Denote by k_1' the smallest integer for which $mE_k < 14\pi k_1^{-1/3}$ and suppose that b_n' has been defined for all $n \leqslant k_1'$. We have $S_{k_1'}(x) > k_1$ at all points of $(0, 2\pi)$ except at points of the set $E_{k_1'}$ of measure $< 14 k_1^{-1/3}$. Thus by the above definition of b_k' for $k = 1, 2, \ldots, k_1'$ we have achieved that the series (17) is $> k_1$ except at a set of measure $< 14\pi k_1^{-1/3}$. We define now j_2 so that $k_2 = [\sqrt{A(j_2)}]$ be greater than k_1^2 and k_1'. There will be further a fixed value of $k_2' > k_2$. We define, for $k_1' < n \leqslant k_2'$, $b_n' - a_n' = h_{j_2}$ and b_n will be defined by an induction similar to the first one. First for $k' < n \leqslant k_1$ the intervals (a_n, b_n) are subject only to the condition not to overlap with the intervals already defined. For these values of n we denote by E_n the set of those values of x for which $S_n(x) < k_2$ so that $E_n = (0, 2\pi)$ for all $k' < n < k_2$. After b_n' has been defined, for an $n \geqslant k_2$, b_{n+1}' can be defined so that the interval (a_{n+1}, b_{n+1}) does not overlap with the intervals that have already been defined and that

$$\int_{E_n} s_{j_2}(x - b'_{n+1})\,dx > \frac{mE_n - 6nh_{j_2}}{2\pi} \int_0^{2\pi} \bar{s}_{j_2}(x)\,dx\,.$$

As before we shall arrive at the value k_2' of n such that $mE_{k_2'} < 14\pi k_2^{-1/3}$. Having defined b_n' for all $n \leqslant k_2'$ we shall have the value of the series (17) $> k_2$ except at a set of points of measure $< 14 k_2^{-1/3}$. Continuing in this way we arrive at a proof of the theorem.

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Orderable spaces

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Introduction. A topological space E will be called *orderable* if there exists a total order relation R on E such that the interval topology of the totally ordered set (E, R) coincides with the topology of the space E. Any total order relation R on an orderable space E which has this property will be called an *order* of the space E. For any relation R on a set E, the dual relation will be denoted by σR. It is clear that if R is one of the orders of an orderable space E then σR is also an order of the space E. The pair $(R, \sigma R)$ will then be called an *order class* of the space. The basic theorem concerning connected orderable spaces is due to Eilenberg ([2]): A connected space E is orderable exactly if it is locally connected and the subset $E \times E - D$ of the product space $(D = \{(x, x) \mid x \in E\}$, the diagonal in $E \times E)$ is not connected; in this case there is exactly one order class of E given by the closures of the components of $E \times E - D$.

This note will mainly be concerned with densely orderable spaces, i.e. with orderable spaces possessing orders R such that the ordered set (E, R) is dense in itself. It will be shown that there is a one-to-one correspondence between the order classes $(R, \sigma R)$ with dense order R of such a space E and its connected orderable compact extension spaces which is, in one direction, given by the passage to the Dedekind completion $\delta(E, R)$ by cuts of the ordered set (E, R). Also, if R is a dense order of a space E, then $\delta(E, R)$ will be described as the completion of E with respect to a certain uniform structure which is defined by means of R. Then, the uniform structures of a densely orderable space arising in this way out of a dense ordering of E will be characterized by a number of properties; this constitutes a criterion for the existence of dense orders on a space in terms of uniform structures. The application of these ideas to topological groups is shown to lead to a characterization of the dense subgroups of the additive group of reals. Finally, the existence of dense orders of a locally connected space is considered.

All concepts of general topology are taken in the sense of N. Bourbaki. The same goes for notions and notations related to totally ordered sets.

1. Extension of densely orderable spaces. If R is an order of a given orderable space E and W a (merely topological) extension space of E, then one can ask whether W is also orderable and, in particular, whether it possesses an order which extends the order R of E. In the case of a dense order R and a connected, locally connected W there is a simple answer to this question.

PROPOSITION 1. *If R is a dense order of the orderable space E and W a connected, locally connected extension space of E, then R can be extended to an order of the space W if and only if the closure of R and of its dual σR in $W\times W$ have only the diagonal of $W\times W$ as their intersection.*

Proof. Suppose that R has an extension to W. Then by [2], the set $P=W\times W-D$, D the diagonal, has exactly two components, say S and σS, which determine the only two orders of W, and one has $S\cap(E\times E)=R-D$ if S is appropriately chosen. Further, since S is open and $E\times E$ dense in $W\times W$, it follows that $\bar{S}=\overline{S\cap(E\times E)}=\overline{R-D}=\bar{R}$, the latter by the fact that R is a dense order of E. Applying σ, the symmetry transformation $(x,y)\to(y,x)$, also gives $\overline{\sigma S}=\sigma\bar{S}=\sigma\bar{R}=\overline{\sigma R}$, and for $\bar{S}=S\cup D$, $S\cap\sigma S=\emptyset$, one obtains $\bar{R}\cap\overline{\sigma R}=D$.

Conversely, let $\bar{R}\cap\sigma\bar{R}=D$ be given and call now S the interior of $\bar{R}$. $R-D$ is open in $E\times E$ and therefore there exists an open $V\subseteq W\times W$ such that $R-D=V\cap(E\times E)$. This leads to $V\subseteq\bar{V}=\overline{V\cap(E\times E)}=\overline{R-D}=\bar{R}$ from which $R-D\subseteq V\subseteq S$ and hence $\bar{R}\subseteq\bar{S}$ is obtained. It now follows that $W\times W=\overline{E\times E}=\bar{R}\cup\overline{\sigma R}\subseteq\bar{S}\cup\overline{\sigma S}$, i.e. that $S\cup\sigma S$ is dense in $W\times W$. Any boundary point of S is therefore also a boundary point of σS. Now by the hypothesis $\bar{R}\cap\sigma\bar{R}=D$ and by $\bar{S}\subseteq\bar{R}$ one also has $\bar{S}\cap\sigma\bar{S}\subseteq D$, hence $S\cap D=\emptyset$, and then $S\cap\sigma\bar{S}\subseteq S\cap D$ gives $S\cap\sigma S=\emptyset$. Finally, $P=W\times W-D=(\bar{S}\cup\sigma\bar{S})-D=S\cup\sigma S$, which shows that P is not connected. By [2] it follows that $\bar{S}=S\cup D$ is an order of the space W, since W is connected and locally connected, and from $\bar{R}=\bar{S}$ and the fact that R is closed in $E\times E$ one sees that $R=\bar{R}\cap(E\times E)=\bar{S}\cap(E\times E)$, hence $\bar{S}$ is an extension of R. This completes the proof.

