Generalized cohomotopy groups as limit groups *

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1. Introduction. K. Borsuk introduced in [2] the notion of a generalized (n, k)-cohomotopy group $\pi_k^n(X)$ of a Δ-ANR space X. These groups are defined for each $k < 2n-1$ and are based on the concept of homotopic k-skeleton of a space. Some elementary properties of the (n, k)-cohomotopy groups were studied in [3]. In this paper we show that the Borsuk (n, k)-cohomotopy groups of a Δ-ANR space are direct limits of the corresponding groups of polyhedra. This will allow us to define the groups $\pi_k^n(X)$ for any compact space X.

2. Preliminaries. We recall here the definition of the Borsuk (n, k)-cohomotopy groups and their fundamental properties.

A closed subspace A of a space X with $\dim A \leqslant k$ is called a *homotopic k-skeleton of* X if every closed subspace of X of dimension $\leqslant k$ can be continuously deformed into A. If X is a simplicial complex then its polyhedral k-skeleton X^k is a homotopic k-skeleton of X in the sense of Borsuk. Borsuk also showed that any compact ANR space with property (Δ) (called here a Δ-ANR *space*, see [1]) contains a homotopic k-skeleton for each $k = 0, 1, \ldots$

Throughout this paper, let S be the n-sphere S^n. If A is a closed subspace of a space X, then $(A \subset X \to S)$ will denote the set of all mappings α: $A \to S$ which are extendable over X and $[A \subset X \to S]$ will denote the set of homotopy classes $[\alpha]$ of mappings $\alpha \in (A \subset X \to S)$. If $A = X$ then we shall write $(A \to S)$ instead of $(A \subset A \to S)$ and $[A \to S]$ instead of $[A \subset A \to S]$.

If $\dim A < 2n-1$, then $[A \to S]$ under a suitable group operation becomes a group called the n-th *cohomotopy group* $\pi^n(A)$ *of* A. If B is a space and f: $B \to A$ is a mapping, then the assignment $\alpha \to \alpha f$, for α: $A \to S$, defines a transformation $f^{\#}$: $[A \to S] \to [B \to S]$. If $\dim A$, $\dim B < 2n-1$, then $f^{\#}$: $\pi^n(A) \to \pi^n(B)$ is a homomorphism.

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* This work was done when the author was supported by the National Science Foundation under NSF-G 14779.

Let (X, A) be a closed pair (i.e., A is a closed subspace of a space X) with $\dim A < 2n-1$. In general, the set $[A \subset X \to S] \subset \pi^n(A)$ does not form a subgroup of $\pi^n(A)$. For example, if $S = S^2$, $X = S \times S$, s_0 is a point of S, $A = S \times s_0 \cup s_0 \times S$ and α, β: $A \to S$ are the mappings defined by $\alpha(x, y) = x$, $\beta(x, y) = y$, for each $(x, y) \in A$, then either of α and β can be extended over X. The sum $[\alpha]+[\beta]$, however, is represented by the mapping γ: $A \to S$ defined by $\gamma(x, s_0) = x$, $\gamma(s_0, y) = y$ and this mapping cannot be extended over X because the 2-sphere S does not admit a continuous multiplication.

Following Borsuk, we denote by $\pi^n(A \subset X)$ the subgroup of $\pi^n(A)$ generated by $[A \subset X \to S]$. The following properties of the groups $\pi^n(A \subset X)$ have been proved in [3] (the notation here will slightly differ from that used in [3]):

(2.1) *Let $(X, A), (Y, B)$ be closed pairs with* $\dim A, \dim B < 2n-1$, *f: $(Y, B) \to (X, A)$ be a mapping, and f_0: $B \to A$ be the partial mapping defined by f. Then the homomorphism $f_0^{\#}$: $\pi^n(A) \to \pi^n(B)$ induced by f_0 maps $[A \subset X \to S]$ into $[B \subset Y \to S]$ and $\pi^n(A \subset X)$ into $\pi^n(B \subset Y)$ and hence defines a homomorphism*

$$\bar{f}: \pi^n(A \subset X) \to \pi^n(B \subset Y)$$

with the following properties:

(i) *If f: $(X, A) \to (X, A)$ is the identity, then $\bar{f}$ is the identity.*

(ii) *If (Z, C) is a closed pair with* $\dim C < 2n-1$ *and g: $(Z, C) \to (Y, B)$ is a mapping, then $\overline{fg} = \bar{g}\bar{f}$.*

(iii) *If f, g: $(Y, B) \to (X, A)$ are two mappings and $f \simeq g$ as mappings of Y into X, then $\bar{f} = \bar{g}$.*

(See [3], Theorems 1 and 3.)

