This completes the proof of the Main Theorem.

Remark. With our definition of the (n, k)-cohomotopy groups as direct limits, theorems 5 and 6 of [3] remain true for any compact space X (1).

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(1) Theorems 5 and 6 of [3] have been proved independently by R. Engelking.

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On automorphisms of relatively free groups

by

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1. This paper deals with automorphisms of relatively free groups, i.e. groups which can be represented as $F(V) = F/V$ where F is a free group and V its fully invariant subgroup. A study of the theory of such groups was initiated by B. H. Neumann [7], who obtained many interesting results, especially for finite relatively free groups. It was continued by P. Hall [2] with special interest in the splitting properties of relatively free groups. Malcev [4] has solved the question what subgroups of nilpotent free groups are nilpotent free groups; see also my paper [6]. The main problem of this paper, i.e. the description of the groups of automorphisms, as far as I know, has not been investigated hitherto. There were known only some theorems, e.g. in Malcev's paper [3] (theorems 5a, 6a, 7a, p. 27), which are marked in this paper as theorems 3 and 4.

In this paper there are investigated (theorems 1 and 2) connections between automorphisms of a relatively free group G and automorphisms of its (abelian free) factor group G/G' (those automorphisms can be described by some matrices), under the assumption that G is either finitely generated or residually nilpotent. The aim of these investigations is a description of the structure of the group of automorphisms for these relatively free groups which are nilpotent. This is given by theorems 5 and 6.

At the end of this brief introduction I wish to express my gratitude to A. L. Szmielkin from Moscow for many helpful suggestions, and to E. Sąsiada from Toruń for his considerations concerning formulation questions.

2. Now we shall give some of the terminology and basic facts. By a base of $F(V)$ we mean a free generating system, i.e. such a set X of generators that every mapping $\mu(X) \leqslant F(V)$ can be extended to an endomorphism φ of $F(V)$ such that $\varphi(x) = \mu(x)$ for $x \in X$. From this definition it follows at once that an endomorphism is uniquely determined by its values on the base of the relatively free group.

For a relatively free group there exists a base, and all bases of a group have the same number of elements, which is called the rank of the group.

In M. Hall's book [1] we find the proof of the following theorem, valid for the free group (theorem 7.3.4, p. 111): all automorphisms of the free group with a finite base $x_1, \ldots, x_r$ are generated by the automorphisms:

(2.1) P_{ij}: $x_i \to x_j\,, \quad x_j \to x_i\,, \quad x_k \to x_k\,, \quad k \neq i, j;$

(2.2) V_i: $x_i \to x_i^{-1}, \quad x_j \to x_j\,, \quad j \neq i;$

(2.3) W_{ij}: $x_j \to x_i \cdot x_j\,, \quad i \neq j\,, \quad x_k \to x_k\,, \quad k \neq j;$

where $i \neq j$ are any of the numbers $1, \ldots, r$.

The proof is based on the Schreier-Nilsen method. It cannot be used in the case of any relatively free group, and the theorem fails in this case. For example when F is the free group and V the subgroup F^n generated by the n-th powers of elements from F, then for any base $x_1, x_2, \ldots$ of $F(F^n)$, and any integers $n_1, n_2, \ldots$ each prime to n, the mapping

(2.4) Q: $x_i \to x^{n_i}$

is an automorphism of $F(F^n)$, which is different from those given above. Some other significant cases of the failure of this theorem are discussed by theorem 3 of this paper. Now we shall prove the following:

(2.5) *For any base X of a group G, the endomorphisms P_{ij}, V_i, and W_{ij}, $i \neq j$, defined by formulas* (2.1)-(2.3) *are automorphisms.*

