In case (1), there are open sets U about a and V about b with $U \cap V = \square$, so that $(I \times U) \times (I \times V)$ is open and misses R. In case (2), using the continuity of g, there are open sets U about a, N about x, and M about y with $g(x', y') < a'$ for each $x' \in N \cap I$, $y' \in M \cap I$, and $a' \in U$. Then $[(N \cap I) \times D] \times [(M \cap I) \times U]$ is open and misses R. Thus R is closed, and it follows that φ is continuous.

Since $I \times [0, 1]$ is a compact topological semigroup and R is a closed congruence, it follows that $I \times [0, 1]/R$ is again a compact topological semigroup. Denote the natural homomorphism by η. We have the diagram

$$I \times [0, 1] \xrightarrow{\eta} I \times [0, 1]/R \xrightarrow{\varphi^*} S$$
$$\xrightarrow{\qquad\qquad \varphi \qquad\qquad}$$

where φ^* is the function induced by φ. It follows from the continuity of φ that φ^* is continuous, and is 1-1 onto and hence a homeomorphism. We identify $I \times [0, 1]/R$ with S so that S is now a topological semigroup and is the continuous homomorphic image of $I \times [0, 1]$. Now let T be the relation on $I \times [0, 1]$ defined by $T = (I \times 0) \times (I \times 0) \cup \Delta$ where Δ is the diagonal of $(I \times [0, 1])^2$. Then T is a closed congruence, $I \times [0, 1]/T$ is a compact topological semigroup, and the natural mapping β is a continuous homorphism. Since $T \subset R$, there is induced a continuous homomorphism α as indicated:

$$I \times [0, 1] \xrightarrow{\beta} I \times [0, 1]/T \xrightarrow{\alpha} I \times [0, 1]/R = S$$
$$\xrightarrow{\qquad\qquad\qquad \eta \qquad\qquad\qquad}$$

Thus S has been realized as the continuous homomorphic image of $I \times [0, 1)/T$, which is the "fan" over I, i.e. the topological semigroup obtained from $I \times [0, 1]$ by shrinking $I \times \{0\}$ to a point.

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On groups of functions defined on Boolean algebras

by

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With an arbitrary set T and abelian group G one may connect the group of all functions x defined on T with values in G. These groups are known as complete direct sums and can be considered also as groups of functions defined on T with values in G that are measurable with respect to the complete Boolean algebra 2^T of all subsets of T. Let us write for an arbitrary function x: $T \to G$ and arbitrary $g \in G$

$$x(g) = \{t \in T;\ x(t) = g\}\,.$$

Then we obviously have

(1) $$x(g) \in 2^T \quad \text{for all} \quad g \in G\,,$$

(2) $$\bigcup_{g \in G} x(g) = T\,,$$

(3) $$x(g) \cap x(g') = 0 \quad \text{for} \quad g \neq g'\,,$$

(4) $$(x+y)(g) = \bigcup_{g' \in G} x(g') \cap y(g-g')\,.$$

Now we get a clear way for generalizations; let G be an arbitrary abelian group of cardinality $\overline{\overline{G}} = \mathfrak{m}$ and $\mathcal{B}$ a Boolean $\mathfrak{m}$-additive algebra with maximal element e. Elements of the group $S(\mathcal{B}, G)$ are functions x defined on G and satisfying the following conditions:

(5) $$x(g) \in \mathcal{B} \quad \text{for all} \quad g \in G\,,$$

(6) $$\bigcup_{g \in G} x(g) = e\,,$$

(7) $$x(g) \cap x(g') = 0 \quad \text{for} \quad g \neq g'\,,$$

the sum $z = x + y$ of two such functions is defined by the equation

(8) $$z(g) = \bigcup_{g' \in G} x(g') \cap y(g-g') \quad \text{for} \quad g \in G\,.$$

Since $\mathcal{B}$ is $\mathfrak{m}$-additive, then the sum in (8) exists and elements $z(g)$ are well-defined and satisfy conditions (5)-(7).

It is easy to see that $S(\mathfrak{B}, G)$ is an abelian group [1] and $S(\mathfrak{B}, G)$ is isomorphic with a complete direct sum of $\overline{\overline{T}}$ copies of the group G in case $\mathfrak{B} = 2^T$.

For each element x of $S(\mathfrak{B}, G)$ we shall write

$$\nu(x) = \bigcup_{0\neq g\in G} x(g) = [x(0)]'\,.$$

Let $\mathcal{I}$ be an arbitrary (finitely additive) ideal in the algebra $\mathfrak{B}$ and $\mathcal{I}_\beta$ the least σ-ideal containing $\mathcal{I}$. We define $S(\mathcal{I})$ as being the set of all elements $x \in S(\mathfrak{B}, G)$ with $\nu(x) \in \mathcal{I}$

$$S(\mathcal{I}) = \{x \in S(\mathfrak{B}, G);\ \nu(x) \in \mathcal{I}\}\,. \tag{9}$$

Since the relation $\nu(x-y) \subset \nu(x) \cup \nu(y)$ holds, the set $S(\mathcal{I})$ is a subgroup of $S(\mathfrak{B}, G)$.

The purpose of this paper is to give the structure of the factor groups $S(\mathcal{I}_\beta)/S(\mathcal{I})$ for the torsion free group G: these groups are torsion free algebraically compact groups in the sense of Kaplansky (see [7] and § 3 below). In the special case of groups $S(2^T, C)$ and ideals $\mathcal{I}^*$ consisting of all subsets of T of cardinality $< \aleph_\varkappa$ all cardinal invariants of those groups are found (by freely using Cantor's Generalized Continuum Hypothesis, i.e. $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all ordinals α). In the second part we present some theorems relating homomorphisms of groups $S(\mathfrak{B}, G)$ in slender groups with measures defined on $\mathfrak{B}$. In the last section, § 9, we present two theorems on groups $S(\mathfrak{B}, G)$ in the case of $\mathfrak{m}$-distributive Boolean algebra $\mathfrak{B}$; if G is an algebraically compact group, then $S(\mathfrak{B}, G)$ is also such a group; if G is complete direct sum $G = \sum_{t\in T}^* G_t$, then

$$S\Big(\mathfrak{B}, \sum_{t\in T}^* G_t\Big) = \sum_{t\in T}^* S(\mathfrak{B}, G_t)\,.$$

Some of the results contained in this paper were communicated in [1].