(2.2) *Let X be a compact Δ-ANR space, $k < 2n-1$, A be a homotopic k-skeleton of X, (Y, B) be a compact pair with* $\dim B \leqslant k$, *and f: $Y \to X$ be a mapping. Then f defines a unique homomorphism*

$$\bar{f}_{A,B}: \pi^n(A \subset X) \to \pi^n(B \subset Y)$$

with the following properties:

(i) *If f: $X \to X$ is the identity, then $\bar{f}_{A,A}$ is the identity.*

(ii) *If Y is a compact Δ-ANR space, B is a homotopic k-skeleton of Y, (Z, C) is a compact pair with* $\dim C \leqslant k$ *and g: $Z \to Y$, then $\bar{f}_{A,C} = \bar{g}_{B,C}\bar{f}_{A,B}$.*

(iii) *If f, g: $Y \to X$ and $f \simeq g$, then $\bar{f}_{A,B} = \bar{g}_{A,B}$.*

(iv) *If $f(B) \subset A$, then $\bar{f}_{A,B} = \bar{f}$.*

(See [3], Theorems 2 and 4).

In particular, we obtain the following corollaries:

(2.3) *If A and B are two homotopic k-skeletons of a compact Δ-ANR space X with $k < 2n-1$, then the identity mapping e: $X \to X$ induces an isomorphism $\bar{e}_{A,B}$: $\pi^n(A \subset X) \approx \pi^n(B \subset Y)$.*

Theorem (2.3) proved orginally by Borsuk in a slightly different way (see [2], Nr. 4) enables us to define, for $k < 2n-1$, the (n, k)-cohomotopy group $\pi_k^n(X)$ of a Δ-ANR space X as the abstract group isomorphic to $\pi^n(A \subset X)$, for a homotopic k-skeleton A of X. If $k = \dim X$, then $\pi_k^n(X) = \pi^n(X)$. We also have

(2.4) *If X, Y are compact Δ-ANR spaces, $k < 2n-1$, and f: $Y \to X$ is a mapping, then f induces a unique homomorphism*

$$f^{\#}: \pi_k^n(X) \to \pi_k^n(Y)$$

with the following properties:

(i) *If f: $X \to X$ is the identity, then $f^{\#}$ is the identity.*

(ii) *If Z is a compact Δ-ANR space and g: $Z \to Y$, then $(fg)^{\#} = g^{\#}f^{\#}$.*

(iii) *If f, g: $Y \to X$ and $f \simeq g$, then $f^{\#} = g^{\#}$.*

(iv) *If* $\dim X, \dim Y \leqslant k$ *then $f^{\#}$ coincides with the homomorphism of the cohomotopy groups induced by f.*

3. Limit groups. We shall consider the direct system of finite open coverings of a compact space X. If σ and τ are two coverings and τ is a refinement of σ, then we write $\sigma < \tau$.

Let σ be a covering of X and $N(\sigma)$ be the nerve of σ. If $V \in \sigma$, then v will denote the vertex of $N(\sigma)$ corresponding to V. A set $v_0, v_1, \ldots, v_k$ of vertices forms a simplex in $N(\sigma)$ if and only if $V_0 \cap V_1 \cap \ldots \cap V_k \neq 0$. If A is a closed subspace of X, then $N(\sigma|A)$ will denote the subcomplex of $N(\sigma)$ consisting of the simplexes $(v_0, v_1, \ldots, v_k)$ of $N(\sigma)$ such that $V_0 \cap V_1 \cap \ldots \cap V_k \cap A \neq 0$.

If σ is a covering of X, then $\varkappa_\sigma$: $X \to N(\sigma)$ will denote a canonical mapping corresponding to σ. It has the property that if $V \in \sigma$ and $x \in V$, then $\varkappa_\sigma(x) \in \operatorname{St} v$. This property defines the canonical mapping uniquelly up to a homotopy. If A is a closed subspace of X, then $\varkappa_\sigma$ maps A into $N(\sigma|A)$; the canonical mapping $\varkappa_\sigma$ restricted to A will be denoted by $\varkappa_{\sigma|A}$.

If σ and τ are two coverings of X and $\sigma < \tau$, then there exists a simplicial mapping $p_{\tau\sigma}$ of $N(\tau)$ into $N(\sigma)$ called a *projection*. It is defined uniquelly up to a homotopy by the condition that if v is a vertex of $N(\tau)$ and $p_{\tau\sigma}(v) = u$, then $V \subset U$. If A is a closed subspace of X, then the projection $p_{\tau\sigma}$ maps $N(\tau|A)$ into $N(\sigma|A)$. The projection $p_{\tau\sigma}$ restricted to the polyhedral k-skeleton $N^k(\tau)$ of $N(\tau)$ will be denoted by $p_{\tau\sigma}^k$. If $p_{\tau\sigma}$ is a projection and $\varkappa_\tau$ is a canonical mapping corresponding to τ, then $\varrho_{\tau\sigma}\varkappa_\tau$ is a canonical mapping corresponding to σ.