Proof. The proof consists in proving that the images are bases. It is quite obvious that the images are generating sets. We prove that they are free sets. According to the definition, we need to prove that every mapping μ, e.g. of $W_{ij}(X)$ into the group G, can be extended into an endomorphism φ. The set X is a base, consequently we can extend a mapping $\mu_1(X)$ such that

$$\mu_1(x_j) = \mu\big(W_{ij}(x_i)\big)^{-1} \cdot \mu\big(W_{ij}(x_j)\big)\,,$$
$$\mu_1(x_k) = \mu\big(W_{ij}(x_k)\big)\,, \qquad k \neq j\,,$$

to an endomorphism φ such that $\varphi(x) = \mu_1(x)$ for $x \in X$. This φ is an extension of $\mu\big(W_{ij}(X)\big)$, since

$$\varphi\big(W_{ij}(x_j)\big) = \varphi(x_i \cdot x_j) = \varphi(x_i) \cdot \varphi(x_j) = \mu_1(x_i) \cdot \mu_1(x_j)$$
$$= \mu\big(W_{ij}(x_i)\big) \cdot \big[\mu\big(W_{ij}(x_i)\big)^{-1} \cdot \mu\big(W_{ij}(x_j)\big)\big] = \mu\big(W_{ij}(x_j)\big)\,,$$

and

$$\varphi\big(W_{ij}(x_k)\big) = \varphi(x_k) = \mu_1(x_k) = \mu\big(W_{ij}(x_k)\big) \qquad \text{for} \qquad k \neq j\,.$$

The proof of $P_{ij}(X)$ and $V_i(X)$ being free sets is quite similar, and therefore we shall omit it.

In the investigations of this paper we shall make a distinction between two possible cases. The first when $V \leqslant F'$, and the second when $V = F^n \cdot (V \cap F')$ for some $n > 0$. In the first case all elements of any base of $F(V)$ are of infinite order, in the second they are all of finite

order equal to n. In the sequel we shall call: the number 0, in the first case, and the number n, in the second case, the exponents of a base.

(2.6) *For any base X of a group G, having the exponent $n > 0$, if n_i are integers prime to n, then the endomorphism Q given by formula* (2.4) *is an automorphism of G.*

Proof. When n is zero, this is trivially true, since only the integers ± 1 are prime to 0, If $n > 0$, then Q has an inverse Q': $x_i \to x_i^{a_i}$, $n_i a_i \equiv 1 \pmod{n}$. This completes the proof.

3. Now we shall investigate the mappings induced in a quotient group. Let N be a normal subgroup of a group G and φ the natural homomorphism with the kernel N. Then for every endomorphism ε of G, which maps N into itself, there exists a uniquely determined endomorphism ε' of G/N, which forms the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & G/N \\ \varepsilon \downarrow & & \downarrow \varepsilon' \\ G & \xrightarrow{\varphi} & G/N \end{array}$$

The mapping φ^* of ε onto ε' is multiplicative, i.e. if $\varepsilon_1, \varepsilon_2$ map N into itself, then $\varphi^*(\varepsilon_1 \cdot \varepsilon_2) = \varphi^*(\varepsilon_1) \cdot \varphi^*(\varepsilon_2)$. In the case when N is a characteristic subgroup of G the mapping φ^* is a homomorphism of the group of automorphisms of G into that of G/N. If G is a relatively free group, then for any normal subgroup N and any endomorphism ε' of G/N there exists an endomorphism ε of G which is mapped by φ^* onto ε'. Note that for ε' to be an automorphism, ε need not be an automorphism, as is shown in example 1 of section 4.

When $G = F(V)$ is a relatively free group and N its fully invariant subgroup, then G/N is a relatively free group $F(U)$, where $U \geqslant V$ is a fully invariant subgroup of F. A base of $F(V)$ is mapped by a natural homomorphism φ onto a base of $F(U)$. It is easy to see that if ϱ is automorphism of $F(V)$ defined by one of the formulas (2.1)-(2.4), then $\varrho' = \varphi^*(\varrho)$ is an automorphism of $F(U)$ defined by the same formulas as those written for the images of the preceding base. Some questions may arise for $\varrho = Q$ defined by formula (2.4). Let us denote the exponent of the base of $F(V)$ by n, and that of $F(U)$ by m. Then $V = F^n \cdot (V \cap F')$ and $U = F^m \cdot (U \cap F')$. But since $U \geqslant V$, the integer m is a divisor of n; then all integers n_i from formula (2.4) prime to n are prime to m.