I wish express my gratitude and thanks to Professor J. Łoś and Dr. E. Sąsiada for their helpful suggestions.

§ 1. Notations and lemmas. All groups that will be considered are abelian ones. If $\{G_t\}_{t\in T}$ is a family of groups, then $\sum_{t\in T}^* G_t$ is the complete direct sum of the family $\{G_t\}$, i.e. the group of all functions x defined on T such that $x(t) \in G_t$ for $t \in T$. The subgroup of the group $\sum_{t\in T}^* G_t$ consisting of all functions x with $x(t) = 0$ for almost all $t \in T$ is the discrete direct sum of the family $\{G_t\}$ and is denoted by $\sum_{t\in T} G_t$. If all groups G_t are identical

[1] The definition of the groups $S(\mathfrak{B}, G)$ is due to J. Łoś.

with a group G, then the group $\sum_{t\in T}^* G_t$ is denoted by G_*^T or $G_*^{\overline{\overline{T}}}$ and the group $\sum_{t\in T} G_t$ by G^T or $G^{\overline{\overline{T}}}$.

The group of rational integers is denoted by C, that of rationals by R, that of p-adic integers by I_p. C_p is a cyclic group of prime order p, P is the set of all primes $P = (p_1, p_2, \ldots)$.

We shall prove three simple lemmas, which will be used in the proof of Theorem 1.

LEMMA 1. *Let G be a group and $\mathfrak{B}$ a $\overline{\overline{G}}$-additive Boolean algebra. Then for $x, y \in S(\mathfrak{B}, G)$ and $n = \pm 1, \pm 2, \ldots$ the following two conditions are equivalent*:

(i) $y = nx$,

(ii) $y(g) = \bigcup_{ng'=g} x(g')$ *for* $g \in nG$ *and* $y(g) = 0$ *for* $g \notin nG$.

Proof. Since an equality $(-y)(g) = y(-g)$ holds, it is sufficient to prove our lemma for positive integers n only. We shall prove the implication (i)$\Rightarrow$(ii) inductively with respect to n; for $n = 1$ it is obvious. Let us suppose that the above implication holds for $n-1$; then

$$y(g) = \bigcup_{g'} x(g-g') \cap [(n-1)x](g') = \bigcup_{g'} \bigcup_{(n-1)g''=g'} x(g-g') \cap x(g'')\,.$$

The element $x(g-g') \cap x(g'')$ is not empty only if $g-g' = g''$. Moreover $(n-1)g'' = g'$ and the two relations imply $g = ng''$ and the double join is reduced to $\bigcup_{ng''=g} x(g'') = y(g)$.

Let x, y be elements of $S(\mathfrak{B}, G)$ satisfying (ii). Then by the result just proved $(nx)(g) = y(g)$ for all $g \in G$ and consequently $y = nx$.

COROLLARY. *Element $y \in S(\mathfrak{B}, G)$ belongs to $nS(\mathfrak{B}, G)$ if and only if $y(g) = 0$ for $g \notin nG$.*

LEMMA 2. *If G is torsion free and $\mathcal{J}$ is a σ-ideal containing $\mathcal{I}$: $\mathcal{J} \supset \mathcal{I}$, χ is the natural homomorphism of $S(\mathcal{J})$ on $S(\mathcal{J})/S(\mathcal{I})$, x is an element of $S(\mathcal{J})$ and for a fixed positive integer n the join of all $x(g)$ with $g \notin nG$ $\bigcup_{g\notin nG} x(g)$ is in $\mathcal{I}$, then $\chi(x) \in n\big(S(\mathcal{J})/S(\mathcal{I})\big)$.*

Proof. It is sufficient to find such an element $y \in S(\mathcal{J})$ that $\nu(ny-x) \in \mathcal{I}$. We define y by the following relations:

$$y(g) = x(ng) \quad \text{for} \quad g \neq 0\,, \qquad y(0) = x(0) \cup \bigcup_{g\notin nG} x(g)\,.$$

At first we prove for $g \neq 0$

$$(ny)(g') \cap x(g'-g) = \begin{cases} 0 & \text{if} \quad g' \neq 0\,, \\ y(0) \cap x(-g) & \text{if} \quad g' = 0\,. \end{cases}$$

In fact, if $g' \notin nG$, then $(ny)(g') = 0$; if $g' = ng'' \neq 0$, then $(ny)(ng'') = y(g'') = x(g')$ and $x(g') \cap x(g'-g) = 0$ because $g \neq 0$. The last case, $g' = 0$, follows by the equality $(ny)(0) = y(0)$. By the definition of addition we have $(ny-x)(g) = y(0) \cap x(-g)$ for $g \neq 0$, and consequently $\nu(ny-x) = y(0) \cap \nu(x) = [x(0) \cup \bigcup_{g\notin nG} x(g)] \cap \nu(x) = \bigcup_{g\notin nG} x(g) \in \mathcal{I}$, whence $ny-x \in S(\mathcal{I})$ holds.

LEMMA 3. *Let $g_1, \ldots, g_n$ be elements of a group G and $p_1, \ldots, p_n$ different primes; then there exists an element $g \in G$ such that $g-g_1 \in p_1^n G, \ldots, g-g_n \in p_n^n G$.*

Proof. Let us denote $r = (p_1 \ldots p_n)^n$; then there exist integers $k_1, \ldots, k_n$ such that

$$\frac{r}{p_i^n} k_i \equiv 1 \ (p_i^n) .$$

If we write

$$m_i = \frac{r}{p_i^n} k_i , \qquad g = m_1 g_1 + \ldots + m_n g_n ,$$

then

$$m_i \equiv 1 \ (p_i^n) , \qquad m_j \equiv 0 \ (p_i^n)$$

for $j \neq i$ and consequently $g - g_i \in p_i^n G$.

