

8. Why four Q**'s?** Let A, B be two planes between Θ_1 and Θ_2 as shown in Figure 10. We suppose that U_1 is so thin that $\overline{U}_1 \cdot A$ (as is $\overline{U}_1 \cdot B$) is the sum of eight mutually exclusive disks such that each disk contains a point of G_1.

One of the things that leads one to believe that Υ may be topologically different from E^3 is the suspicion that if h is any homeomorphism of E^3 onto itself that is fixed outside U_1, then for each integer n there is a component V of G_n such that $h(V)$ intersects both A and B. If this is shown, it follows from either Theorem 9, 10, or 11 that Υ is topologically different from E^3.

We mention why we use four Q's in describing each component of G_i instead of two or three.

Why not two Q**'s?** If we used only two Q's in describing G_1, there would be a homeomorphism h of E^3 onto itself which is fixed except near G_1 as shown in Figure 11 such that $h[P(i), Q(j)]$ $(i = 1, 2;\ j = 1, 2)$ does not intersect both A and B. Hence there is a homeomorphism h of E^3 onto itself fixed except near G_1 such that for each component V of G_2, $h(V)$ does not intersect both A and B.

Why not three Q**'s?** If there were three Q's in G_1 instead of two, it could be shown that if h is a homeomorphism of E^3 onto itself fixed except near G_1, then for some component V of G_2, $h(V)$ intersects both A and B. However, $h(V)$ might fail, to intersect A and B in as essential a fashion as does G_1. For example, if V is a component of G_2 replacing $[P(1), Q(1)]$ of G_1 and $h(V)$ intersects A and B as shown in Figure 12, then h can be such for that for no component V' of G_3 near V does $h(V')$ intersect both A and B.

With three Q's there is a very effective entwining of the $G(i, j)$'s near the Q's because, as shown in Figures 7 and 9, if any $Q(i, j, m)$ is moved upward to shorten $[P(1, j, 1), Q(1, j, m)] + [P(1, j, 2), Q(1, j, m)]$ then each $[P(2, j, n), Q(2, j, n)] + [P(2, j, 2), Q(2, j, n)]$ is stretched.

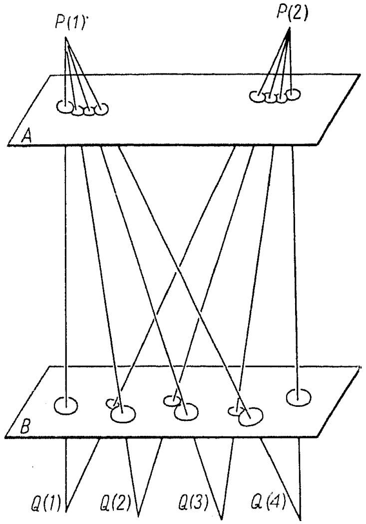

Fig. 10

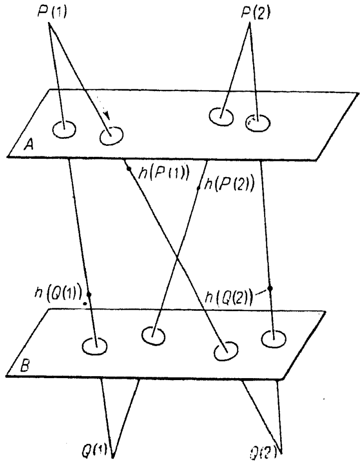

Fig. 11

However, the entwining is not so effective near the P's because, as shown in Figure 6, $P(1, 1, 1)$ may be pulled down to shorten $[P(1, 1, 1), Q(1, 1, 1)] + [P(1, 1, 1), Q(1, 1, 2)]$ without altering any of $[P(1, 2, 1), Q(1, 2, 2)]$, $[P(1, 2, i), Q(1, 2, 3)]$, $[P(1, 3, i), Q(1, 2, 2)]$, or $[P(1, 2, i), Q(1, 2, 3)]$. This is a reason that we use more than three Q's. There is a homeomorphism h such that h pulls $P(1, 1, i)$, $P(1, 2, i)$, and $P(2, 3, i)$ $(i = 1, 2)$ below A to shorten the upper half of $[P(1, 1, i), Q(1, 1, 1)]$, $[P(1, 1, i), Q(1, 1, 2)]$, $[P(1, 2, i), Q(1, 2, 2)]$, $[P(1, 2, i), Q(1, 2, 3)]$, $[P(2, 3, i), Q(2, 3, 2)]$, $[P(2, 3, i), Q(2, 3, 3)]$ and pulls each $Q(i, j, k)$ above B to shorten the lower half of each $[P(2, 1, r), Q(2, 1, s)]$, $[P(2, 2, r), Q(2, 2, s)]$, and $[P(1, 3, r), Q(1, 3, s)]$. Although for each component V of G_2, $h(V)$ intersects each of A and B, each intersects in an innessential fashion as depicted in Figure 12.