If for a homomorphism φ of a relatively free group $F(V)$ the kernel is $V \cdot F'/V$, then for every automorphism α of $F(V)$ the induced automorphism $A = \varphi^*(\alpha)$ is represented by a matrix $r \times r$:

$$A = \begin{bmatrix} a_{11}, \ldots, a_{1r} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{r1}, \ldots, a_{rr} \end{bmatrix}$$

with integral coefficients, or with coefficients which are integers modulo n. The coefficients of A are defined, by choosing a base X of $F(V)$, by formulas

$$A(\bar{x}_i) = \prod_{j=1}^{r} \bar{x}_j^{a_{ij}}, \qquad i = 1, \ldots, r,$$

for the image $\bar{X} = \varphi(X)$. (Abelian groups are written multiplicatively.) For instance the automorphisms P_{ij}, V_i, W_{ij}, $i \neq j$, are matrices of elementary operations: e.g.

$$P_{12} = \begin{bmatrix} 0 & 1 & & & \\ 1 & 0 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}, \quad V_1 = \begin{bmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}, \quad W_{12} = \begin{bmatrix} 1 & 1 & & & \\ 0 & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}.$$

By multiplying the matrix A on the left and on the right on these matrices we can transform A into a diagonal matrix $E = [e_{ij}]$; $e_{ij} = 0$ for $i \neq j$. (This is also true for rectangular matrices.)

Suppose that A is an automorphism. Since E, being a product of automorphisms, is an automorphism, the coefficients e_{ii} are unities. This proves the existence of an automorphism Q given by formula (2.4), such that $Q \cdot E = 1$ is the identity automorphism. Summarising the results so far: we have proved for any automorphism A of $F(V \cdot F')$ the existence of automorphisms A_1, A_2, Q of $F(V \cdot F')$. These automorphisms have preimages which are automorphisms of $F(V)$; moreover $A_1 \cdot A \cdot A_2 = E$, $Q \cdot E = 1$. This proves that $A = A_1^{-1} \cdot Q^{-1} \cdot A_2^{-1}$ has a preimage which is an automorphism of $F(V)$. This directly implies the following theorem:

THEOREM 1. *Let $F(V)$ be a relatively free group of finite rank* [1], *and let $\mathfrak{A}(F(V))$ be a group of automorphisms. Then, for the natural homomorphism φ onto $F(V \cdot F')$, the induced homomorphism φ^* is onto $\mathfrak{A}(F(V \cdot F'))$.*

4. Now we shall investigate endomorphisms of a relatively free group $F(V)$ inducing automorphisms of the abelian relatively free group $F(V \cdot F')$. There is a lemma of P. Hall ([2], lemma 1, p. 349) which states that every endomorphism of a residually nilpotent group G which induces an identical automorphism of G/G' is an injection. We shall give a stronger result in the case of relatively free groups which are residually nilpotent.

Let a be an endomorphism of $F(V)$ which induces an automorphism A of $F(V \cdot F')$ and let (x_i) be a countable base of $F(V)$ and write

[1] The theorem is true also for countable rank; this can be deduced by similar argument as in section 4.

$H_s = \{x_1, \ldots, x_s\}$. Then for $s \leqslant r$ $H_s \leqslant H_r$ and $F(V) = \bigcup_{i=1}^{\infty} H_i$. The matrix A is row-finite. We can write it in the following form:

$$A = \begin{bmatrix} a_{11}, \ldots, a_{1n}, 0, \ldots, 0 \\ \ldots\ldots\ldots\ldots \\ a_{s1}, \ldots, a_{sn}, 0, \ldots, 0 \\ a_{s+1,1}, \ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots \end{bmatrix}.$$

By the same method as that used in section 3 we can prove that the following matrices exist: matrix A_s, acting on n columns of A; matrix A_1, acting on s rows of A, (both being products of elementary matrices), and matrix Q, given by the formula (1.4), such that

$$I = Q \cdot A_1 \cdot A \cdot A_s$$

has the form

$$I = \begin{bmatrix} \begin{array}{|ccc|} 1 & & \\ & 1 & 0 \\ 0 & & 1 \end{array} \\ \varrho_{s+1,1}, \ldots\ldots \\ \ldots\ldots\ldots \end{bmatrix}.$$