If X and Y are compact spaces, σ is a covering of Y, and $f\colon X\to Y$ is a mapping, then let $f^{-1}\sigma$ denote the covering of X made up of the inverse-images of the elements of σ. The mapping f induces then a simplicial mapping $f_\sigma\colon N(f^{-1}\sigma)\to N(\sigma)$ such that if τ is a refinement of σ then $f^{-1}\tau$ is a refinement of $f^{-1}\sigma$ and the diagram

$$\begin{array}{ccc} N(f^{-1}\sigma) & \xrightarrow{f_\sigma} & N(\sigma) \\ {\scriptstyle p_{f^{-1}\tau f^{-1}\sigma}}\uparrow & & \uparrow{\scriptstyle p_{\tau\sigma}} \\ N(f^{-1}\tau) & \xrightarrow{f_\tau} & N(\tau) \end{array}$$

is commutative up to a homotopy.

If σ and τ are two coverings of X, $k<2n-1$ and $\sigma<\tau$, then a projection of $N(\tau)$ into $N(\sigma)$ induces, by (2.4), a unique homomorphism $q_{\sigma\tau}\colon \pi^n_k(N(\sigma))\to\pi^n_k(N(\tau))$ such that if $\sigma<\varrho<\tau$ then $q_{\varrho\sigma}q_{\tau\varrho}=q_{\tau\sigma}$. Hence $\{\pi^n_k(N(\sigma)),q_{\sigma\tau}\}$ forms a direct system of groups over the directed system of all open finite coverings of X. We define

$$\overline{\pi}^n_k(X)=\varinjlim \pi^n_k(N(\sigma))\,.$$

If $k\geqslant \dim X$, then the family of all open finite coverings of X contains a cofinal subfamily whose nerves are of dimension $\leqslant k$. Therefore in this case the group $\overline{\pi}^n_k(X)$ is isomorphic to the n-th cohomology group of X.

(3.1) *If X and Y are compact spaces, $k<2n-1$ and $f\colon Y\to X$ is a mapping, then f induces a homomorphism*

$$\bar f^{\#}\colon \overline{\pi}^n_k(X)\to\overline{\pi}^n_k(Y)$$

with the following properties:

(i) *If $f\colon X\to X$ is the identity, then $\bar f^{\#}$ is the identity.*

(ii) *If X,Y,Z are compact spaces, $f\colon Y\to X$ and $g\colon Z\to Y$, then $\overline{(fg)}^{\#}=\bar g^{\#}\bar f^{\#}$.*

(iii) *If $f,g\colon Y\to X$ and $f\simeq g$, then $\bar f^{\#}=\bar g^{\#}$;*

(iv) *If $k\geqslant\dim X,\dim Y$, then $\bar f^{\#}$ coincides with the homomorphism $f^{\#}$ of the cohomotopy groups induced by f.*

Proof. If Σ is a cofinal subfamily of finite open coverings of X, then the coverings $f^{-1}\sigma$, $\sigma\in\Sigma$ form a cofinal subfamily of the family of finite open coverings of Y. The simplicial mappings $f_\sigma\colon N(f^{-1}\sigma)\to N(\sigma)$ induce, by (2.4), homomorphisms $f^{\#}_\sigma\colon \pi^n_k(N(\sigma))\to\pi^n_k(N(f^{-1}\sigma))$ which form a map of the direct system $\{\pi^n_k(N(f^{-1}\sigma))\}$ into $\{\pi^n_k(N(\sigma))\}$. We define $\bar f^{\#}\colon\overline{\pi}^n_k(X)\to\overline{\pi}^n_k(Y)$ to be the limit of the homomorphisms $f^{\#}_\sigma$.

The properties (i), (ii) and (iv) are obvious. To prove property (iii), we first show that, if $\varphi_0,\varphi_1\colon Y\to Y\times I$ are the inclusion mappings of Y into the lower and upper base of $Y\times I$, respectively, then $\bar\varphi_0^{\#}=\bar\varphi_1^{\#}\colon \overline{\pi}^n_k(Y\times I)\to\overline{\pi}^n_k(Y)$.

Let $\{\mathfrak{a}\}$ be an element of $\overline{\pi}^n_k(Y\times I)$ represented by $\mathfrak{a}\in\pi^n_k(N(\sigma))$, where σ is a covering of $Y\times I$. We can assume that the covering σ is "stacked", i.e. has the form

$$\sigma=\{V^i_{\cdot}\times I^i\}\,,\qquad i=1,\ldots,r;\; j=0,\ldots,s\,,$$

where $\tau=\{V^i\}$ is a covering of Y, $I^0=[0,t_1)$, $I=(t_{j-1},t_{j+1})$, for $1\leqslant j\leqslant s-1$, and $I^s=(t_{s-1},1]$, where $0=t_0<t_1<\ldots<t_s=1$. The stacked coverings form a cofinal subfamily of the family of all finite open coverings of $Y\times I$. We observe that

