tude, introduite par nous [2], a été utilisée dans le cas particulier des objets à une composante. Nous disons que les objets Ω_1 et Ω_2 sont semblables (équivalents) s'il existe une fonction $\Phi(u)$ strictement monotone et telle que la relation

$$\Omega_2 = \Phi(\Omega_1) \tag{15}$$

subsiste dans tous les systèmes de coordonnées. En écrivant cette relation pour le système $\bar{\xi}$ nous devons avoir

$$\bar{\Omega}_2 = \Phi(\bar{\Omega}_1) \,. \tag{16}$$

Mais on a (d'apres (4) et (7))

$$\bar{\Omega}_1 = \Omega_1 + a_1(\xi - \bar{\xi}) \,, \qquad a_1 \neq 0 \,, \tag{17}$$

$$\bar{\Omega}_2 = \Omega_2 + a_2(\xi - \bar{\xi}) \,, \qquad a_2 \neq 0 \,. \tag{18}$$

En substituant (17), (18) dans (16) et en tenant compte de (15) nous obtenons

$$\Phi(\Omega_1) + a_2(\xi - \bar{\xi}) = \Phi[\Omega_1 + a_1(\xi - \bar{\xi})] \,.$$

Il est évident qu'en posant

$$\Phi(u) \stackrel{\mathrm{df}}{=} \frac{a_2}{a_1} u$$

on obtient pour Φ une fonction strictement monotone et en même temps l'identité de (15) par rapport aux variables $\Omega_1, \xi, \bar{\xi}$. Donc les deux objets géométriques Ω_1 et Ω_2 admettant les règles de transformation (17) et (18) sont équivalents et, par conséquent, il est permis de dire que les relations correspondantes R sont „équivalents".

De ce qui précède on peut tirer comme conséquence:

THÉORÈME 4. *Si la fonction f est linéaire, il existe une seule relation essentielle (en faisent abstraction des objets équivalents) R satisfaisant aux propriétés* I, II, III.

Travaux cités

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On the functional equation $f(x+y)=f(x)+f(y)$

by

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One of the most important and best known functional equations is the functional equation of Cauchy

$$f(x+y) = f(x) + f(y) \,. \tag{1}$$

In the present paper we consider equation (1) for functions $f(x)$ whose domain and range are multidimensional spaces.

H. Kestelman ([2], th. 2) has proved a theorem which after taking into account the recent results of S. Kurepa [3] may be formulated as follows:

If a function $f(x)$ whose domain and range are an n-dimensional and an m-dimensional euclidean space respectively satisfies functional equation (1) *and is bounded on a set V of positive Lebesgue measure, i.e.*

$$\|f(x)\| \leqslant a \quad \text{for} \quad x \in V \,, \qquad |V| > 0 \,, \tag{2}$$

then for every $x(\xi^1, \ldots, \xi^n)$ we have

$$f(x) = \sum_{j=1}^{n} \xi^j f(u^j) \,, \tag{3}$$

where u^j $(j = 1, \ldots, n)$ are the unit vectors of the n-dimensional space which is the domain of the function $f(x)$.

As is well known ([2], [5]), when the domain as well as the range of the function $f(x)$ is the space of real numbers, a stronger theorem can be proved. Namely, condition (2) may be replaced by a weaker one:

$$f(x) \leqslant a \quad \text{for} \quad x \in V \,, \qquad |V| > 0 \,. \tag{4}$$

(Thus in this case we assume that $f(x) \leqslant a$ instead of $|f(x)| \leqslant a$.) The question arises whether an analogical weakening of hypothesis (2) in the theorem of H. Kestelman is also possible in the multidimensional case. The purpose of the present note is to give an answer to this question.

Let $\mathcal{H}$ be a Hilbert space (with finite or infinite dimension), i.e. a vector space in which there is defined a scalar product of two elements. The elements of the space $\mathcal{H}$ will be denoted by the initial letters of the Latin alphabet, the scalar product of elements from $\mathcal{H}$ will be denoted by a dot. Further let $\mathcal{V}^n$ be an n-dimensional normalized vector space. The elements of the space $\mathcal{V}^n$ will be denoted by the final small letters of the Latin alphabet, and the subsets of the space $\mathcal{V}^n$ by the corresponding capital letters of the Latin alphabet. For arbitrary two subsets X, Y of the space $\mathcal{V}^n$, we shall denote by $X+Y$, resp. $X-Y$, the set of vectors of the form $x+y$, resp. $x-y$, where $x \in X$ and $y \in Y$. The real numbers will be denoted by letters of the Greek alphabet. Moreover, we shalls denote by $u^1, \ldots, u^n$ the unit vectors of the space $\mathcal{V}^n$, i.e. the vectors with the coordinates $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$ respectively.