Note that the assumption that the induced mapping of $F(V, F')$ is an automorphism has been used here to prove that the first s elements of the diagonal can be turned into *unities*. Denoting by a_1, a_s and β the automorphisms of $F(V)$ which are preimages of A_1, A_2 and Q, and defining $\eta = \beta \cdot a_1 \cdot a \cdot a_s$, we obtain $\varphi^*(\eta) = I$. This proves that $\eta(x_1) = x_1 \cdot k_1, \ldots, \eta(x_s) = x_s \cdot k_s$ for some $k_1, \ldots, k_s \in F(V)'$. Now the endomorphism ε defined as follows: $\varepsilon(x_1) = \eta(x_1), \ldots, \varepsilon(x_s) = \eta(x_s)$ and $\varepsilon(x_i) = x$ for $i \neq 1, \ldots, i \neq s$ is by the lemma of P. Hall an injection. Here we use the assumption that the group is residually nilpotent. For $x \in H_s$ we have the equality $\varepsilon(x) = \eta(x)$. Then an equality

$$a \cdot a_s(x) = a_1^{-1} \cdot \beta^{-1} \cdot \varepsilon(x)$$

is valid for $x \in H$. This proves that $a \cdot a_s$ restricted to H is a composition of one-to one mappings, and consequently a one-to-one mapping on H. For $r > s$ we can construct in the same manner a_r, so that $a_r(x) = a_s(x)$ for $x \in H_s$, and prove that $a \cdot a_r$ is on H_r a one-to-one mapping. Then $\bigcup_{i=1}^{\infty} a_i = \gamma$ is an automorphism of $F(V)$, and $a \cdot \gamma$ is an injection. This proves the following theorem.

THEOREM 2. *An endomorphism of a relatively free and residually nilpotent group G of a finite or countable rank which induces an automorphism of G/G' is an injection.*

Now we shall give some counter examples connected with this theorem. The first example proves that the endomorphism dealt with in the theorem does not have to be an automorphism of the group G.

EXAMPLE 1. Let F be a free group with two generators a and b. By Magnus's theorem it is residually nilpotent. Let us define

$$\alpha(a) = a^{-1}\cdot b^{-1}\cdot a\cdot b\cdot a = a\cdot(a, b\cdot a),$$
$$\alpha(b) = b^{-1}\cdot a^{-1}\cdot b\cdot a\cdot b = b\cdot(b, a\cdot b).$$

The image $\alpha(F)$ is not the complete F since the length of any non-empty word belonging to $\alpha(F)$ is $\geqslant 5$. This proves that α is not an automorphism of F, but it induces an identical automorphism of F/F'.

The second example shows that theorem 2 fails when the assumption is weaker, i.e. when the induced mapping is one-to-one.

EXAMPLE 2. Let F be a free group with two generators a and b. Denote by V a fully invariant subgroup generated by words $((x_1, x_2), x_3)$ and $(x_1, x_2)^2$. The group $F(V)$ is metabelian and therefore residually nilpotent. The endomorphism defined by $\alpha(a) = a^2$, $\alpha(b) = b$ induces in $F(V, F')$ an isomorphism. To prove that α is not a one-to-one mapping we notice that the derived subgroup $F(V)'$ consists of two elements, 1 and $(a, b) \neq 1$. It is mapped by α onto 1 since

$$\alpha((a, b)) = (\alpha(a), \alpha(b)) = (a^2, b) = (a, b)^2 = 1.$$

Now we shall give the results of theorem 2 under a condition on the group G:

(4.1) *For every subgroup B if $B\cdot G' = G$ then $B = G$.*

This condition is always meet when G is nilpotent. (Cf. [1], corollary 10.3.3.)

We shall now prove the following theorem:

THEOREM 3. *If a relatively free and residually nilpotent group G of a finite or countable rank satisfies condition (4.1), then every endomorphism of G is an automorphism if and only if it induces an automorphism of G/G'.*

Proof. Evidently every automorphism of any group G induces in G/G' an automorphism. Conversely, if α induces an automorphism in G/G', then, by theorem 2, it is a one-to-one mapping. For $B = \alpha(G)$, we have $B\cdot G' = G$. By (4.1) it follows that $B = G$, which proves that α is onto G and therefore an automorphism.