$$\varphi_0^{-1}\sigma=\tau=\varphi_1^{-1}\sigma\,.$$

Consider now the geometrical realizations $|N(\tau)|$ of $N(\tau)$ and $|N(\sigma)|$ of $N(\sigma)$. We easily see that there exists an inclusion mapping $\vartheta\colon |N(\tau)|\times I\to|N(\sigma)|$ such that, if $\psi_0,\psi_1\colon |N(\tau)|\to|N(\tau)|\times I$ are the inclusion mappings of $|N(\tau)|$ onto the lower and upper base of $|N(\tau)|\times I$, respectively, then $\varphi_{0\sigma}=\vartheta\psi_0$, $\varphi_{1\sigma}=\vartheta\psi_1$. Evidently, $\psi_0\simeq\psi_1$ and, therefore, by (2.4), we have

$$\varphi^{\#}_{0\sigma}\mathfrak{a}=(\vartheta\psi_0)^{\#}\mathfrak{a}=\psi_0^{\#}\vartheta^{\#}\mathfrak{a}=\psi_1^{\#}\vartheta^{\#}\mathfrak{a}=(\vartheta\psi_1)^{\#}\mathfrak{a}=\varphi^{\#}_{1\sigma}\mathfrak{a}\,.$$

It follows that $\bar\varphi_0^{\#}=\bar\varphi_1^{\#}$.

Now let $f,g\colon Y\to X$ be two mappings and let $h\colon Y\times I\to X$ be a homotopy connecting f to g. Then $f=h\varphi_0$, $g=h\varphi_1$ and, since $\bar\varphi_0^{\#}=\bar\varphi_1^{\#}$, we have $\bar f^{\#}=\bar g^{\#}$.

Remark. We observe that the limit group $\overline{\pi}^n_k(X)$ is isomorphic to the limit $\widetilde{\pi}^n_k(X)$ of the direct system of groups $\{\pi^n(N^k(\sigma)\subset N(\sigma))\}$, where σ is a covering of X and the homomorphisms

$$q_{\sigma\tau}=\bar p_{\tau\sigma}\colon \pi^n(N^k(\sigma)\subset N(\sigma))\to\pi^n(N^k(\tau)\subset N(\tau))$$

induced by the projections

$$p_{\tau\sigma}\colon (N(\tau),N^k(\tau))\to(N(\sigma),N^k(\sigma))\,,\qquad\text{where } \sigma<\tau,$$

according to (2.1). Each group $\pi^n(N^k(\sigma)\subset N(\sigma))$ is generated by the set $[N^k(\sigma)\subset N(\sigma)\to S]$ and the projection $p_{\tau\sigma}$ maps $[N^k(\tau)\subset N(\tau)\to S]$ into $[N^k(\sigma)\subset N(\sigma)\to S]$, so that the system of sets $\{[N^k(\sigma)\subset N(\sigma)\to S]\}$ forms a direct subsystem of sets of the direct system $\{\pi^n(N^k(\sigma)\subset N(\sigma))\}$ of groups. Evidently, its limit

$${}^n_k[X]=\varinjlim[N^k(\sigma)\subset N(\sigma)\to S]$$

generates the limit group $\widetilde{\pi}^n_k(X)$.

The definition of the homomorphism $\tilde f\colon\widetilde{\pi}^n_k(X)\to\widetilde{\pi}^n_k(Y)$ induced by a mapping by aid of the direct system $\{\pi^n(N^k(\sigma)\subset N(\sigma))\}$ is also obvious.

4. The groups $\overline{\pi}_k^n$ of Δ-ANR spaces.

(4.1) MAIN THEOREM. *If X is a compact metric Δ-ANR space, then there exists an isomorphism i: $\overline{\pi}_k^n(X) \approx \pi_k^n(X)$.*

This section will be devoted to a proof of the Main Theorem.

Since X is compact and metric, we can assume that it lies in the Hilbert cube I^ω. We shall consider finite ε-coverings of X by means of open spherical regions in I^ω whose radii are less than $\varepsilon > 0$. If σ is a covering of X, then by picking a point v in each spherical region V of σ so that the finite set $\{v\}$ is in a general position in I^ω, we can realize the nerve $N(\sigma)$ as a finite simplicial polyhedron in I^ω. We observe that

(4.1) *If σ is an ε-covering of X, then the simplexes of $N(\sigma)$ have diameters less than 4ε.*

(4.2) *If σ is an ε-covering of X, then $|\varkappa_\sigma(x)-x| < 10\varepsilon$.*

(4.3) *If σ is an ε-covering of X, then $N(\sigma)$ lies in the 6ε-neighbourhood of X.*