It is seen that the extension of R, if it exists, is merely the closure $\bar{R}$ of R in $W\times W$. In any case, $\bar{R}$ can be seen to be reflexive and transitive and is antisymmetric if and only if $\bar{R}\cap\sigma\bar{R}=D$. Hence one can say that $\bar{R}$ *is an order of the space W exactly if it is a partial order relation.*

COROLLARY. *If the interior of $\bar{R}$ is connected and W is orderable, then R can be extended to W.*

For suppose there exists a point $u\in\bar{S}\cap\sigma\bar{S}$, $u\notin D$. Then $S\cup\sigma S\cup\{u\}$ is connected, and since this is dense in P, P must be connected which contradicts the fact that W is orderable. Hence $\bar{S}\cap\sigma\bar{S}=D$, thus $\bar{R}\cap\sigma\bar{R}=D$ and R can be extended to W.

Remark. The fact that the interior of $\bar{R}$ is connected does not imply that W is orderable. Take, for example, E as the set of all rational points except the origin on the T-shaped set given on the coordinate axes in the plane by the points $(-1,0)$, $(1,0)$ and $(0,-1)$ and W as its closure in the plane. E can be ordered by taking the natural order on its horizontal part, the natural order ("going down") on the vertical part and defining each point of the latter to be less than any point of the former. This gives an order R of the space E. The closure of R in $W\times W$ is homeomorphic to a set consisting of the product of a T with a closed line segment and a closed circular disc which has the vertical stroke of the T as a diameter. The interior of this in $W\times W$ is obtained by omitting the boundary points that this set has as part of the space consisting of three intersecting planes; obviously, this is a connected set. However, the pair $\big((0,0),(-1,0)\big)$, being the limit of $\big((0,-1/n),(1/n-1,0)\big)\in R$, as well as of $\big((1/n,0),(1/n-1,0)\big)\in\sigma R$, belongs to $\bar{R}\cap\sigma\bar{R}$ and hence W is not orderable.

If R is any order of an orderable space E, then a particular extension space of E defined by means of R is the Dedekind completion $\delta(E,R)$ of the ordered set (E,R) by cuts, taken in its order topology. $\delta(E,R)$ is always compact. A property whose occurrence in $\delta(E,R)$ might be of interest is connectedness. With respect to this one has the following result.

PROPOSITION 2. *If R is an order of the orderable space E, then $\delta(E,R)$ is connected exactly if R is dense.*

Proof. The usual notations $\leqslant$, $<$ etc. will be used with respect to R and its extension to $\delta(E,R)$. Now, suppose R is not dense, i.e. there are $a,b\in E$ such that $a<b$ and the open interval $]a,b[$ in (E,R) is void. Then, if for $u\in\delta(E,R)$ one has $a\leqslant u<b$, $u=a$ follows from the basic property of the Dedekind completion that $y=\sup_{\delta(E,R)}\{x\mid x\in E,\ x\leqslant y\}$ for all $y\in\delta(E,R)$. Hence, the ordered set $\delta(E,R)$ is not dense in itself, which contradicts the fact that $\delta(E,R)$ is connected.

Conversely, let R be dense and $u<v$ in $\delta(E,R)$. If u and v belong to E, there exist elements $x\in E$ between them since (E,R) is dense in itself. Further, if u or v does not belong to E it follows from the relation $y=\sup_{\delta(E,R)}\{x\mid x\in E,\ x\leqslant y\}=\inf_{\delta(E,R)}\{x\mid x\in E,\ x\geqslant y\}$ which hold in any Dedekind completion that there must be $x\in E$ between u and v. It is thus seen that $\delta(E,R)$ is also dense in itself. Now, suppose $\delta(E,R)=A\cup B$ is a decomposition into disjoint non-void closed sets and take $a\in A$, $b\in B$ with $a<b$, say. Since A is open, there must be a c such that the closed interval $[a,c]$ in $\delta(E,R)$ belongs to A, and one has $c<b$. Let c_0 be the supremum of all these c; this lies in A since A is closed. By definition of c_0 there exists, for any x with $c_0<x\leqslant b$, some $x'\in B$

with $c_0 < x' \leqslant x$. Thus c_0 is the infimum of a subset of B and must also belong to B, which is a contradiction.

Proposition 2 raises the question whether there exist, for a dense order R of an orderable space E, connected orderable extension spaces W of E whose order extends R but which are different from $\delta(E, R)$. A complete description of all such W is given by

PROPOSITION 3. *If R is a dense order of an orderable space E and W a connected orderable extension space of E such that R can be extended to an order of W, then W is equal to $\delta(E, R)$ or to a space obtained from this by deleting extremal points which do not belong to E, and conversely.*

Proof. It is clear that $\delta(E, R)$ and its subspaces which are obtained in the manner indicated are all extension spaces of E of the kind in question. Also, it is obvious that any connected subspace X of $\delta(E, R)$ containing E must be of this kind. Hence, it has only to be shown that any W as described is a subspace of $\delta(E, R)$. Now, take any $u \in W - E$. Then, for any $v, w \in W$ with $v < u < w$ (where the symbol $<$ refers to the order of W which extends R) there exist $x, y \in E$ with $v < x < u$ and $u < y < w$, the reason being that W is densely ordered because it is connected and E is a dense subspace of W. One thus has $u = \sup_W\{x \mid x < u, x \in E\} = \inf_W\{y \mid y > u, y \in E\}$. It follows, again from the fundamental property of Dedekind completions, that W can be mapped order isomorphically into $\delta(E, R)$ such that $E \subseteq W$ is carried identically onto $E \subseteq \delta(E, R)$. Therefore, W as an ordered set can be regarded as a subset of the ordered set $\delta(E, R)$; but since $E \subseteq W$, W is topologically dense in $\delta(E, R)$, and it follows from this that W is also a subspace of the ordered space $\delta(E, R)$. By what has already been said this proves the proposition.