This theorem can be stated in another form as well.

THEOREM 4. *If a group G satisfies the assumptions of the preceding theorem, and if N is a normal subgroup contained in G', then, for every automorphism γ of the factor group G/N, there exists an automorphism of G which induces γ.*

Proof. The existence of the endomorphism of G which has these properties was noted in section 3 on page 405. Since $N \leqslant G'$, then this endomorphism induces an automorphism of G/G'. By theorem 3 it is an automorphism of G.

5. Now we shall consider the structure of the group of automorphisms for a relatively free and nilpotent group $F(V)$ of a finite or countable rank. By the nilpotency of $F(V)$ we can find an c such that F_{c+1}, and not F_c, is contained in V. Here F_c is the c-th term of the lower central series, i.e. $F_1 = F$, $F_2 = (F_1, F) = F'$, $F_{c+1} = (F_c, F)$.

Denote by $\varphi_c, \ldots, \varphi_2$ a sequence of natural homomorphisms,

$$F(V) \xrightarrow{\varphi_c} F(V\cdot F_c) \xrightarrow{\varphi_{c-1}} \ldots \xrightarrow{\varphi_2} F(V\cdot F')$$

with kernels $\mathrm{Ker}\varphi_i = V\cdot F_i/V\cdot F_{i+1}$, $i = 2, \ldots, c$. Moreover, denote the groups of automorphisms as $\mathfrak{A}_i = \mathfrak{A}(F(V\cdot F_i))$, $i = 2, \ldots, c$, $\mathfrak{A}_{c+1} = \mathfrak{A}(F(V))$.

Now we can state the following theorem [2].

THEOREM 5. *For $i = 2, \ldots, c$ the homomorphism φ_i^* maps $\mathfrak{A}_{i+1}$ onto $\mathfrak{A}_i$. The kernel $\mathrm{Ker}\varphi_i^*$ is an unrestricted direct product P of as many copies of $\mathrm{Ker}\varphi_i$ as a rank of $F(V)$.*

Proof. The first half of the theorem, that φ_i^* is a mapping onto $\mathfrak{A}_i$, follows directly from theorem 4. To prove the second half of the theorem we choose a base $X = (x_\lambda)_{\lambda\in\Lambda}$ of $F(V\cdot F_{i+1})$. If $\gamma \in \mathrm{Ker}\varphi_i^*$ then for $\lambda \in \Lambda$:

$$\gamma(x_\lambda) = x_\lambda\cdot g_\lambda(x_{\lambda_1}, \ldots, x_{\lambda_{r(\lambda)}}),$$

where $g_\lambda(x_{\lambda_1}, \ldots, x_{\lambda_{r(\lambda)}}) = g_\lambda(x) \in \mathrm{Ker}\varphi_i$. Conversely, every mapping γ given by this formula is, by theorem 3, an automorphism, and therefore it belongs to $\mathrm{Ker}\varphi_i^*$. It establishes the one-to-one correspondence between the element $g = g_\lambda(x)$; $\lambda \in \Lambda$, of the product P and the automorphism γ of $\mathrm{Ker}\varphi_i^*$. It remains to be proved that this correspondence is multiplicative.

Let $\eta \in \mathrm{Ker}\varphi_i^*$, then $\eta(x_\lambda) = x_\lambda\cdot h_\lambda(x)$, where $h_\lambda(x) \in \mathrm{Ker}\varphi_i$, for $\lambda \in \Lambda$. Then

$$\gamma\eta(x_\lambda) = x_\lambda\cdot g_\lambda(x)\cdot h_\lambda(x\cdot g) \quad \text{for} \quad \lambda \in \Lambda.$$

Now, since for every $\lambda \in \Lambda$, $g_\lambda(x)$, $h_\lambda(x) \in \mathrm{Ker}\varphi_i = V\cdot F_i/V\cdot F_{i+1}$, it is easy to find that

$$h_\lambda(x\cdot g) = h_\lambda(x_{\mu_1}\cdot g_{\mu_1}, \ldots, x_{\mu_{s(\lambda)}}\cdot g_{\mu_{s(\lambda)}}) = h_\lambda(x_{\mu_1}, \ldots, x_{\mu_{s(\lambda)}}) = h_\lambda(x)$$

by using formulas 10.2.1, p. 150 of book [1], or formulas P4, p. 261 of [6]. This proves the multiplicativity:

$$\gamma\eta(x_\lambda) = x_\lambda\cdot g_\lambda(x)\cdot h_\lambda(x), \quad \lambda \in \Lambda,$$

and completes the proof.