If X is an ANR space, there exists a compact neighbourhood M of X in I^ω (containing X in its interior) and a retraction r: $M \to X$. Given an $\varepsilon > 0$, there exists an $\eta > 0$ such that, for each $x, y \in M$, $|x-y| < 6\eta$ implies that $|r(x)-r(y)| < \frac{1}{2}\varepsilon$. By (4.3), there exists a $\delta > 0$ such that, if σ is a δ-covering of X then $N(\sigma) \subset M$. Define $r_\sigma = r|N(\sigma)$ and $r_\sigma^k = r|N^k(\sigma)$. Assume that $\eta < \delta$ and $\eta < \frac{1}{12}\varepsilon$ and let σ be an η-covering of X. By (4.3), if y is a point of the polyhedron $N(\sigma)$, then there exists a point $x \in X$ such that $|x-y| < 6\eta < \frac{1}{2}\varepsilon$. It follows that $|y-r(y)| \leqslant |y-x|+|x-r(y)| = |y-x|+|r(x)-r(y)| < \frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon = \varepsilon$. Hence we have

(4.4) *If X is an ANR space and $\varepsilon > 0$, then there exists an $\eta > 0$ such that if σ is an η-covering of X, then $|y-r_\sigma(y)| < \varepsilon$, for every $y \in N(\sigma)$.*

(4.5) *If X is an ANR space and σ is a covering of X, then there exists a refinement τ of σ such that the mapping $\varkappa_\sigma r_\tau$: $N(\tau) \to N(\sigma)$ is homotopic to the projection $p_{\tau\sigma}$: $N(\tau) \to N(\sigma)$.*

Proof. Let 12ε be the Lebesgue number of the covering σ. By (4.4), there exists an η-covering τ such that $\sigma < \tau$ and, for each $y \in N(\tau)$, $|y-r_\tau(y)| < \varepsilon$. We can assume that, in addition, $\eta < \varepsilon$. Let $y \in N(\tau)$, let v be a vertex of $N(\tau)$ such that $y \in \mathrm{St}\, v$ and let V be the element of τ corresponding to v. Hence V is a spherical region $V(p_0, \eta)$ with the center $p_0 \in I^\omega$ and radius $\eta < \varepsilon$. By (4.1), the simplexes of $N(\tau)$ have diameters less than 4η. It follows that

$$|r_\tau(y)-p_0| \leqslant |r_\tau(y)-y|+|y-v|+|v-p_0| < \varepsilon+5\eta .$$

Therefore $r_\tau(y)$ lies in the spherical region $V(p_0, \varepsilon+5\eta)$ whose diameter is less than 12ε. It follows that there exists an element U of the covering σ containing $V(p_0, 6\eta)$ and hence $\varkappa_0$ maps $r_\tau(y)$ into the star $\mathrm{St}\, u$ in $N(\sigma)$

of the vertex u corresponding to U. Since $V = V(p_0, \eta) \subset V(p_0, 6\eta) \subset U$, we can assume that the projection $p_{\tau\sigma}$ maps v into u. Therefore

$$\varkappa_\sigma r_\tau(\mathrm{St}\, v) \subset \mathrm{St}\, p_{\tau\sigma}(u)$$

which means that $\varkappa_\sigma r_\tau \simeq p_{\tau\sigma}$.

If A is a subset of X with $\dim A \leqslant k$, then there exists a cofinal subfamily Σ of the family of all spherical coverings of X in I^ω such that, for each $\sigma \in \Sigma$, $N(\sigma|A) \subset N^k(\sigma)$. Throughout the following part of this section we shall consider only coverings of the family Σ.

(4.6) *Let A be a closed subspace of X with $\dim A \leqslant k < 2n-1$ and let $a \in [A \subset X \to S]$. Then there exists a covering σ of X and a mapping β: $(N^k(\sigma) \subset N(\sigma) \to S]$ such that $a = \overline{\varkappa}_\sigma[\beta]$* (see 2.1).

Proof. Let α: $A \to S$ ba a mapping representing a and let α': $X \to S$ be an extension of α. By lemma 3.4 of [4], there exists a covering σ of X and a mapping β: $N(\sigma) \to S$ such that $\beta\varkappa_\sigma = \alpha'$. Let $\beta^k = \beta|N^k(\sigma)$. Since $\varkappa_\sigma(A) \subset N^k(\sigma)$, it follows that $\beta^k\varkappa_{\sigma|A} \simeq \alpha$. Then, by (2.1), $\overline{\varkappa}_\sigma[\beta] = a$.

(4.7) *If is a Δ-ANR space, A is a homotopic k-skeleton of X and σ is a covering of X, then there exists a refinement τ of σ such that the mapping r_τ: $N(\tau) \to X$ is homotopic to a mapping r': $N(\tau) \to X$ with $r'(N^k(\tau)) \subset A$.*

Proof. By (4.5), there exists a refinement τ of σ such that $\varkappa_\sigma r_\tau \simeq p_{\tau\sigma}$. By Lemma 1 of [3], r_τ is homotopic to a mapping r': $N(\tau) \to X$ with $r'(N^k(\tau)) \subset A$.