§ 2. p-complete groups. If G is an arbitrary group and p a prime, then the subgroup of all elements of infinite p-height is defined as the meet: $p^\infty G = \bigcap_{n=1}^{\infty} p^n G$. If $p^\infty G = \{0\}$, then the group G admits p-adic (Hausdorff) topology: a complete set of neighbourhoods of 0 consists of subgroups $p^n G$. The group G (with $p^\infty G = \{0\}$) is called *p-complete* if it is complete in its p-adic topology, i.e. every Cauchy sequence is convergent (a sequence $\{g_n\}$ of elements of G is a Cauchy sequence if for each positive integer k there exists an integer $N(k)$ such that, for each $n \geqslant N(k)$, $g_n - g_{N(k)} \in p^k G$).

§ 3. Algebraically compact groups. Let us recall the definition of an algebraically compact group (see [7], p. 56): a group A is *algebraically compact* if and only if it has a form $A = A_0 + \sum_{p\in P}^{*} A_p$ and A_0 is a divisible group, A_p (for $p \in P$) has no element of infinite p-height and is p-complete.

The groups A_p are isomorphic to $A/p^\infty A$ and A_0 is the maximal divisible subgroup of A.

If A is a torsion free group, then $A_0 = \bigcap_{p\in P} p^\infty A$ and it is easy to verify the following

LEMMA 4. *A torsion free group A is algebraically compact if and only if it possesses two properties:*

(i) *$A/p^\infty A$ is p-complete for each prime p,*

(ii) *for each sequence $\{a_k\}$ of elements of A there exists such an element $a \in A$ that $a - a_k \in p_k^\infty A$ for all $k = 1, 2, \ldots$*

Kaplansky gave an algebraic characterization of groups without elements of infinite p-height and p-complete (see [7], p. 51). If such a group G is torsion free then it takes the form $G = \operatorname{compl} I_p^\Gamma$, i.e. the completion in p-adic topology of the group I_p^Γ (for a suitable set Γ). It may be considered as the subgroup of the group of all functions defined on Γ with p-adic integer values and of at most countable support. For each countable subset Γ_0 of Γ there exists an element of G with support identical with Γ_0.

§ 4. Factor groups $S(I_\beta)/S(I)$. We shall prove the following

THEOREM 1. *If G is a torsion free group of cardinality $\mathfrak{m}$ and $\mathcal{I}$ is a finitely additive ideal in a Boolean $\mathfrak{m}$-additive algebra $\mathcal{B}$, then the factor group $A = S(\mathcal{I}_\beta)/S(\mathcal{I})$ is a torsion free algebraically compact group.*

Proof. By Lemma 1, for $n \neq 0$, $\nu(x) = \nu(nx)$; hence, if $x \in S(\mathcal{I}_\beta)$ and $nx \in S(\mathcal{I})$ for some $n \neq 0$, then $x \in S(\mathcal{I})$ and A is torsion free.

Let us denote by χ the natural homomorphism of $S(\mathcal{I}_\beta)$ on A: $\chi(S(\mathcal{I}_\beta)) = A$; furthermore $S_p = \chi^{-1}(p^\infty A)$ for $p \in P$ and χ_p the natural homomorphism of $S(\mathcal{I}_\beta)$ on $S(\mathcal{I}_\beta)/S_p$. Then we have

$$A/p^\infty A \approx S(\mathcal{I}_\beta)/S(\mathcal{I}) \big/ S_p/S(\mathcal{I}) \approx S(\mathcal{I}_\beta)/S_p$$

and by Lemma 4 it is sufficient to prove that

(10) *the groups $A_p = S(I_\beta)/S_p$ are p-complete;*

(11) *for each sequence $\{x_k\}$ of elements of $S(\mathcal{I}_\beta)$ there exists such an element $x \in S(\mathcal{I}_\beta)$ that $\chi_{p_k}(x) = \chi_{p_k}(x_k)$ for $k = 1, 2, \ldots$*

Let us suppose that $\{a_n\}$ ($a_n \in A_p$ for $n = 1, 2, \ldots$) is a Cauchy sequence. Then for each positive integer k there exists such an $N(k)$ that

(12) $$a_{N(k)} - a_n \in p^k A_p \quad \text{for} \quad n \geqslant N(k) .$$

Without any restriction we may suppose that $N(1) < N(2) < \ldots$ We shall prove that there exists a sequence $\{z_n\}$ of elements of $S(\mathcal{I}_\beta)$ such that

(13) $$\chi_p(z_n) = a_{N(n)} \quad \text{and} \quad z_k - z_n \in p^k S(\mathcal{I}_\beta) \quad \text{for} \quad n \geqslant k .$$

Let $\{y_n\}$ be an arbitrary sequence of elements of $S(\mathcal{I}_\beta)$ such that $\chi_p(y_n) = a_{N(n)}$ for all n. We define $z_1 = y_1$; relation (12) implies $y_k - y_n \in p^k S(\mathcal{I}_\beta) + S_p$ for $n \geqslant k$ and all k. Therefore $y_1 - y_n = pt_n + u_n$ for some $t_n \in S(\mathcal{I}_\beta)$, $u_n \in S_p$ and $n > 1$. If we put $y_1' = y_1$, $y_n' = y_n + u_n$ for $n > 1$, then $y'_1 - y'_n \in pS(\mathcal{I}_\beta)$ and for $n \geqslant k > 1$ $y'_k - y'_n = y_k - y_n + u_k - u_n \in p^k S(\mathcal{I}_\beta) + S_p$, and we define $z_2 = y'_2$. Taking $k = 2$ in the above relation we have $y'_2 - y'_n = p^2 v_n + w_n$ for some $v_n \in S(\mathcal{I}_\beta)$, $w_n \in S_p$ and $n > 2$. If we put $y''_1 = y'_1$, $y''_2 = y'_2$, $y''_n = y'_n + w_n$ for $n > 2$, then $y'_2 - y'_n = (y'_2 - y'_1) + (y'_1 - y'_n) \in pS(\mathcal{I}_\beta)$ and $w_n \in pS(\mathcal{I}_\beta)$; consequently $y''_1 - y''_n = y'_1 - y'_n - w_n \in pS(\mathcal{I}_\beta)$ for $n > 2$ and $y''_2 - y''_n = y'_2 - y'_n - w_n = p^2 v_n \in p^2 S(\mathcal{I}_\beta)$, and we define $z_3 = y''_3$. This procedure may be continued, and finally we get the sequence $\{z_n\}$ that satisfies (13).