COROLLARY 1. *If R is a dense order of an orderable space E, then there exists exactly one connected compact orderable extension of E to which R can be extended, namely $\delta(E, R)$.*

COROLLARY 2. *If R and S are two dense orders of an orderable space E such that $\delta(E, R)$ and $\delta(E, S)$ are homeomorphic over E, then $R = S$ or $R = \sigma S$.*

Here, two extensions of the same space E are called *homeomorphic over E* if they are homeomorphic such that E is carried identically onto itself by some homeomorphism between them. If $f\colon \delta(E, R) \to \delta(E, S)$ is such a homeomorphism, then by [2] f either preserves order or inverts it and the same holds for the restriction of f to E. Hence $R = S$ or $R = \sigma S$.

These two corollaries taken together immediately lead to

COROLLARY 3. *For any densely orderable space E there is a one-to-one correspondence between the classes of dense orders R of E and the connected compact orderable extension spaces W of E, given by $R \to W = \delta(E, R)$.*

Here, of course, two extension spaces of E which are homeomorphic over E are regarded as identical.

2. Order and uniform structure.

The fact that $\delta(E, R)$ for a dense order R of an orderable space E is a certain compact extension of E means that there exists a precompact uniform structure on E such that $\delta(E, R)$ is the corresponding completion of E. In the following, a simple description of this uniform structure will be given directly in terms of R. The usual notations will be employed with respect to the order relation R.

For any ascending sequence ξ of an even number of points $x_i \in E$ let $\mathfrak{U}(\xi)$ denote the finite open covering $\{[\to, x_2[,]x_1, x_4[,]x_3, x_6[, \ldots,]x_{2n-3}, x_{2n}[,]x_{2n-1}, \to]\}$ and $U_\xi \subseteq E \times E$ the binary relation $\bigcup U \times U\big(U \in \mathfrak{U}(\xi)\big)$. Further, call a similar sequence η a *refinement* of ξ (in symbols: $\eta < \xi$) if for x_i, y_i belonging to ξ and η respectively one has $x_{2(k+1)} = y_{4(k+1)}$ and $x_{2k+1} = y_{4k+1}$ for all $k \geqslant 0$ for which these terms are defined. It will first be shown that $\eta < \xi$ implies $U_\eta \circ U_\eta \subseteq U_\xi$.

Let $2n$ be the length of the sequence ξ. Then, the refinement η of ξ has the length $4n$. Now, take any $U, V \in \mathfrak{U}(\eta)$ with non-void intersection; suppose that U lies farther down than V and let y_{2k} be the upper end point of U. The set $U \cup V$ is either (i) $[\leftarrow, y_{2k+2}[$ or (ii) $]y_{2k-3}, \to]$ or (iii) $]y_{2k-3}, y_{2k+2}[$ depending on how far up or down y_{2k} lies in η. Case (i) implies that $k = 1$, and then one has $2k+2 = 4$, hence $y_{2k+2} = x_2$ and $U \cup V \subseteq [\leftarrow, x_2[$. Case (ii) only occurs if $2k = 4n$, and it follows that $2k-3 = 4n-3 = 4(n-1)+1$; hence $y_{2k-3} = x_{2n-1}$ and $U \cup V \subseteq]x_{2n-1}, \to]$. In case (iii) one has to consider even and odd k separately. If $k = 2m$, then $2k-3 = 4(m-1)+1$ and $y_{2k-3} = x_{2m-1}$. Also, $2k+2 = 4m+2$ and $y_{2k+2} = y_{4m+2}$ cannot be the last point of η since this has the index $4n$. Therefore, $y_{4(m+1)} = x_{2(m+1)}$ exists and $U \cup V \subseteq]x_{2m-1}, x_{2(m+1)}[$. If $k = 2m+1$, one only has to deal with $m \geqslant 1$, the case $m = 0$, i.e. $k = 1$, already being settled. Then, $2k-3 = 4m-1$ and $y_{4m-3} = x_{2m-1}$ exists, whilst $2k+2 = 4(m+1)$ and thus $y_{2k+2} = x_{2(m+1)}$, hence $U \cup V \subseteq]x_{2m-1}, x_{2(m+1)}[$. In all, it is shown that any two sets in $\mathfrak{U}(\eta)$ which meet have their union contained in a set of $\mathfrak{U}(\xi)$. This, however, implies that $U_\eta \circ U_\eta \subseteq U_\xi$.

Now, let $\mathcal{U}$ be the collection of all finite intersections $U_{\xi_1,\ldots,\xi_k} = \bigcap_{i=1}^{k} U_{\xi_i}$. $\mathcal{U}$ is closed under finite intersections and if $\eta_1 < \xi_1, \ldots, \eta_k < \xi_k$, then $U_{\eta_1,\ldots,\eta_k} \circ U_{\eta_1,\ldots,\eta_k} = \{(x, y) \mid (x, z), (z, y) \in U_{\eta_1,\ldots,\eta_k}$ for some $z\} \subseteq \{(x, y) \mid (x, z_i), (z_i, y) \in U_{\eta_i}$ for some $z_i;\ i = 1, \ldots, k\} = \bigcap_{i=1}^{k} U_{\eta_i} \circ U_{\eta_i} \subseteq \bigcap_{i=1}^{k} U_{\xi_i} = U_{\xi_1,\ldots,\xi_k}$. Hence, $\mathcal{U}$ is the basis of some uniform structure. Also, for any $x \in E$, $U_\xi(x)$ is an open interval containing x, hence the $U_{\xi_1,\ldots,\xi_k}(x)$ are open neighbourhoods of x. Furthermore, for any neighbourhood V of x there obviously exists a ξ such that $U_\xi(x) \subseteq V$ since R is a dense order of E. Hence, $\mathcal{U}$ is the basis of a uniform structure of the space E, i.e. compatible with the topology of E. This uniform structure will be denoted by $\mathcal{U}(R)$. From the way $\mathcal{U}$ is defined one sees immediately that $\mathcal{U}(R)$ is precompact, and hence the completion of E with respect to $\mathcal{U}(R)$ is a compact space.