Suppose that U is a component of U_i and h is a homeomorphism of E^3 onto itself. We would like to find a definition of an essential intersection of $h(U)$ with A and B such that if $h(U)$ intersects A and B in this effective way, then for some component U' of U_{i+1} in U, $h(U')$ intersects A and B in this essential fashion. We could then use induction, along with Theorem 10, to show that Υ is topologically different from E^3. In the next three paragraphs we make a guess as to what an essential intersection might be.

An arc is said to have oscillation k with respect to A and B if it contains k points of $A+B$ such that no two of these points which are adjacent on the arc belong to the same one of A and B. Note that if an arc has oscillation k, it has oscillation j for any positive integer j less than k. A simple closed curve has oscillation k if it contains an arc of oscillation k with respect to A and B.

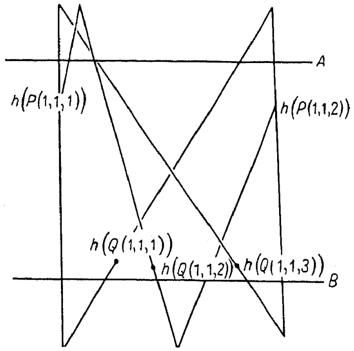

Fig. 12

Property X. An image G' of G_1 has Property X if either some simple closed curve in G' has oscillation 6 or G' contains a θ curve such that each simple closed curve in the θ curve has oscillation 4.

Essential intersection. Suppose that U is a component of U_i and h is a homeomorphism of E^3 onto itself. $h(U)$ has an essential intersection with A and B if for each homeomorphism h' of E^3 onto itself that agrees with h on E^3-U, $h'(U\cdot G_i)$ has Property X.

Suppose that the G's are defined with four Q's as suggested in Section 6 and h is a homeomorphism of E^3 onto itself that is fixed outside U_1. Then $h(U_1)$ intersects A and B is an essential fashion. It follows from Theorem 10 that Υ is topologically different from E^3 if the following conjecture is true.

CONJECTURE. *For each component U of U_i such that $h(U)$ intersects A and B in an essential fashion there is a component U' of $U\cdot U_{i+1}$ such that $h(U')$ intersects A and B in an essential fashion.*

While we do not prove the conjecture, we can prove the following theorem.

THEOREM 12. *If $h(G_1)$ has Property X, then there are an i, j, and k such that $h([P(i),Q(j)]+[P(i),Q(k)])$ has oscillation 3.*

The above theorem is proved by considering where the various points of $A\cdot h(G_1)$ and $B\cdot h(G_1)$ might be.

Here is how one might use Theorem 12 to try to establish the conjecture. Suppose $U=U_1$. Consider the two intermediate open sets in U_1

as depicted in Figure 9, and let W be the one of these containing $[P(i),Q(j)]+[P(i),Q(k)]$ of Theorem 12. Try to show from Theorem 12 that we have a situation something like that depicted in Figure 13 where $A\cdot h(\overline{W})$ contains two disks separating the loops of $h(W)$ associated with $h(Q(i))$ and $h(Q(j))$ while $B\cdot h(\overline{W})$ contains a disk between these two disks as shown. If we were to get a situation exactly like that depicted