Now we define an element $z \in S(\mathcal{I}_\beta)$ with a property $\chi_p(z) = \lim a_n$. The join $b = \bigcup_{n=1}^{\infty} \nu(z_n)$ belongs to $\mathcal{I}_\beta$ and may be represented as a join of disjoint elements of $\mathcal{I}$: $b = \bigcup_{m=1}^{\infty} b_m$ $(b_m \in \mathcal{I},\ b_m \cap b_n = 0$ for $m \neq n)$. Element z is defined by the equalities

$$z(g) = \bigcup_{m=1}^{\infty} b_m \cap z_m(g) \quad \text{for} \quad g \in G,\ g \neq 0\,,$$
$$z(0) = [\bigcup_{0 \neq g \in G} z(g)]'\,.$$

Let us consider the difference $z - z_k$; by the definition of z we have

$$z(0) = [\bigcup_{g' \neq 0} z(g')]' = [\bigcup_{g' \neq 0} \bigcup_{m=1}^{\infty} b_m \cap z_m(g')]' = [\bigcup_{m=1}^{\infty} b_m \cap \nu(z_m)]' = \bigcap_{m=1}^{\infty} [b'_m \cup z_m(0)]$$

and furthermore

$$z(0) \cap b = \bigcap_{m=1}^{\infty} [b'_m \cup z_m(0)] \cap \bigcup_{n=1}^{\infty} b_n = \bigcup_{n=1}^{\infty} \bigcap_{m=1}^{\infty} [b'_m \cup z_m(0)] \cap b_n$$
$$\subset \bigcup_{n=1}^{\infty} [b'_n \cup z_n(0)] \cap b_n = \bigcup_{n=1}^{\infty} z_n(0) \cap b_n\,.$$

Since $\nu(z_k) \subset b$, by the above inclusion we get for $g \neq 0$

$$(z - z_k)(g) = \bigcup_{g'} z_k(g' - g) \cap z(g')$$
$$\subset \bigcup_{g' \neq 0} [z_k(g' - g) \cap \bigcup_{m=1}^{\infty} b_m \cap z_m(g')] \cup z_k(-g) \cap \bigcup_{n=1}^{\infty} z_n(0) \cap b_n$$
$$= \bigcup_{m=1}^{\infty} b_m \cap [\bigcup_{g'} z_m(g') \cap z_k(g' - g)] = \bigcup_{m=1}^{\infty} b_m \cap (z_m - z_k)(g).$$

Since by (13) $z_m - z_k \in p^k S(\mathcal{I}_\beta)$ for $m \geqslant k$, we have by Lemma 1 $(z_m - z_k)(g) = 0$ for $g \notin p^k G$. Then for those g we have $(z - z_k)(g) \subset \bigcup_{m=1}^{k-1} b_m$ and consequently $\bigcup_{g \notin p^k G} (z - z_k)(g) \subset \bigcup_{m=1}^{k-1} b_m \in \mathcal{I}$. By Lemma 2, element $\chi_p(z - z_k)$ belongs to $p^k A_p$; furthermore we have, for $i \geqslant N(k)$,

$$\chi_p(z) - a_i = [\chi_p(z) - a_{N(k)}] + [a_{N(k)} - a_i] = \chi_p(z - z_k) + a_{N(k)} - a_i \in p^k A_p\,.$$

This implies $\chi_p(z) = \lim a_n$ and the proof of (10) is finished.

Let $\{x_k\}$ be a sequence of elements of $S(\mathcal{I}_\beta)$. The element $c = \bigcup_{k=1}^{\infty} \nu(x_k)$ belongs to $\mathcal{I}_\beta$ and may be represented as a join of a countable set of disjoint elements of I: $c = \bigcup_{i=1}^{\infty} c_i$ $(c_i \in \mathcal{I},\ c_i \cap c_j = 0$ for $i \neq j)$. By Lemma 3 for each n-tuple of elements of G $g_1, \ldots, g_n$ there exists an element $g \in G$ such that $g - g_1 \in p_1^n G, \ldots, g - g_n \in p_n^n G$; we denote by $\varphi(g_1, \ldots, g_n)$ one of those elements such that $g \neq 0$ (it exists because G is torsion free).

We shall prove that element $x \in S(\mathcal{I}_\beta)$ defined by equalities

$$x(g) = \bigcup_{n=1}^{\infty} \bigcup x_1(g_1) \cap \ldots \cap x_n(g_n) \cap c_n \quad \text{for} \quad g \neq 0\,,$$
$$x(0) = [\bigcup_{0 \neq g \in G} x(g)]'$$

(the second join is taken over all $g_1, \ldots, g_n \in G$ such that $\varphi(g_1, \ldots, g_n) = g$) satisfies (11). It is easy to verify that $x(0) = c'$.

Let us take an arbitrary prime p_k and an integer $m \geqslant k$; then, since for $g' \neq 0$, $x(0) \cap x_k(-g') \subset x(0) \cap c = 0$, we get

$$(x - x_k)(g') = \bigcup_{g} x(g) \cap x_k(g - g')$$
$$= \bigcup_{g \neq 0} \bigcup_{n=1}^{\infty} \bigcup x_1(g_1) \cap \ldots \cap x_n(g_n) \cap c_n \cap x_k(g - g')$$
$$\subset c_1 \cup \ldots \cup c_{m-1} \cup \bigcup_{g \neq 0} \bigcup_{n=m}^{\infty} \bigcup x_k(g_k) \cap x_k(g - g')\,.$$

The element $x_k(g_k) \cap x_k(g - g')$ is not empty only in the case of $g_k = g - g'$; if $n \geqslant m$, then $g' = g - g_k \in p_k^n G \subset p_k^m G$ because $g = \varphi(g_1, \ldots, g_n)$. Therefore, if $g' \notin p_k^m G$, then $(x - x\ )(g') \subset c_1 \cup \ldots \cup c_{m-1} \in \mathcal{I}$, and consequently $\bigcup_{g' \notin p_k^m G} (x - x_k)(g') \in \mathcal{I}$. By Lemma 2, $\chi_{p_k}(x - x_k) \in p_k^m A_{p_k}$ for all $m \geqslant k$; since $p_k^\infty A_{p_k} = \{0\}$, we have $\chi_{p_k}(x) = \chi_{p_k}(x_k)$ and the proof of (11) is finished.