Proposition 4. *The completion of an orderable space E with respect to the uniform structure $\mathcal{U}(R)$ associated with a dense order R of E is $\delta(E, R)$.*

Proof. Let $K = \delta(E, R)$. Any sequence ξ in E of the type considered above determines a finite open covering $\mathfrak{U}^*(\xi)$ of K by the intervals in K which are obtained from ξ in the same way the sets of $\mathfrak{U}(\xi)$ were obtained in E. If U_ξ^* is the binary relation on K given by $\mathfrak{U}^*(\xi)$, one again has $U_\eta^* \circ U_\eta^* \subseteq U_\xi^*$ if $\eta < \xi$, and as above it follows that the finite intersections of the U_ξ^* form the basis $\mathcal{U}^*$ for a uniform structure. Again, for any $y \in K$ and ξ, $U^*(y)$ is an open interval containing y and hence a neighbourhood of y in K. Also, since E is dense in K, those open intervals containing y whose end points belong to E form a neighbourhood basis for y in K. Thus $\mathcal{U}^*$ is a basis of the unique uniform structure of K. In other words: The uniform structure $\mathcal{U}(R)$ of E is the restriction of the uniform structure of K to E. From this it follows immediately that $K = \delta(E, R)$ is the completion of E with respect to $\mathcal{U}(R)$.

Since it is seen that total order relations determine uniform structures in the manner described above, one can introduce the notion of an orderable uniform space. A uniform space $(E, \mathcal{U})$ will be called *orderable* if E is orderable with orders R such that $\mathcal{U} = \mathcal{U}(R)$. These orders R will correspondingly be referred to as the orders of the uniform space $(E, \mathcal{U})$. Similarly, a uniform structure $\mathcal{U}$ of a space E will be called orderable if the uniform space $(E, \mathcal{U})$ is orderable. With these notions, one has the following results as consequences of proposition 4.

Corollary 1. *A densely orderable uniform space possesses only one order class.*

Proof. If $(E, \mathcal{U})$ is the orderable uniform space and R and S two of its dense orders, then it follows from $\mathcal{U}(R) = \mathcal{U} = \mathcal{U}(S)$ that $\delta(E, R) = \delta(E, S)$ and by the second corollary of proposition 3, R and S belong to the same order class.

Corollary 2. *There is a one-to-one correspondence between the dense order classes of an orderable space and its densely orderable uniform structures, given by $R \to \mathcal{U}(R)$.*

Remark. It can easily be seen, by means of examples on the real line, that corollary 1 is no longer true if one considers merely orderable uniform spaces instead of densely orderable ones.

Given the definition of orderable uniform spaces one is led to the question as to how these can be characterized. This will be considered in the following for the case of dense orders.

Proposition 5. *A precompact uniform space $(E, \mathcal{U})$ is densely orderable if and only if $\mathcal{U}$ contains a basis $\mathcal{V}$ with the following properties:*

(i) *For each $V \in \mathcal{V}$, $\bigcup_{n \geq 1} V^n = E \times E$, where $V^1 = V$ and $V^n = V \circ V^{n-1}$.*

(ii) *If $x_0 = x, x_1, \ldots, x_n = y$ and $y_0 = y, y_1, \ldots, y_n = x$ are two sequences in E and $V_0, V_1, \ldots, V_n$ a sequence in $\mathcal{V}$ such that all $V_i(x_i) \cap V_{i+1}(x_{i+1})$ and $V_i(y_i) \cap V_{i+1}(y_{i+1})$ are non-void for $i = 0, 1, \ldots, n-1$, then $V_j(x_j) \cap V_j(y_j) \neq \emptyset$ for some j.*

Proof. Let the uniform structure $\mathcal{U}$ satisfy the stated hypotheses and W denote the compact completion of the uniform space $(E, \mathcal{U})$. Then, by [1], chapt. II § 3 the closures $\overline{V}$ of $V \in \mathcal{U}$ in $W \times W$ form a basis for the uniform structure of the space W. Now, given any $w \in W$, the set $\overline{V}^*(w) = \bigcup_{n \geq 1} \overline{V}^n(w)$ contains points $x \in E$ and thus $\bigcup_{n \geq 1} V^n(x)$, which is equal to E by (i); it follows that $W = \overline{V}(E) \subseteq \overline{V}^*(w)$ and by [1], chapt. II, § 4 this shows that W is connected.

For the next part of the proof the following obvious statement will be used: Let X be a space and $\mathfrak{B}$ a basis for its open sets. For any open $P \subseteq X$ and $a \in P$, let $C_P(a)$ denote the set of all $x \in P$ such that there exists a sequence $B_i \subseteq P$, $i = 1, \ldots, n$, in $\mathfrak{B}$ with $a \in B_1$, $x \in B_n$ and $B_i \cap B_{i+1} \neq \emptyset$ for all $i = 1, \ldots, n-1$. Then $P = \bigcup C_P(a)$ $(a \in P)$; any two distinct $C_P(a)$, $C_P(a')$ are disjoint and each $C_P(a)$ is open.