[2] This theorem was announced at the Second Hungarian Mathematical Congress. See [5].

The theorem just proved shows that the group $\mathfrak{A}(F(V))$ is an extension of the solvable group of c-th degree by the group $\mathfrak{A}(F(V\cdot F'))$. It is worth noticing that the group $\mathfrak{A}(F(V\cdot F'))$ is a group of all row-finite regular matrices over either the ring of rational integers, or the ring of integers modulo n. Now let us prove a stronger result.

THEOREM 6. *If a relatively free group $F(V)$ of a finite or countable rank is nilpotent and has nil-c, then the group of automorphisms $\mathfrak{A}(F(V))$ is an extension of a nilpotent group $\mathfrak{N}$ having nil-$(c-1)$ by the group $\mathfrak{A}(F(V\cdot F'))$.*

Proof. We have to prove that the kernel $\mathfrak{N}$ of the homomorphism φ^* of $\mathfrak{A}(F(V))$ onto $\mathfrak{A}(F(V\cdot F'))$ is a nilpotent group having nil-$(c-1)$. Here φ^* denotes the homomorphism which is induced by the natural homomorphism φ of $F(V)$ onto $F(V\cdot F')$. The φ^* is onto $\mathfrak{A}(F(V\cdot F'))$ by theorem 3.

Let us choose a base $(x_\lambda)_{\lambda\in\Lambda}$ of $F(V)$. Denote by $\mathfrak{N}_i$; $i=1,\dots,c$, sets of all $\gamma\in\mathfrak{A}(F(V))$ which are of the form

$$\gamma(x_\lambda)=x_\lambda\cdot g_\lambda(x)\,,\qquad \lambda\in\Lambda\,.$$

where $g_\lambda(x)\in F_{i+1}\cdot V/V$ for $\lambda\in\Lambda$. It is easy to see that $\mathfrak{N}_i$ is precisely the $\mathrm{Ker}(\varphi_c^*\dots\varphi_{i+1}^*)$, for $i=1,\dots,c-1$, and $\mathfrak{N}_c$ is the unity subgroup; the φ_i^* are defined in the same way as in theorem 5.

The proof will be complete when we prove that the sequence of subgroups

$$\mathfrak{N}=\mathfrak{N}_1\geqslant\mathfrak{N}_2\geqslant\dots\geqslant\mathfrak{N}_c=\{1\}$$

is a central series of $\mathfrak{N}$, i.e. that $(\mathfrak{N},\mathfrak{N}_i)\leqslant\mathfrak{N}_{i+1}$ for $i=1,\dots,c-1$.

Let $\eta\in\mathfrak{N}$ and $\gamma\in\mathfrak{N}_i$. We have to prove that $\eta^{-1}\cdot\gamma^{-1}\cdot\eta\cdot\gamma\in\mathfrak{N}_{i+1}$. By the definition of the subgroups $\mathfrak{N}_i$ we have

$$\gamma(x_\lambda)=x_\lambda\cdot g_\lambda(x);\qquad \gamma^{-1}(x_\lambda)=x_\lambda\cdot\tilde{g}(x);\qquad \lambda\in\Lambda\,.$$

where $g_\lambda(x)$, $\tilde{g}_\lambda(x)\in F_{i+1}\cdot V/V$ for $\lambda\in\Lambda$. And

$$\eta(x_\lambda)=x_\lambda\cdot h_\lambda(x);\qquad \eta^{-1}(x_\lambda)=x_\lambda\cdot\tilde{h}_\lambda(x);\qquad \lambda\in\Lambda\,,$$

where $h_\lambda(x)$, $\tilde{h}_\lambda(x)\in F'\cdot V/V$ for $\lambda\in\Lambda$. Find

$$\gamma\eta(x_\lambda)=x_\lambda\cdot g_\lambda(x)\cdot h_\lambda(x\cdot g);\qquad \lambda\in\Lambda\,.$$