§ 5. Special cases. Let $\mathcal{B} = 2^T$ be a Boolean algebra of all subsets of a set T and let G be an arbitrary group. Then the group $S(\mathcal{B}, G)$ may be identified with the complete direct sum G_*^T and, for $x \in G_*^T$, $\nu(x)$ is the set $\{t \in T;\ x(t) \neq 0\}$. For an arbitrary finitely additive ideal $\mathcal{I}$ in $\mathcal{B}$ the group $S(\mathcal{I})$ consists of all elements $x \in G_*^T$ with $\nu(x) = \mathcal{I}$. By Theorem 1 we immediately get

THEOREM 2. *If G is a torsion free group and $\mathcal{I}$ is a finitely additive ideal of subsets of T, then the group $S(\mathcal{I}_\beta)/S(\mathcal{I}) = \{x \in G_*^T;\ \nu(x) \in \mathcal{I}_\beta\}/\{x_\beta \in G_*^T;\ \nu(x) \in \mathcal{I}\}$ is a torsion free algebraically compact group.*

The direct proof of this theorem is simpler from the technical point of view than that of Theorem 1. In that case all the functions under consideration can be defined explicitly, not by their counterimages, as in the proof of Theorem 1.

From now on we shall consider the factor groups mentioned in Theorem 2 for $G = C$.

Let T be a set of cardinality $\aleph_\lambda$ and $\mathcal{I}^\varkappa$ the ideal consisting of all subsets T_0 of T with $\overline{\overline{T}}_0 < \aleph_\varkappa$. The ideal $\mathcal{I}^\varkappa_\beta$ is different from $\mathcal{I}^\varkappa$ if and only if $\aleph_\varkappa$ is the sum of a countable number of alephs less than $\aleph_\varkappa$. In that case the ideal $\mathcal{I}^\varkappa_\beta$ is identical with $\mathcal{I}^{\varkappa+1}$.

Consistently with the notation used in [8] we shall denote by $S^\varkappa_\lambda$ the subgroup $S(\mathcal{I}^\varkappa)$ of $C_*^{\aleph_\lambda}$. Using Theorem 2 we shall study the structure of groups $S^{\varkappa+1}_\lambda/S^\varkappa_\lambda$ for $\varkappa$ confinal with ω_0 (i.e. $\mathrm{cf}\,\varkappa = 0$).

Let us recall the following useful

DEFINITION. For an arbitrary ordinal α the symbol $\mathrm{cf}\,\alpha$ denotes the least ordinal γ such that ω_α is confinal with ω_γ.

Tarski proved (admitting Cantor's Generalized Continuum Hypothesis) the following propositions:

(14) $$\aleph_\lambda^{\aleph_\varkappa} = \begin{cases} \aleph_\lambda & \text{for} \quad \varkappa < \mathrm{cf}\,\lambda, \\ \aleph_{\lambda+1} & \text{for} \quad \varkappa \geqslant \mathrm{cf}\,\lambda. \end{cases}$$

(15) *If $\varkappa \leqslant \lambda$ then the family of all subsets of cardinality $\aleph_\varkappa$ ($< \aleph_\varkappa$) contained in a set T with $\overline{\overline{T}} = \aleph_\varkappa$ is of cardinality $\aleph_\lambda(\aleph_\lambda)$ in the case of $\mathrm{cf}\,\lambda > \aleph$, $\aleph_{\lambda+1}(\aleph_\lambda)$ in the case of $\mathrm{cf}\,\lambda = \varkappa$ and $\aleph_{\lambda+1}(\aleph_{\lambda+1})$ in the case of $\mathrm{cf}\,\lambda < \varkappa$* (see [13]).

We shall deduce, as an easy consequence of the preceding propositions, the following

LEMMA 5. *Let T be an arbitrary set of cardinality $\aleph_\lambda$, and let $\varkappa$ be an ordinal with $\aleph \leqslant \lambda$. Then there exists a family $\mathcal{K}_\varkappa$ of subsets of T having the properties*

(i) *If $X \in \mathcal{K}_\varkappa$ then $X \subset T$ and $\overline{\overline{X}} = \aleph_\varkappa$.*

(ii) *If $X, Y \in \mathcal{K}_\varkappa$ and $X \neq Y$ then $\overline{\overline{X \dot{-} Y}} = \aleph_\varkappa$.*

(iii) *$\overline{\overline{\mathcal{K}}}_\varkappa = \aleph_\lambda$ in the case of $\mathrm{cf}\,\lambda > \varkappa$, and $\overline{\overline{\mathcal{K}}}_\varkappa = \aleph_{\lambda+1}$ in the case of $\mathrm{cf}\,\lambda \leqslant \varkappa$.*

Proof. By the equality $\aleph_\lambda = \aleph_\lambda \cdot \aleph_\varkappa$ the set T may be considered as a cartesian product $T = T_1 \times T_2$ with $\overline{\overline{T}}_1 = \aleph_\lambda$ and $\overline{\overline{T}}_2 = \aleph_\varkappa$. We define $\mathcal{K}_\varkappa$ as the family of sets of the form $X = X_1 \times T_2$ with $X_1 \subset T_1$ and $\overline{\overline{X}}_1 = \aleph_\varkappa$. By (15) the family $\mathcal{K}_\varkappa$ possesses all the desired properties.

We shall prove two lemmas that give the cardinal invariants of factor groups $S^{\varkappa+1}_\lambda/S^\varkappa_\lambda$.