As a consequence of (4.5) and (4.7) we have

(4.8) *If X is a Δ-ANR space, A is a homotopic k-skeleton of X and σ is a covering of X, then there exists a refinement τ of σ such that the projection $p_{\tau\sigma}$: $N(\tau) \to N(\sigma)$ is homotopic to a mapping h: $N(\tau) \to N(\sigma)$ with $h(N^k(\tau)) \subset N(\sigma|A) \subset N^k(\sigma)$.*

(4.9) *Let X be a Δ-ANR space, A be a homotopic k-skeleton of X, and σ be a covering of X. Let α: $N^k(\sigma) \to S$ be a mapping such that the mapping $\alpha\varkappa_{\sigma|A}$: $A \to S$ can be extended over X. Then there exists a refinement τ of σ such that the mapping $\alpha p_{\tau\sigma}^k$: $N^k(\tau) \to S$ can be extended over $N(\tau)$.*

Proof. Choosing a covering τ as in (4.7), we can replace $p_{\tau\sigma}^k$ by a mapping $\varkappa_{\sigma|A}r'$: $N^k(\tau) \to N^k(\sigma)$. If β: $X \to S$ is an extension of $\alpha\varkappa_{\sigma|A}$, then $\beta r'$: $N(\tau) \to S$ is an extension of $\alpha\varkappa_{\sigma|A}r'$. It follows that $\alpha p_{\tau\sigma}^k$ can be extended over $N(\tau)$.

(4.10) *If X is a Δ-ANR space, is a homotopic k-skeleton of X, σ is a covering of X and α: $N^k(\sigma) \to S$ is a mapping such that $\alpha\varkappa_{\sigma|A}$: $A \to S$ is homotopic to a constant $s_0 \in S$, then there exists a refinement τ of σ such that the mapping $\alpha p_{\tau\sigma}$: $N^k(\tau) \to S$ is homotopic to a mapping γ: $N^k(\tau) \to S$ with $\gamma(N(\tau|A)) \subset s_0$.*

Proof. Let S be the boundary of the $(n+1)$-simplex with the vertices $s_0, s_1, \ldots, s_{n+1}$. The mapping $\alpha\varkappa_{\sigma|A}$ being homotopic to a constant mapping can be extended over X to a mapping $f\colon X\to S$. It follows by (4.9) that there exists a refinement ϱ of σ such that the mapping $ap_{\varrho\sigma}\colon N^k(\varrho)\to S$ can be extended over $N(\varrho)$ to a mapping $\beta\colon N(\varrho)\to S$. Evidently we can assume that $\varkappa_\sigma = p_{\varrho\sigma}\varkappa_\varrho$. Next, there exists a homotopy $F\colon X\times I\to S$ such that $F(x,0)=f(x)$, for each $x\in X$, $F(x,0) = a\varkappa_{\sigma|A}(x) = ap^k_{\varrho\sigma}\varkappa_{\varrho|A}(x)$, for each $x\in A$, and $F(x,1)=s_0$, for each $x\in A$.

Consider the covering of $X\times I$ with the open sets $F^{-1}(\mathrm{St}\,s_0)$, $F^{-1}(\mathrm{St}\,s_1), \ldots, F^{-1}(\mathrm{St}\,s_{n+1})$ and choose a refinement $\tau=\{V^i\}$ of ϱ and a subdivision $0=t_0<t_1<\ldots<t_m=1$ of I such that, for each $V^i\in\tau$ and $0<j\leqslant m$, there exists a ν_{ij} such that $V^i\times[t_{j-1},t_j]\subset F^{-1}(\mathrm{St}\,s_{\nu_{ij}})$. Define mappings $\gamma_j\colon N(\tau)\to S$ as follows. Let $\gamma_0=\beta p_{\tau\varrho}$ and let $s_{\nu_{i0}}=\gamma_0(v^i)$, where v^i is the vertex of $N(\tau)$ corresponding to V^i; define γ_j on the vertices by putting $\gamma_j(v^i)=s_{\nu_{ij}}$. We easily see that γ_j is a simplical mapping and that the mappings γ_{j-1} and γ_j are "adjacent" for $0<j\leqslant m$. It follows that $\gamma_0\simeq\gamma_m$.

Moreover, since F maps $A\times\{I\}$ into s_0, then the only element of the covering $\{F^{-1}(\mathrm{St}\,s_0), F^{-1}(\mathrm{St}\,s_1), \ldots, F^{-1}(\mathrm{St}\,s_{n+1})\}$ that intersects (and contains) $A\times I$ is $F^{-1}(\mathrm{St}\,s_0)$. If follows that $\gamma_m(v^i)=s_0$ for every i. Thus γ_m maps $N(\tau|A)$ into s_0 and we can put $\gamma=\gamma_m$.