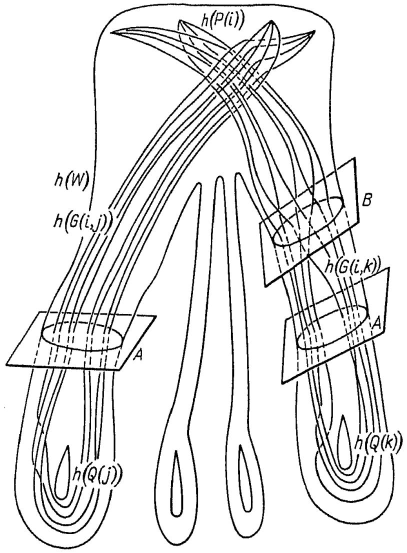

Fig. 13

in Figure 13, it could probably be shown that if h' is a homeomorphism of E^3 onto itself that agrees with h on E^3-W, then one of $h'(G(i,j))$, $h'(G(i,k))$ has Property X. It would be desirable to recognize a situation that is enough like that shown in Figure 13 so that it could be shown that one of $h'(G(i,j))$, $h'(G(i,k))$ has Property X but is general enough so that it can be shown that the situation occurs.

The more Q's we use in defining the G's, the more complicated the decomposition space Υ appears. The reason we stop at four is that we feel that the conjecture may be true.

9. Questions. Is the decomposition space Υ disussed in the preceding three sections topologically different from E^3? Is the conjecture of Section 8 true?

Suppose that T is a double torus as shown in Figure 14; J_1, J_2 are two simple closed curves in $\mathrm{Int}\, T$ as shown; $A \cdot T$ contains two disks; and $B \cdot T$ contains a disk between these two disks as shown. If h is a homeomorphism of E^3 onto itself that is fixed on $E^3 - T$, need one of $h(J_1)$, $h(J_2)$ have oscillation 4 with respect to A and B? An affirmative answer might be obtained by considering a universal covering space of T (see proof of Theorem 5). An affirmative answer might help in

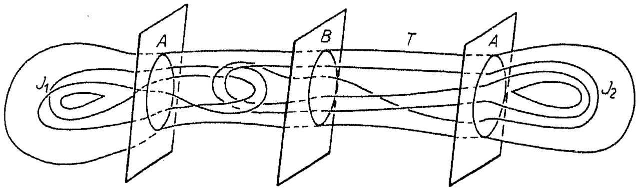

Fig. 14

showing that if $G(i, j)$, $G(i, k)$ are as shown in Figure 13 and h' is a homeomorphism of E^3 onto itself that agrees with h on $E^3 - W$, then one of $h'(G(i, j))$, $h'(G(i, k))$ has Property X.

In each of the following questions we suppose that H' is a decomposition of E^3 whose nondegenerate elements are straight line segments and Υ' denotes the resulting decomposition space.

Suppose that the sum of the nondegenerate elements of H' is the sum of two closed sets X_1, X_2 each of which is the sum of elements of H' and H_i is used to denote the decomposition whose nondegenerate elements are the elements of H' in X_i. Is Υ topologically E^3 if each of Υ_1, Υ_2 is? An answer even in the case where $X_1 \cdot X_2 = 0$ would be interesting. McAuley's result [14] is one phase of considering the case where there are infinitely many X's.

If Υ' is topologically different from E^3, does H' contain a Cantor set of segments such that if H_3 is the decomposition of E^3 whose nondegenerate elements are elements of this Cantor set of segments, then Υ_3 is topologically different from E^3?

For each planar circular disk D and each pair of positive numbers ε, δ let $H(D, \varepsilon, \delta)$ denote the elements of H' with diameters greater than or equal to ε which intersect D within δ of its center and whose directions differ from the normal to D by no more than δ degrees. If H' is topologically different from E^3, does there exist a planer circular disk D and a positive number ε such that for each $\delta > 0$, the decomposition space whose only nondegenerate elements are those of $H(D, \varepsilon, \delta)$ is topologically different from E^3?

Does a point-like decomposition of E^3 yield E^3 if it yields a 3-manifold?

Does each point of the decomposition space Γ discussed in Sections 3-5 have an arbitrarily small simply connected neighborhood?

Does one obtain Euclidean 4-space if one takes the cartesian product of a Euclidean line with either the decomposition space Γ of Sections 3-5 or the decomposition space Υ of Sections 6-8? This question is motivated by [5].

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