LEMMA 6. *For an arbitrary prime p and ordinals $\varkappa, \lambda$ with $\varkappa \leqslant \lambda$ and $\mathrm{cf}\,\varkappa = 0$ the group $A_p = S^{\varkappa+1}_\lambda/S_p$ satisfies: $\overline{\overline{A}}_p = \aleph_\lambda$ in the case of $\mathrm{cf}\,\lambda > \varkappa$ and $\overline{\overline{A}}_p = \aleph_{\lambda+1}$ in the case of $\mathrm{cf}\,\lambda \leqslant \varkappa$* (2).

(2) The group S_p is the same one as has been defined in the proof of Theorem 1, i.e. $S_p = \chi^{-1}[p^\infty(S^{\varkappa+1}_\lambda/S^\varkappa_\lambda)]$, where χ is the natural homomorphism of $S^{\varkappa+1}_\lambda$ onto $S^{\varkappa+1}_\lambda/S^\varkappa_\lambda$.

Proof. Let T be a set of cardinality $\aleph_\lambda$ and $\mathcal{K}_\varkappa$ a family of subsets of T with all the properties of Lemma 5. For each X of $\mathcal{K}_\varkappa$ a function defined by relations:

$$x(t) = \begin{cases} 1 & \text{for} \quad t \in X, \\ 0 & \text{for} \quad t \notin X \end{cases}$$

belongs to $S^{\varkappa+1}_\lambda$, and it is easy to prove that the images of those functions by homomorphisms χ_p (defined in the proof of Theorem 1) are all different; then $\overline{\overline{A}}_p \geqslant \overline{\overline{\mathcal{K}}}_\varkappa$. On the other hand $\overline{\overline{A}}_p \leqslant \overline{\overline{S}}{}^{\varkappa+1}_\lambda$ and the cardinality of the group $S^{\varkappa+1}_\lambda$ is not greater than that of the family of all subsets of T of cardinality $\leqslant \aleph_\varkappa$ multiplied by $2^{\aleph_\varkappa}$. The lemma follows by the properties of $\mathcal{K}_\varkappa$ and by (15).

LEMMA 7. *If $\varkappa \leqslant \lambda$ and $\mathrm{cf}\,\varkappa = 0$, then the cordinality of the maximal divisible subgroup of the group $S^{\varkappa+1}_\lambda/S^\varkappa_\lambda$ is $\aleph_\lambda$ in the case of $\mathrm{cf}\,\lambda > \varkappa$ and is $\aleph_{\lambda+1}$ in the case of $\mathrm{cf}\,\lambda \leqslant \varkappa$.*

The proof may be carried out in the same way as preceding one, by defining for arbitrary $X \in \mathcal{K}_\varkappa$ a decomposition $X = \bigcup_{n=1}^{\infty} X_n$ with $\overline{\overline{X}}_n < \aleph_\varkappa$ and then the elements

$$x(t) = \begin{cases} n! & \text{for} \quad t \in X_n, \\ 0 & \text{for} \quad t \notin X. \end{cases}$$

LEMMA 8. *Let G be a torsion free algebraically compact group with $p^\infty G = \{0\}$ and $\overline{\overline{G}} = \aleph_\alpha$. Then $\mathrm{cf}\,\alpha > 0$ and the group is of the form:*

$G = \mathrm{compl}(I_p^{\aleph_\alpha})$ *in the case of a limit ordinal α,*

$G = I_{p^*}^{\aleph_{\alpha-1}}$ *in the case of α not being a limit ordinal and $\mathrm{cf}(\alpha-1) > 0$,*

$G = \mathrm{compl}(I_p^{\aleph_{\alpha-1}})$ *or $I_{p^*}^{\aleph_{\alpha-1}}$ in the case of α not being a limit ordinal, $\alpha > 1$ and $\mathrm{cf}(\alpha-1) = 0$,*

$G = I_p, I_p^2, \ldots, \mathrm{compl}(I_p^{\aleph_0})$ *or $I_{p^*}^{\aleph_0}$ in the case of $\alpha = 1$*

according to whether G (in the p-adic topology) contains a dense subset of cardinality $\aleph_{\alpha-1}$ or not.

Proof. By the theorem of Kaplansky, the group G has the form $G = \mathrm{compl}(I_p^{\mathfrak{m}})$ and using the same method as in the proof of Lemma 5 one may find

$$\overline{\overline{\mathrm{compl}(I_p^{\aleph_\beta})}} = \begin{cases} \aleph_\beta & \text{in the case of} \quad \mathrm{cf}\,\beta > 0, \\ \aleph_{\beta+1} & \text{in the case of} \quad \mathrm{cf}\,\beta = 0. \end{cases}$$

It follows that no group G has power $\aleph_\alpha$ with $\mathrm{cf}\,\alpha = 0$ (it can be proved independently of Kaplansky's characterization, by using the method of Baire's category theorem).

The group $I_{p^*}^{\aleph_\gamma}$ is algebraically compact and has cardinality $\aleph_{\gamma+1}$. The factor group $I_{p^*}^{\aleph_\gamma}/pI_{p^*}^{\aleph_\gamma}$ is isomorphic to $(I_p/pI_p)_*^{\aleph_\gamma} \approx (C_p)_*^{\aleph_\gamma}$ and then is of cardinality $\aleph_{\gamma+1}$. Since the cardinality of a dense subset (in p-adic topology) of a group H is at least $\overline{\overline{H/pH}}$, then the group $I_{p^*}^{\aleph_\gamma}$ does not contain a dense subset of cardinality $\aleph_\gamma$.

Let α be a limit ordinal with $\operatorname{cf}\alpha > 0$; then $G = \operatorname{compl}(I_p^{\aleph_\alpha})$.

Suppose that α is not a limit ordinal and $\operatorname{cf}(\alpha-1) > 0$. Then the cardinality of $\operatorname{compl}(I_p^{\aleph_{\alpha-1}})$ is $\aleph_{\alpha-1}$ and $G = \operatorname{compl}(I_p^{\aleph_\alpha})$. The group $I_{p^*}^{\aleph_{\alpha-1}}$ is of cardinality $\aleph_\alpha$ and, being algebraically compact, it is isomorphic to $\operatorname{compl}(I_p^{\aleph_\alpha})$. Consequently $G = I_{p^*}^{\aleph_{\alpha-1}}$.