Remark. The proof of (4.10) is a modification of the proof of Theorem 13.4 in [4].

(4.11) *Let Σ be a directed set and $\{H_\sigma, h_{\sigma\tau}\}$, $\sigma,\tau\in\Sigma$, be a direct system of Abelian groups indexed by Σ with the limit H. Let, for each $\sigma\in\Sigma$, $\mathcal{B}_\sigma$ be a subset of H_σ such that $h_{\sigma\tau}(\mathcal{B}_\sigma)\subset\mathcal{B}_\tau$, for every $\sigma<\tau$ (i.e. $\{\mathcal{B}_\sigma, h_{\sigma\tau}|\mathcal{B}_\sigma\}$ is a direct subsystem of sets of the direct system $\{H_\sigma, h_{\sigma\tau}\}$ of groups). Let $\mathcal{B}=\varinjlim\mathcal{B}_\sigma$. Let G be an abelian group generated by a subset $\mathcal{A}\subset G$ and, for each σ, let $i_\sigma\colon H_\sigma\to G$ be a homomorphism such that $i_\tau h_{\sigma\tau}=i_\sigma$, for every $\sigma<\tau$. If the homomorphism $i\colon H\to G$ defined by $\{i_\sigma\}$ maps $\mathcal{B}$ onto $\mathcal{A}$, then i is an epimorphism.*

Proof. Let g be an element of G with $g=m_1a_1+\ldots+m_sa_s$, where m_j are integers and $a_j\in\mathcal{A}$. Let $b_j\in\mathcal{B}$ be such that $a_j=i(b_j)$. Let, for each $j=1,\ldots,s$, $b^j_{\sigma_j}$ be an element of $\mathcal{B}_{\sigma_j}$ representing b_j and let τ be such that $\sigma_j<\tau$, for each $j=1,\ldots,s$. Then $b_\tau=\sum_{j=1}^{s}m_jh_{\sigma_j\tau}(b^j_{\sigma_j})$ is an element of H_τ and $i_\tau(b_\tau)=g$ showing that i is an epimorphism.

Proof of the Main Theorem. Let $\{\mathfrak{a}\}$ be an element of $\overline{\pi}^n_k(X)$ represented by an element $\mathfrak{a}\in\pi^n_k(N(\sigma))$, where σ is a covering of X. The canonical mapping $\varkappa_\sigma\colon X\to N(\sigma)$ induces, by (2.4), a homomorphism

$$\varkappa_\sigma^{\#}\colon \pi^n_k(N(\sigma))\to\pi^n_k(X)\,.$$

We define $i\{\mathfrak{a}\}$ to be $\varkappa_\sigma^{\#}(\mathfrak{a})$. Then i is a homomorphism.

By the remark of section 3, the limit group $\overline{\pi}^n_k(X)$ is isomorphic with the limit $\widetilde{\pi}^n_k(X)$ of the direct system of groups $\{\pi^n(N^k(\sigma)\subset N(\sigma))\}$, $\sigma\in\Sigma$ and the homomorphisms

$$q_{\sigma\tau}\colon \pi^n(N^k(\sigma)\subset N(\sigma))\to\pi^n(N^k(\tau)\subset N(\tau))$$

induced by the projections $p_{\tau\sigma}\colon (N(\tau), N^k(\tau))\to(N(\sigma), N^k(\sigma))$, where σ is a covering of X and τ is a refinement of σ. Then the homomorphism $\tilde{i}\colon \widetilde{\pi}^n_k(X)\to\pi^n_k(X)$ corresponding to i can be defined as follows. Let $\tilde{\mathfrak{a}}$ be the element of $\widetilde{\pi}^n_k(X)$ corresponding to $\{\mathfrak{a}\}\in\overline{\pi}^n_k(X)$ and represented by an element $a\in\pi^n(N^k(\sigma)\subset N(\sigma))$. Let A be a homotopic k-skeleton of X. The canonical mapping $\varkappa_\sigma\colon X\to N(\sigma)$ maps A into $N^k(\sigma)$ and induces by (2.1) a homomorphism

$$\overline{\varkappa}_\sigma\colon \pi^n(N^k(\sigma)\subset N(\sigma))\to\pi^n(A\subset X)$$

such that, if $\sigma<\tau$, then the diagram

$$\begin{array}{ccc}\pi^n(N^k(\sigma)\subset N(\sigma)) & \xrightarrow{q_{\sigma\tau}} & \pi^n(N^k(\tau)\subset N(\tau))\\ & {\scriptstyle\overline{\varkappa}_\sigma}\searrow \quad \swarrow{\scriptstyle\overline{\varkappa}_\tau} & \\ & \pi^n(A\subset X) & \end{array}$$

is commutative. Then $\tilde{i}(\tilde{\mathfrak{a}})$ is the element of $\pi^n_k(X)$ represented by $\overline{\varkappa}_\sigma(a)$. We also recall that, if $\alpha\colon N^k(\sigma)\to S$ represents a, then $\overline{\varkappa}_\sigma(a)$ is represented by the mapping $\alpha\varkappa_{\sigma|A}\colon A\to S$.