Now, take $X = W$, $P = W \times W - D$ and $\mathfrak{B}$ as the collection of sets $V^0(x) \times V^0(y) \subseteq W \times W$ where $x, y \in E$ and V^0 denotes for each $V \in \mathcal{V}$ the interior of its closure in $W \times W$. From the definition of W as the completion of $(E, \mathcal{U})$ it follows that the $V^0(x)$, $x \in E$ and $V \in \mathcal{V}$, form a basis for W; hence $\mathfrak{B}$ is a basis for $W \times W$. Now suppose $(y, x) \in C_P(y, x)$ for any $(y, x) \in P \cap (E \times E)$. Then, if $B_i = V_i^0(x_i) \times V_i^0(y_i) \subseteq P$, $i = 1, \ldots, n$, is the corresponding sequence in $\mathfrak{B}$, one obtains from this $x_0 = x, x_1, \ldots, x_n, x_{n+1} = y$ and $y_0 = y, y_1, \ldots, y_n, y_{n+1} = x$, such that for $V_0, V_1, \ldots, V_n$, $V_{n+1} = V_0$, with V_0 chosen such that $V_0^0(x) \times V_0^0(y), V_0^0(y) \times V_0^0(x) \subseteq P$, the relations $V_i(x_i) \cap V_{i+1}(x_{i+1}) \neq \emptyset$ and $V_i(y_i) \cap V_{i+1}(y_{i+1}) \neq \emptyset$ follow for all $i = 0, 1, \ldots, n$. Therefore by (ii) there exists a j such that $V_j(x_j) \cap V_j(y_j) \neq \emptyset$. This, however, implies $V_j^0(x_j) \cap V_j^0(y_j) \cap D \neq \emptyset$, contradicting the fact that $V_i^0(x_i) \times V_i^0(y_i) \subseteq P$ for $i = 0, 1, \ldots, n+1$. It follows that $(y, x) \notin C_P(x, y)$ and therefore $C_P(x, y) \neq C_P(y, x)$. The remark in the preceding paragraph now leads to the conclusion that P is not connected. By [2], this means that there exists a total order relation S on the set W such that the open intervals in (W, S) are open sets in W. Then, the topology given by S on W is a Hausdorff topology which has no more open sets than W has in its own topology, and since W is compact, the two topologies must coincide. Thus, W is seen to be orderable, and since it is connected, S is dense. Furthermore, the restriction R of S to E is a dense order of E and $W = \delta(E, R)$ by corollary 1 of proposition 3. Finally, by proposition 4, $\mathcal{U} = \mathcal{U}(R)$, and this completes the first part of the proof.

Conversely, let $(E, \mathcal{U})$ be densely orderable. Then, its completion W is orderable and connected; furthermore it is locally connected. Therefore, one can take as a basis for the uniform structure those entourages of D which are defined by finite coverings of W by connected open sets. Its restriction $\mathcal{V}$ to E is then a basis of $\mathcal{U}$, consisting of symmetric relations, which clearly satisfies (i). Now, consider $V_i(x_i), V_i(y_i)$ as described in (ii). Since the interiors $V_i^0(x_i), V_i^0(y_i)$ of the closures of the $V_i(x_i), V_i(y_i)$ in W are connected in W, since $P = W \times W - D$ is disconnected and since (x, y) and (y, x) belong to different components of P, $V_i^0(x_i) \times V_i^0(y_i)$ cannot completely lie in P for otherwise the two components of P would be linked by a connected set. Hence $D \cap \bigcup V_i^0(x_i) \times V_i^0(y_i) \neq \emptyset$ which gives $V_j^0(x_j) \cap V_j^0(y_j) \neq \emptyset$ for some j and thus also $V_j(x_j) \cap V_j(y_j) \neq \emptyset$. In all, it is seen that the basis $\mathcal{V}$ of $\mathcal{U}$ satisfies the conditions stated in proposition 5.

3. An application to topological groups. The techniques of the preceding section can be employed to obtain the following characterization of the dense subgroups of the reals in terms of properties of the neighbourhoods of the unit element:

PROPOSITION 6. *A topological group G is isomorphic to a dense subgroup of the additive group of real numbers if and only if there exists in G a neighbourhood basis $\mathfrak{V}$ of the unit $e \in G$, consisting of symmetric open neighbourhoods of e, which satisfies the following conditions:*

(i) *Each $V \in \mathfrak{V}$ is a set of generators of G.*

(ii) *For any $U, V \in \mathfrak{V}$, U is covered by finitely many sets Vx, $x \in G$.*

(iii) *If $z_0 = z, z_1, \ldots, z_n = z^{-1}$ is a sequence of elements of G and $V_0, V_1, \ldots, V_n$ a sequence in $\mathfrak{V}$ such that $V_i z_i V_i \cap V_{i+1} z_{i+1} V_{i+1} \neq \emptyset$ for all $i = 0, 1, \ldots, n-1$, then $z_j \in V_j^2$ for some j.*

Remark. If V ranges over the neighbourhoods in $\mathfrak{V}$, then the relations S_V, defined by "$(x, y) \in S_V$ if and only if $x \in VyV$" are symmetric and form the basis $\mathcal{S}$ of a uniform structure on G whose topology coincides with that of G. The intuitive meaning of (iii) is that if in a sequence of points, going from one element of G to its inverse, successive elements are close to each other in the sense of having $\mathcal{S}$-neighbourhoods which meet, then the sequence must contain a term so close to the unit e that its assigned $\mathcal{S}$-neighbourhood contains e.