Suppose that α is not a limit ordinal, $\alpha > 1$ and $\operatorname{cf}(\alpha-1) = 0$. Then the groups $\operatorname{compl}(I_p^{\aleph_{\alpha-1}})$ and $\operatorname{compl}(I_p^{\aleph_\alpha})$ are of cardinality $\aleph_\alpha$. The first one contains a dense subset of cardinality $\aleph_{\alpha-1}$ and since $I_{p^*}^{\aleph_{\alpha-1}}$ is of cardinality $\aleph_\alpha$ and does not contain such a set, it is isomorphic to $\operatorname{compl}(I_p^{\aleph_\alpha})$.

The last case, $\alpha = 1$, can be discussed in a similar manner.

Now we are able to give a simple proof of

THEOREM 3. *If $\varkappa$ and λ are ordinals such that $\varkappa \leqslant \lambda$ and $\operatorname{cf}\varkappa = 0$, then the factor group $S_\lambda^{\varkappa+1}/S_\lambda^\varkappa$ is algebraically compact and*

(i) *$S_\lambda^{\varkappa+1}/S_\lambda^\varkappa \approx R^{\aleph_\lambda} + \sum_{p\in P}^* I_{p^*}^{\aleph_{\lambda-1}}$ in the case of λ not being a limit ordinal and $\lambda > 0$,*

(ii) *$S_\lambda^{\varkappa+1}/S_\lambda^\varkappa \approx R^{\aleph_\lambda} + \sum_{p\in P}^* \operatorname{compl}(I_p^{\aleph_\lambda})$ in the case of λ being a limit ordinal and $\operatorname{cf}\lambda > \varkappa$,*

(iii) *$S_\lambda^{\varkappa+1}/S_\lambda^\varkappa \approx R^{\aleph_{\lambda+1}} + \sum_{p\in P}^* I_{p^*}^{\aleph_\lambda}$ in the case of λ being a limit ordinal and $\operatorname{cf}\lambda \leqslant \varkappa$.*

The groups of (i) *and* (iii) *admit compact topologies.*

Proof. By Theorem 2 the group $G = S_\lambda^{\varkappa+1}/S_\lambda^\varkappa$ is a torsion free algebraically compact group. By Lemma 7 the divisible part of G is of the form described.

Let us suppose that $\operatorname{cf}\lambda \leqslant \varkappa$. Then $\overline{\overline{G}}_p = \aleph_{\lambda+1}$ and using the results of Lemma 8 we shall consider two cases:

a) $\operatorname{cf}\lambda > 0$; then $G_p = I_{p^*}^{\aleph_\lambda}$.

b) $\operatorname{cf}\lambda = 0$; we can take into consideration the functions $x(t)$ defined in the proof of Lemma 6. The difference of two such functions $x_1 - x_2$ takes values 1 or -1 on a set of cardinality $\aleph_\varkappa$ and then (since the elements of $pS_\lambda^{\varkappa+1} + S_p$ take values not divisible by p only on a set of cardinality $< \aleph_\varkappa$) $\chi_p(x_1) - \chi_p(x_2) \notin pG_p$. Consequently $\overline{\overline{G_p/pG_p}} = \aleph_{\lambda+1}$ and G_p has no dense subset of cardinality $\aleph_\lambda$. By Lemma 8, $G_p = I_{p^*}^{\aleph_\lambda}$.

Let us suppose that $\operatorname{cf}\lambda > \varkappa$. Then $\overline{\overline{G}}_p = \aleph_\lambda$ and using the results of Lemma 8 we shall consider three special cases:

c) *λ is a limit ordinal*; then $G_p = \operatorname{compl}(I_p^{\aleph_\lambda})$,

d) *λ is not a limit ordinal and* $\operatorname{cf}(\lambda-1) > 0$; then $G_p = I_{p^*}^{\aleph_{\lambda-1}}$,

e) *λ is not a limit ordinal and* $\operatorname{cf}(\lambda-1) = 0$; there exist $\aleph_\lambda$ functions in $S_\lambda^{\varkappa+1}$ with disjoint supports of cardinality $\aleph_\varkappa$ and taking only the values 0 and 1. By the same reasoning as before the images of those functions by χ_p are all different modulo pG_p; then $\overline{\overline{G_p/pG_p}} = \aleph_\lambda$ and $G_p = I_{p^*}^{\aleph_{\lambda-1}}$.

To prove the last part of the theorem it is sufficient to know that the complete direct sum of groups I_p admits compact topology and the groups $R^{\aleph_{\mu+1}} \approx (R^{\aleph_1})_*^{\aleph_\mu}$ are isomorphic to the complete direct sums of groups of real numbers which admit compact topology also (see [6]).

The assumption $\operatorname{cf}\varkappa = 0$ in Theorem 3 is essential; if $\operatorname{cf}\varkappa > 0$ then in the group $S_\lambda^{\varkappa+1}/S_\lambda^\varkappa$ the only element of infinite p-height (for some p) is 0. In fact, if element $x + S_\lambda^\varkappa$ is of infinite p-height, then every set $T_k = \{t \in T;\ p^k \nmid x(t)\}$, $k = 1, 2, \ldots$, is of cardinality $< \aleph_\varkappa$ and since $\operatorname{cf}\varkappa > 0$, we have $\overline{\overline{\bigcup_{k=1}^{\infty} T_k}} < \aleph_\varkappa$. If $t \notin \bigcup_{k=1}^{\infty} T_k$ then $p^k | x(t)$ for all k and consequently $x \in S_\lambda^\varkappa$.

§ 6. Groups $\mathcal{S}_\lambda^\varkappa$. J. Łoś in paper [8] has considered groups $\mathcal{S}_\lambda^\varkappa = S_\lambda^{\lambda+1}/S_\lambda^\varkappa$ and proved some interesting properties of those groups for $\varkappa$ and λ which are not limit ordinals. We shall prove Theorem 4 concerning the structure of groups $\mathcal{S}_\lambda^\varkappa$ with $\varkappa \leqslant \lambda$ such that $\operatorname{cf}\varkappa = 0$.