Of course, it is not quite evident what condition should be assumed as an analogue of condition (4) for $f(x) \in \mathcal{H}$. If $\mathcal{H} = \mathcal{E}^m$ is the m-dimensional euclidean space, a natural generalization of condition (4) is the requirement that, for $x \in V$, $f(x)$ should lie in a halfspace, i.e. on one side of an $(n-1)$-dimensional hyperplane. Analytically such a condition can be written in the form

$$a \cdot f(x) \leqslant \alpha \quad \text{for} \quad x \in V, \tag{5}$$

where $a \in \mathcal{H}$, and α is a real number. Condition (5), however, has a meaning also for an arbitrary Hilbert space $\mathcal{H}$, and thus it may be admitted as a generalization of condition (4) in the case $f(x) \in \mathcal{H}$.

Now, it turns out that if in the theorem of Kestelman we replace condition (2) by condition (5), then formula (3) need not be valid. It is obvious from the following example:

Let $\mathcal{H} = \mathcal{E}^2$ be the euclidean plane, $\mathcal{V}^n = \mathcal{E}^1$ the space of real numbers, and $\delta(x)$ an arbitrary discontinuous real function, satisfying Cauchy's functional equation

$$\delta(x+y) = \delta(x) + \delta(y)$$

(cf. [1]). Let us define $f(x) \in \mathcal{E}^2$ by the formula

$$f(x) = \big(\delta(x), 0\big).$$

The function $f(x)$ satisfies equation (1) and fulfils the condition

$$a_0 \cdot f(x) = 0 \quad \text{for} \quad x \in \mathcal{E}^1$$

(a_0 denotes here the vector with the coordinates $(0, 1)$).

On the other hand, relation (3), which in this case would have the form $f(x) = cx$ (where $c = f(1) = \big(\delta(1), 0\big)$ is a fixed vector) evidently is not fulfilled, since the function $\delta(x)$ is not continuous.

However, we shall prove the following

Theorem 1. *If a function $f(x)$ whose domain and range are the spaces $\mathcal{V}^n$ and $\mathcal{H}$ respectively satisfies equation* (1) *and fulfils (for some $a \in \mathcal{H}$ and real number α) the condition*

$$a \cdot f(x) \leqslant \alpha \quad \text{for} \quad x \in S, \tag{6}$$

where $S \subset \mathcal{V}^n$ is an n-dimensional sphere, then for every $x(\xi^1, \ldots, \xi^n) \in \mathcal{V}^n$ the relation

$$a \cdot \Big[f(x) - \sum_{j=1}^{n} \xi^j f(u^j)\Big] = 0 \tag{7}$$

holds.

Proof. Suppose that there exists a vector $x_0(\xi_0^1, \ldots, \xi_0^n) \in \mathcal{V}^n$ for which relation (7) does not hold. We write shortly

$$d(x_0) = f(x_0) - \sum_{j=1}^{n} \xi_0^j f(u^j). \tag{8}$$

Thus we have

$$a \cdot d(x_0) \neq 0. \tag{9}$$

Let ϱ_ν be an arbitrary sequence of rational numbers such that

$$\lim_{\nu\to\infty} \varrho_\nu a \cdot d(x_0) = +\infty. \tag{10}$$

According to (9) such a sequence certainly exists. Further, let us choose n sequences of rational numbers σ_ν^j $(j = 1, \ldots, n)$ such that

$$\lim_{\nu\to\infty} (\varrho_\nu \xi_0^j - \sigma_\nu^j) = 0, \quad j = 1, \ldots, n, \tag{11}$$

which, of course, is also always possible. It follows from (11) that

$$\lim_{\nu\to\infty} \Big(\varrho_\nu x_0 - \sum_{j=1}^{n} \sigma_\nu^j u^j\Big) = 0. \tag{12}$$