One can easily prove that the right sides are, for $\lambda\in\Lambda$, modulo $F_{i+2}\cdot V/V$ equal to $x_\lambda\cdot h_\lambda(x)\cdot g_\lambda(x)$. Therefore there exists elements $\tilde{f}_\lambda(x)\in F_{i+2}\cdot V/V$ such that

$$\gamma\eta(x_\lambda)=x_\lambda\cdot h_\lambda(x)\cdot g_\lambda(x)\cdot\tilde{f}_\lambda(x)$$

for $\lambda\in\Lambda$. Now

$$\eta^{-1}\cdot\gamma\cdot\eta(x_\lambda)=x_\lambda\cdot\tilde{h}_\lambda(x)\cdot h_\lambda(x\cdot\tilde{h})\cdot g_\lambda(x\cdot\tilde{h})\cdot\tilde{f}_\lambda(x\cdot\tilde{h})$$

for $\lambda\in\Lambda$. The first factor is

$$x_\lambda\cdot\tilde{h}_\lambda(x)\cdot h_\lambda(x\cdot\tilde{h})=\eta^{-1}\cdot\eta(x_\lambda)=x_\lambda\qquad\text{for}\qquad\lambda\in\Lambda\,.$$

The second factor $g_\lambda(x\cdot\tilde{h})\cdot\tilde{f}_\lambda(x\cdot h)$, modulo $F_{i+2}\cdot V/V$, is equal to $g_\lambda(x)$ for $\lambda\in\Lambda$. Consequently there exist elements $\tilde{\tilde{f}}_\lambda(x)\in F_{i+2}\cdot V/V$ such that

$$\eta^{-1}\cdot\gamma\cdot\eta(x_\lambda)=x_\lambda\cdot g_\lambda(x)\cdot\tilde{\tilde{f}}_\lambda(x)\qquad\text{for}\qquad\lambda\in\Lambda\,.$$

For

$$\gamma^{-1}\cdot\eta^{-1}\cdot\gamma\cdot\eta(x_\lambda)=x_\lambda\cdot\tilde{g}(x)\cdot g_\lambda(x\cdot\tilde{g})\cdot\tilde{\tilde{f}}_\lambda(x\cdot\tilde{g})\,,\qquad\lambda\in\Lambda\,,$$

we have

$$x_\lambda\cdot\tilde{g}_\lambda(x)\cdot g_\lambda(x\cdot\tilde{g})=\gamma^{-1}\cdot\gamma(x_\lambda)=x_\lambda\,,\qquad\lambda\in\Lambda\,.$$

Moreover

$$f_\lambda(x)=\tilde{\tilde{f}}_\lambda(x\cdot\tilde{g})\in F_{i+2}\cdot V/V\qquad\text{for}\qquad\lambda\in\Lambda\,.$$

The equality

$$\gamma^{-1}\cdot\eta^{-1}\cdot\gamma\cdot\eta(x_\lambda)=x_\lambda\cdot f_\lambda(x)\,,$$

for $\lambda\in\Lambda$, proves that $\gamma^{-1}\cdot\eta^{-1}\cdot\gamma\cdot\eta\in\mathfrak{N}_{i+1}$, which completes the proof of the theorem.

As a corollary to this theorem and theorem 4 we have the following:

THEOREM 7. *Let G be a relatively free nilpotent group of a finite or countable rank, having nil-c, and let N be the normal subgroup which is contained in G'. Then the group $\mathfrak{A}(G)$ is an extension of a nilpotent group having nil-$(c-1)$ by the group $\mathfrak{A}(G/N)$.*

Proof. By theorem 4 the homomorphism induced by the natural homomorphism of G onto G/N maps $\mathfrak{A}(G)$ onto $\mathfrak{A}(G/N)$. The kernel is a subgroup of the group $\mathfrak{N}$ which is defined in the proof of theorem 6. Consequently it is nilpotent having nil-$(c-1)$. This concludes the proof.

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