Proof. It is obvious that for any dense subgroup of the reals the collection of all open intervals, symmetric around zero, satisfies all the stated conditions. Conversely, let G be a group as described, denote by $\mathcal{U}$ the uniform structure of G defined by the relations $\{(x, y) \mid xy^{-1} \in U\}$, U ranging over all neighbourhoods of e, and call $\mathcal{V}$ the basis of $\mathcal{U}$ consisting of those of these relations which are given by the $V \in \mathfrak{V}$. The first thing to be shown is that $\mathcal{V}$ satisfies condition (ii) of proposition 6. Consider, then, two points $x, y \in G$ and sequences $x_0 = x, x_1, \ldots, x_n = y$, $y_0 = y, y_1, \ldots, y_n = x$ such that $V_i x_i \cap V_{i+1} x_{i+1} \neq \emptyset$, $V_i y_i \cap V_{i+1} y_{i+1} \neq \emptyset$ for suitable $V_i \in \mathfrak{V}$, $i = 0, 1, \ldots, n-1$. It follows that $v_i x_i = v_{i+1} x_{i+1}$ and $y_i^{-1} v_i' = y_{i+1}^{-1} v_{i+1}'$ with suitable factors from V_i and V_{i+1} respectively. Hence $v_i x_i y_i^{-1} v_i' = v_{i+1} x_{i+1} y_{i+1}^{-1} v_{i+1}'$ or $V_i x_i y_i^{-1} V_i \cap V_{i+1} x_{i+1} y_{i+1}^{-1} V_{i+1} \neq \emptyset$ for all i, and since $x_n y_n^{-1} = yx^{-1} = (xy^{-1})^{-1} = (x_0 y_0^{-1})^{-1}$, one can use condition (iii) for $\mathfrak{V}$ to obtain $x_j y_j^{-1} \in V_j^2$ or $V_j x_j \cap V_j y_j \neq \emptyset$ for some j. By the proof of proposition 6, this implies that the completion W of G with respect to the uniform structure $\mathcal{U}$ has the property that $W \times W - D$, D the diagonal, is disconnected.

Now, condition (ii) means that G is locally precompact, and this is known to imply that the group operations can be extended from G to W; hence W, besides being a space, is a locally compact group. Since the closures $\overline{V}$ of the $V \in \mathfrak{V}$ in W form a fundamental system of neighbourhoods for the unit in W it follows from (i) by $\overline{V}G \subseteq \bigcup_{n \geq 1} \overline{V}^n$ that each neighbourhood of W generates W. This in turn is known to imply that W is connected, and the fact that $W \times W - D$ is not connected then means that there exists a dense total order relation S on W such that all open intervals of (W, S) are open sets of W.

The next thing to be shown is that this order relation S is in fact an order of the space W. However, since W need not (indeed, never will be) compact one has to use an argument different from the one employed in the preceding section to obtain this.

First, it is easily seen that any closed interval $[a, b]$ in (W, S) is a connected set in W: It is clear that it is closed, and if $[a, b] = A \cup B$ with non-void closed disjoint A and B, where either $a \in A$, $b \in B$ or $a, b \in A$, then either $W = (A \cup [\leftarrow, a]) \cup (B \cup [b, \rightarrow])$ or $W = (A \cup \cup [\leftarrow, a] \cup [b, \rightarrow]) \cup B$ is a decomposition of W into non-void closed disjoint sets, which is a contradiction since W is connected. Now, if X is any compact neighbourhood of a point w in W and $Y \subseteq X$ an open neighbourhood of w which does not contain any closed neighbourhood of w of the form $[a, b]$, then $[a, b] \cap (X - Y) \neq \emptyset$ for all these $[a, b]$ since they are connected. By the compactness of $X - Y$ this implies that $\bigcap [a, b] \cap (X - Y)$, $a < w < b$, is non-void, but this contradicts $\{w\} = \bigcap [a, b]$ $(a < w < b)$. It follows that the neighbourhoods $[a, b]$ of w

form a fundamental system, and this means that W is orderable, S being one of its orders.

The group G will now be considered as a transformation group of W, the transformation given by $a \in G$ being the right translation $T_a: x \to xa$. T_a being a homeomorphism of W onto itself must either preserve the order S or invert it ([2]). Clearly, those $a \in G$ for which T_a is order preserving form a subgroup H of G, and if $a, b \notin H$ then $ab \in H$ such that H is of index 2 in G unless $H = G$. Moreover, H is closed: Let $c_a \in H$ be a general sequence of elements (a ranging over some directed set) converging to $c \in G$ and take any fixed pair $x, y \in W$ with $x < y$; then, by the continuity of the mapping $a \to wa$ of G into W for any fixed $w \in W$ it follows that xc_a and yc_a converge to xc and yc respectively, and since $xc_a \leqslant yc_a$ for all a one has $xc \leqslant yc$, i.e. $c \in H$. Thus, if $H \neq G$; H is not only closed but being of finite index also open, and therefore a neighbourhood of e. This, however, is impossible by (i) since it means that no $V \in \mathfrak{V}$ contained in H can generate G. In particular, it is now proved that the translations $x \to ax$ on G preserve the order R of G which is induced by S.

By exactly the same argument it is seen that the left translations also preserve the order R of G, and therefore G has turned out to be an ordered group, with its topology given by its order. Moreover, the condition (i) implies that G is archimedean as an ordered group, and by Hölder's theorem it finally follows that G is isomorphic, as an ordered and hence also as a topological group, to a subgroup of the additive group of real numbers, the latter being dense in the reals since G is densely ordered. This completes the proof of the proposition.

As an immediate consequence of the above considerations one also has:

Corollary. *A locally compact, connected and locally connected group in which the connected symmetric neighbourhoods of the unit satisfy condition* (iii) *of proposition* 7 *is isomorphic to the additive group of real numbers.*

4. Dense orders in locally connected orderable spaces.

A disconnected orderable space will always have more than one order class. Moreover, it is easy to see that there are orderable spaces some of whose orders are dense whilst others are not. Since it is exactly the dense orders which lead to a connected Dedekind completion and thereby to the possibility of applying the results in [2] in the manner carried out above, the dense orders are of particular interest. In this section, topological conditions for the existence of dense orders will be obtained for a particular type of spaces, namely locally connected ones.