It follows from relation (1) that for an arbitrary rational number λ and for an arbitrary $x \in \mathcal{V}^n$ we have $f(\lambda x) = \lambda f(x)$. Let s denote the centre of the sphere S. Thus we have

$$f\Big(s + \varrho_\nu x_0 - \sum_{j=1}^{n} \sigma_\nu^j u^j\Big) = f(s) + \varrho_\nu f(x_0) - \sum_{j=1}^{n} \sigma_\nu^j f(u^j),$$

which, after using (8), can be written in the form

$$f\Big(s + \varrho_\nu x_0 - \sum_{j=1}^{n} \sigma_\nu^j u^j\Big) = f(s) + \sum_{j=1}^{n} [\varrho_\nu \xi_0^j - \sigma_\nu^j] f(u^j) + \varrho_\nu d(x_0)$$

and finally

$$(13)\qquad a\cdot f\Big(s+\varrho_\nu x_0-\sum_{j=1}^{n}\sigma_\nu^j u^j\Big)=a\cdot f(s)+\sum_{j=1}^{n}[\varrho_\nu\xi_0^j-\sigma_\nu^j]\,a\cdot f(u^j)+\varrho_\nu a\cdot d(x_0)\,.$$

According to (12), for large ν

$$\Big(s+\varrho_\nu x_0-\sum_{j=1}^{n}\sigma_\nu^j u^j\Big)\in S\,,$$

and consequently, on account of assumption (6), the left-hand side of relation (13) is bounded from above as $\nu\to\infty$. Meanwhile, as follows from (11) and (10), the right-hand side of relation (13) increases boundlessly as $\nu\to\infty$. Thus we have got a contradiction, which proves that relation (7) is valid for every $x\in\mathcal{V}^n$.

The above theorem implies immediately the following

THEOREM 2. *If a function $f(x)$ whose domain and range are the spaces $\mathcal{V}^n$ and $\mathcal{H}^m$* (1) *respectively satisfies equation* (1) *and fulfils for a system of m linearly independent vectors $a_i\in\mathcal{H}^m$ and m real numbers α_i the conditions*

$$(14)\qquad a_i\cdot f(x)\leqslant\alpha_i\quad\text{for}\quad x\in S\,,\quad i=1,\dots,m\,,$$

where $S\subset\mathcal{V}^n$ is an n dimensional sphere, then for every $x(\xi^1,\dots,\xi^n)\in\mathcal{V}^n$ relation (3) *holds.*

Remark 1. If $\mathcal{V}^n=\mathcal{E}^n$ is the n-dimensional euclidean space, then in conditions (6) and (14) the sphere S may be replaced by an arbitrary set V with positive (Lebesgue) measure. This follows from the fact that if condition (6) is fulfilled in a certain set X, then in the set $X+X$ an analogical condition

$$a\cdot f(x)\leqslant 2a$$

is fulfilled, and, as S. Kurepa [3] has proved, if $X\subset\mathcal{E}^n$ and $|X|>0$, then the set $X+X$ (as well as $X-X$) contains an n-dimensional sphere.

Remark 2. In the above remark the condition $|V|>0$ may be made still weaker. Namely, it is enough to assume that the set $V+V$ (or even an arbitrary finite sum $V+V+\dots+V$) has a positive measure. We need only to repeat several times the argument described in remark 1.

Remark 3. What has been proved implies in particular the following theorem:

If a real function $f(x)$ of a single real variable satisfies equation (1) *and is bounded from one side on a set V such that $|V+V|>0$, then $f(x)$ is continuous (and thus, as is well known, of the form $f(x)=cx$).*

(1) $\mathcal{H}^m$ denotes here an m-dimensional Hilbert space.

This theorem is a particular case of a more general theorem of S. Kurepa [4]. However, the question arises whether in the above theorem the condition $|V+V|>0$ may be replaced by the condition $|V-V|>0$ (2), as well as whether the analogical theorem (with the condition $|V-V|>0$) is true for an arbitrary convex function $f(x)$ (3). Unfortunately, at present we are not able to answer this question.

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(2) It is so when we assume that $f(x)$ is bounded from both sides on V (cf. [2]).

(3) Under the supposition that $|V+V|>0$, an analogical theorem about convex functions has been proved by S. Kurepa [4]. (In [4] it is supposed that $f(x)$ is bounded from both sides, but the proof remains valid without change also under the hypothesis that $f(x)$ is bounded only from above.)

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