1. *i is a monomorphism:*

Let a be an element of $\pi^n(N^k(\sigma)\subset N(\sigma))$ represented by the mapping $\alpha\colon N^k(\sigma)\to S$ and suppose that $\tilde{i}\{a\}$ represents zero in $\pi^n_k(X)$. Then the mapping $\alpha\varkappa_{\sigma|A}\colon A\to S$ is homotopic to a constant and therefore it can be extended over X. By (4.9), there exists a refinement τ_1 of σ such that $\alpha_1=\alpha p^k_{\tau_1\sigma}\colon N^k(\tau_1)\to S$ can be extended over $N(\tau_1)$; but α_1 also represents a.

By (4.10), there exists a refinement τ_2 of τ_1 such that the mapping $\alpha_1p^k_{\tau_2\tau_1}\colon N^k(\tau_2)\to S$ is homotopic to a mapping $\alpha_2\colon N^k(\tau_2)\to S$ with $\alpha_2(N(\tau_2|A))\subset s_0$. The mapping α_2 also represents a.

By (4.8), there exists a refinement τ_3 of τ_2 such that the projection $p_{\tau_3\tau_2}\colon N(\tau_3)\to N(\tau_2)$ is homotopic to a mapping $h\colon N(\tau_3)\to N(\tau_2)$ with $h(N^k(\tau_3))\subset N(\tau_2|A)$. It follows that the element $\tilde{\mathfrak{a}}=\{a\}\in\widetilde{\pi}^n_k(X)$ corresponding to a can be represented by a mapping $\alpha_3\colon N^k(\tau_3)\to S$ which is homotopic to a constant. This shows that $\tilde{i}$ is a monomorphism.

2. *$\tilde{i}$ is an epimorphism:*

To prove this we apply (4.11) to the case when $H_\sigma=\pi^n(N^k(\sigma)\subset N(\sigma))$, $\mathcal{B}_\sigma=[N^k(\sigma)\subset N(\sigma)\to S]$, $G=\pi^n(A\subset X)$ and $\mathcal{A}=[A\subset X\to S]$. By (4.6), $\tilde{i}$ maps the limit $\mathcal{B}=\varinjlim\mathcal{B}_\sigma$ onto $\mathcal{A}$ and $\tilde{i}$ is an epimorphism.

This completes the proof of the Main Theorem.

Remark. With our definition of the (n, k)-cohomotopy groups as direct limits, theorems 5 and 6 of [3] remain true for any compact space X (1).

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(1) Theorems 5 and 6 of [3] have been proved independently by R. Engelking.

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Reçu par la Rédaction le 1. 2. 1961

On automorphisms of relatively free groups

by

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1. This paper deals with automorphisms of relatively free groups, i.e. groups which can be represented as $F(V) = F/V$ where F is a free group and V its fully invariant subgroup. A study of the theory of such groups was initiated by B. H. Neumann [7], who obtained many interesting results, especially for finite relatively free groups. It was continued by P. Hall [2] with special interest in the splitting properties of relatively free groups. Malcev [4] has solved the question what subgroups of nilpotent free groups are nilpotent free groups; see also my paper [6]. The main problem of this paper, i.e. the description of the groups of automorphisms, as far as I know, has not been investigated hitherto. There were known only some theorems, e.g. in Malcev's paper [3] (theorems 5a, 6a, 7a, p. 27), which are marked in this paper as theorems 3 and 4.

In this paper there are investigated (theorems 1 and 2) connections between automorphisms of a relatively free group G and automorphisms of its (abelian free) factor group G/G' (those automorphisms can be described by some matrices), under the assumption that G is either finitely generated or residually nilpotent. The aim of these investigations is a description of the structure of the group of automorphisms for these relatively free groups which are nilpotent. This is given by theorems 5 and 6.

At the end of this brief introduction I wish to express my gratitude to A. L. Szmielkin from Moscow for many helpful suggestions, and to E. Sąsiada from Toruń for his considerations concerning formulation questions.

2. Now we shall give some of the terminology and basic facts. By a base of $F(V)$ we mean a free generating system, i.e. such a set X of generators that every mapping $\mu(X) \leqslant F(V)$ can be extended to an endomorphism φ of $F(V)$ such that $\varphi(x) = \mu(x)$ for $x \in X$. From this definition it follows at once that an endomorphism is uniquely determined by its values on the base of the relatively free group.

For a relatively free group there exists a base, and all bases of a group have the same number of elements, which is called the rank of the group.