The following observation will be of use later on:

Lemma. *If E is an orderable space and R one of its orders, then any connected set $A \subseteq E$ is a convex set in (E, R). Further, if A and B are disjoint connected sets in E, then either $a < b$ or $b < a$ for all $a \in A$ and $b \in B$.*

Proof. It is understood that the usual notations relating to order are used with respect to R. If $a_1, a_2 \in A$, $a_1 < a_2$ and $c \notin A$ for some c with $a_1 < c < a_2$, then $A = (A \cap [\leftarrow, c]) \cup (A \cap [c, \to])$ is a decomposition of A into two non-void disjoint closed sets which contradicts the connectedness of A. Next, if A and B are disjoint connected sets and $a_0 < b_0$ for some $a_0 \in A$, $b_0 \in B$, say, then $b \leqslant a_0$ for $b \in B$ would imply $a_0 \in B$ against the assumption; thus $a_0 < b$ for all $b \in B$. Similarly, no $a \geqslant a_0$ in A can satisfy $a \geqslant b$ with any $b \in B$ since $a_0 < b \leqslant a$ would imply $b \in A$.

Corollary 1. *If E is an orderable space, then any connected set $A \subseteq E$ is again an orderable space and any order R of E induces on A an order R_A of the subspace A of E.*

Proof. By the lemma, the intersections of open intervals of (E, R) with A are all of one the types $]a, b[$, $[\leftarrow, a[$ or $]a, \to]$ in (A, R_A), and hence the subspace topology of A coincides with the order topology given on A by R_A.

To be able to formulate easily another consequence of the lemma, the following notions will be employed: The topological sum $\sum_{a \in I} E_a$ of a family of spaces E_a, $a \in I$, is the space whose points are the elements (x, a) with $x \in E_a$, $a \in I$, and whose open sets are defined by taking as a basis the sets $\{(x, a) \mid x \in V_a\}$ where V_a ranges over all open sets of E_a and a over I ([1], chapt. I, § 8). Further, if each E_a and I are totally ordered by relations R_a and $S = \leqslant$ respectively, then the *set* $\sum_{a \in I} E_a$ is totally ordered by the relation $R = \bigcup_{a \in I} (R_a \cup \bigcup_{\beta < a} E_\beta \times E_a)$ which will be called the ordered sum $\sum_S R_a$ of the order relations R_a, $a \in I$, with respect to S. If the spaces E_a are orderable and R_a is an order of E_a for each a, then it may happen that for suitable S on I $\sum_S R_a$ is an order of the space $\sum_{a \in I} E_a$.

With these concepts one now has:

Corollary 2. *If E is a locally connected orderable space, E_a, $a \in I$, the family of its connected components, R one of the orders of E and R_a for each $a \in I$ the restriction of R to E_a, then $R = \sum_S R_a$ with a suitable total order relation S on I.*

Proof. It is clear that $E = \sum_{a \in I} E_a$ since it is locally connected. Also, each E_a is orderable by corollary 1 and R_a an order of the subspace E_a of E.

Further, the lemma implies that I can be ordered by S defined by "$\alpha < \beta$ if and only if $x < y$ for all $x \in E_\alpha$, $y \in E_\beta$". Finally, for any $x, y \in E$ one has $x \leqslant y$, i.e. $(x, y) \in R$ if and only if $(x, y) \in R_\alpha$ for some α or $(x, y) \in E_\beta \times E_\alpha$ with $\beta < \alpha$.

According to corollary 2, the question whether a locally connected space is orderable and in particular densely orderable is equivalent to the question whether in a given family E_α, $\alpha \in I$, of connected orderable spaces an order R_α can be chosen from the two orders of each E_α and a total order relation S on I such that $\sum\limits_S R_\alpha$ is an order or, in particular, a dense order of the space $\sum\limits_{\alpha \in I} E_\alpha$.

In the following, a point e of a connected orderable space X will be called an *extremal* point of X if it is maximal or minimal in one of the orders of X. It is clear that if X has two different extremal points or contains only one point it is compact and conversely. Also, of course, the extremal points e of X are topologically characterized by the condition that $X - \{e\}$ is connected.

PROPOSITION 7. *The topological sum of a family of (at least two) connected orderable spaces E_α, $\alpha \in I$, is densely orderable if and only if no E_α is compact and at most two E_α have one extremal point. In this case, the dense orders of $\sum\limits_{\alpha \in I} E_\alpha$ are exactly the relations $\sum\limits_S R_\alpha$ for any choice of orders R_α of E_α such that the possible extremal points are of different kind and any total order relation S on I such that $\alpha \in I$ is maximal (minimal) if (E_α, R_α) has a maximum (minimum).*

Proof. Suppose that $E = \sum\limits_{\alpha \in I} E_\alpha$ is orderable, R one of its orders, and assume that E_α has a maximum a with respect to R. Then, since E_α is open, there exist $b, c \in E$ such that $a \in]b, c[\subseteq E_\alpha$ unless a is maximal in (E, R). If the former case holds it follows that any $x \in E$ with $a \leqslant x < c$ is equal to a, i.e., that no elements lie between a and c. If R is dense this cannot occur and then it follows that only one E_α can have a maximum and this will be the maximum of (E, R). Similarly, with dense R, only one E_α can have a minimum which will be the minimum in (E, R). Now, since there are at least two elements in I, the E_α for which these two properties hold must be distinct. Finally, since the extremal points of any E_α must be either the maximum or the minimum of E_α in (E, R), it follows that the E_α satisfy the given condition. Further, by the second corollary of the lemma one has $R = \sum\limits_S R_\alpha$ where each R_α is one of the orders of E_α, and by the above definition of S and the remarks just made about the occurrence of E_α with maximum or minimum in (E, R) it follows that R_α and S are of the described kind.

Conversely, let E_α, $\alpha \in I$, be a family of spaces as stated and R_α and S order relations satisfying the above requirements. Then $R = \sum\limits_S R_\alpha$ obviously induces R_α on each E_α, and any sufficiently small open interval in (E, R) containing an $x \in E$ is also an open interval in (E_α, R_α). Therefore, E is orderable with order R. Furthermore, each R_α is dense and only at the ends of (E, R) can a maximum or minimum of an E_α occur; this implies that R is also dense.

COROLLARY. *If E , $\alpha \in I$, is a family of non-compact connected orderable spaces such that at most one E_α has an extremal point, then all orders of the space $\sum\limits_{\alpha \in I} E_\alpha$ are dense. If, however, there are two E_α with extremal point, then $\sum\limits_{\alpha \in I} E_\alpha$ also has orders which are not dense.*

Proof. If all E_α are without extremal points then there is no restriction on the total order relation S on I and the choices of the R_α on E_α and hence all possible $\sum\limits_S R_\alpha$ are dense orders of $\sum\limits_{\alpha \in I} E_\alpha$, and by the second corollary of the lemma these will be all orders of $\sum\limits_{\alpha \in I} E_\alpha$. Otherwise, let $\alpha_0 \in I$ the index such that E_{α_0} has an extremal point e. Then by the first part of the proof of proposition 8 e must be maximal or minimal with respect to any order R of $\sum\limits_{\alpha \in I} E$, and then the R_α and the order S on I such that $R = \sum\limits_S R_\alpha$ automatically satisfy the conditions given in proposition 8. Hence R is dense.

Finally, if two of the E_α, E_{α_0} and E_{α_1} say, have extremal points e_0 and e_1, one can choose R_{α_0} and R_{α_1} such that e_0 is maximal with respect to R_{α_0} and e_1 minimal with respect to R_{α_1} and order I in such a way that $\alpha_0 < \alpha_1$ and no α lies between these two. It is easily seen that with any total order relation S on I of this kind and any choice of R_α for $\alpha \neq \alpha_0$, $\sum\limits_S R_\alpha$ is an order of $\sum\limits_{\alpha \in I} E_\alpha$, but not a dense one. This proves the corollary.

Considerations similar to the preceding ones easily lead to a description of all orders of a sum $\sum\limits_{\alpha \in I} E_\alpha$ of connected orderable spaces, although this will be less clear intuitively than the one obtained for dense orders. If a point a in an ordered set which has an immediate successor a^* (predecessor a_*) is called right (left) isolated, then one has the following general statement whose proof follows the above arguments:

PROPOSITION 8. *Let E_α, $\alpha \in I$, be a family of connected orderable spaces, R_α an order for each E_α and S any total order relation on I such that whenever (E_α, R_α) has a greatest (least) element, then α is right (left) isolated and $(E_{\alpha^*}, R_{\alpha^*})$ has a least ($(E_{\alpha_*}, R_{\alpha_*})$ a greatest) element. Then $\sum\limits_S R_\alpha$ is an order of $\sum\limits_{\alpha \in I} E_\alpha$, and any order of $\sum\limits_{\alpha \in I} E_\alpha$ is of this type.*

References

[1] N. Bourbaki, *Topologie générale*, Act. sci. ind., Hermann et Cie., Paris.
[2] S. Eilenberg, *Ordered topological spaces*, Amer. J. Math. 63 (1941), pp. 39-45.

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On isomorphic free algebras

by

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1. Preliminaries and summary. Given a class $\mathfrak{A}$ of abstract algebras which have the same operations, let $\mathcal{A}_m$ denote the free $\mathfrak{A}$-algebra with m generators (cf. [1], p. viii). In this paper we consider the class of all free algebras $\mathcal{A}_m$ with finite m. Assuming that two of these algebras are isomorphic, i.e. $\mathcal{A}_k \simeq \mathcal{A}_l$, $k \neq l$, we investigate the distribution of pairs of isomorphic algebras in the sequence $A_1, A_2, \ldots$

THEOREM 1. *If k is the smallest integer such that $\mathcal{A}_k \simeq \mathcal{A}_m$ holds for some $m \neq k$, and $l = k+d$ is the smallest integer satisfying $\mathcal{A}_k \simeq \mathcal{A}_l$, $l \neq k$, then $\mathcal{A}_m \simeq \mathcal{A}_n$ holds for $m \neq n$ if and only if $m \equiv n \pmod{d}$, and $m, n \geqslant k$.*

Theorem 1 implies that, for any fixed free $\mathfrak{A}$-algebra $\mathcal{A}_m$, the indices j for which $\mathcal{A}_j$ is isomorphic to $\mathcal{A}_m$ form an arithmetic progression. Let in particular $\{\mathcal{A}\}$ be the class consisting of a single abstract algebra $\mathcal{A}$ and suppose that $\mathcal{A}$ has a finite basis (set of independent (¹) generators). Then the above consequence of Theorem 1 yields a theorem of E. Marczewski (cf. also [2], Theorem 5) which says that the finite ranks (cardinals of bases) of $\mathcal{A}$ form an arithmetic progression. The transition from our results to this theorem follows by observing that r is a rank of $\mathcal{A}$ if and only if $\mathcal{A}$ is isomorphic to the free $\{\mathcal{A}\}$-algebra $\mathcal{A}_r$.

THEOREM 2. *Given any integers $0 < k < l$, there exists a class of algebras $\mathfrak{A}_{(k,l)}$ satisfying the assumptions of Theorem 1, i.e. with the property that k and l are the smallest integers such that $\mathcal{A}_k \simeq \mathcal{A}_l$.*

For proving Theorem 2 we use the class $\mathfrak{A}_{(k,l)}$ of all algebras having the following $k+l$ operations

$$(1)\qquad \varphi_i(x_1, \ldots, x_k), \quad \omega_j(x_1, \ldots, x_l), \quad i = 1, \ldots, l;\ j = 1, \ldots, k,$$

which satisfy the axioms (²)

$$(2)\qquad \varphi_i\big(\omega_1(x_1, \ldots, x_l), \ldots, \omega_k(x_1, \ldots, x_l)\big) = x_i; \quad i = 1, \ldots, l,$$

$$(3)\qquad \omega_j\big(\varphi_1(x_1, \ldots, x_k), \ldots, \varphi_l(x_1, \ldots, x_k)\big) = x_k; \quad j = 1, \ldots, k.$$

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(¹) In the sense of E. Marczewski's definition (see [5]; cf. also Def. 5 below).

(²) Axioms of this kind were considered in [4].