THEOREM 4. *Let $\varkappa$, λ be ordinals with $\varkappa \leqslant \lambda$ and $\operatorname{cf}\varkappa = 0$. Then $\mathcal{S}_\lambda^\varkappa \approx S_\lambda^{\varkappa+1}/S_\lambda^\varkappa + \mathcal{S}_\lambda^{\varkappa+1}$.*

The structure of the first summand is fully known as a result of Theorem 3.

Proof. The group $\mathcal{S}_\lambda^\varkappa$ contains a subgroup $G = S_\lambda^{\varkappa+1}/S_\lambda^\varkappa$ and the factor group $\mathcal{S}_\lambda^\varkappa/G$ is isomorphic to $S_\lambda^{\lambda+1}/S_\lambda^{\varkappa+1} = \mathcal{S}_\lambda^{\varkappa+1}$, which is torsion free. Then the algebraically compact group G is pure in $\mathcal{S}_\lambda^\varkappa$ and by a theorem of [2] it is a direct summand of $\mathcal{S}_\lambda^\varkappa$.

§ 7. Factor groups of groups $S(\mathcal{B}, G)$. We shall consider factor groups of the form $S(\mathcal{B}, G)/S(\mathcal{I})$ for some ideal $\mathcal{I}$ of $\mathcal{B}$.

THEOREM 5. *Let G be a group of cardinality $\mathfrak{m}$, $\mathcal{B}$ an $\mathfrak{m}$-additive Boolean algebra, and $\mathcal{I}$ its $\mathfrak{m}$-additive ideal. Then the factor group $S(\mathcal{B}, G)/S(\mathcal{I})$ is isomorphic to $S(\mathcal{B}/\mathcal{I}, G)$* [3].

Proof. Let φ be the natural homomorphism of $\mathcal{B}$ onto $\mathcal{B}/\mathcal{I}$; we define the mapping $\overline{\varphi}$ of $S(\mathcal{B}, G)$ in $S(\mathcal{B}/\mathcal{I}, G)$ as follows:

$$\overline{\varphi}(x) = \overline{x} \text{ if and only if } \overline{x}(g) = \varphi\big(x(g)\big) \text{ holds for all } g \in G.$$

It is sufficient to prove three propositions:

(i) *$\overline{\varphi}$ is a homomorphic mapping,*

(ii) $\overline{\varphi}^{-1}(0) = S(\mathcal{I})$,

[3] Theorem 5 was communicated to me by Professor J. Łoś.

(iii) $\bar{\varphi}\big(S(\mathcal{B}, G)\big) = S(\mathcal{B}/\mathcal{I}, G)$.

If $\bar{x} = \bar{\varphi}(x)$, $\bar{y} = \bar{\varphi}(y)$, $z = x+y$ and $\bar{z} = \bar{\varphi}(z)$, then

$$\bar{z}(g) = \varphi\big(z(g)\big) = \varphi[\bigcup_{g'\in G} x(g') \cap y(g-g')] = \bigcup_{g'\in G} \varphi\big(x(g')\big) \cap \varphi\big(y(g-g')\big) = \bigcup_{g'\in G} \bar{x}(g') \cap \bar{y}(g-g')$$

and consequently

$$\bar{\varphi}(x+y) = \bar{z} = \bar{x}+\bar{y} = \bar{\varphi}(x)+\bar{\varphi}(y) .$$

If x is an element of $\bar{\varphi}^{-1}(0)$, then for each $g \in G$, $g \neq 0$, element $x(g)$ belongs to $\mathcal{I}$ and consequently $\nu(x) \in \mathcal{I}$ by the $\mathfrak{m}$-additivity of $\mathcal{I}$.

Let $\bar{x}$ be an arbitrary element of $S(\mathcal{B}/\mathcal{I}, G)$; then there exists such a decomposition $\{x(g)\}_{g\in G}$ of the unit of $\mathcal{B}$ that $\varphi\big(x(g)\big) = \bar{x}(g)$ and it is equivalent with $\bar{\varphi}(x) = \bar{x}$.

It is known that each Boolean σ-algebra $\mathcal{B}$ can be represented as a factor algebra $\mathcal{B} = \mathcal{F}/\mathcal{I}$ of σ-field $\mathcal{F}$ of subsets of a set T by some σ-ideal $\mathcal{I}$ (see [11]). If G is a countable group, then by Theorem 5, $S(\mathcal{B}, G) \approx S(\mathcal{F}, \mathcal{G})/S(\mathcal{I})$. Since $F \subset 2^T$, then the group $S(\mathcal{F}, G)$ is a subgroup of the complete direct sum G_*^T and consequently each group $S(\mathcal{B}, G)$ with countable G is isomorphic with a factor group of some subgroup of the complete direct sum of sufficiently great numbers of copies of the group G.

Theorem 6. *If $\mathcal{B}$ is an $\mathfrak{m}$-additive Boolean algebra, $\mathcal{I}$ is a finitely additive ideal in $\mathcal{B}$, and G is a torsion free group of cardinality $\mathfrak{m}$, then the group $S(\mathcal{I}_\beta)/S(\mathcal{I})$ is a direct summand of $S(\mathcal{B}, G)/S(\mathcal{I})$. If $\mathfrak{m} = \aleph_0$ then $S(\mathcal{B}, G)/S(\mathcal{I}) \approx S(\mathcal{I}_\beta)/S(\mathcal{I}) + S(\mathcal{B}/\mathcal{I}_\beta, G)$.*

Proof. The proof is similar to that of Theorem 4. The group $S(\mathcal{I}_\beta)/S(\mathcal{I})$ is pure in $S(\mathcal{B}, G)/S(\mathcal{I})$ and by Theorem 1 it is algebraically compact. Then by a result of [2] it is a direct summand. The group $S(\mathcal{B}, G)/S(\mathcal{I})$ contains the subgroup $S(\mathcal{I}_\beta)/S(\mathcal{I})$ and the factor group $S(\mathcal{B}, G)/S(\mathcal{I}) \big/ S(\mathcal{I}_\beta)/S(\mathcal{I}) \approx S(\mathcal{B}, G)/S(\mathcal{I}_\beta)$ is, by Theorem 5 (if $\mathfrak{m} = \aleph_0$), isomorphic with $S(\mathcal{B}/\mathcal{I}_\beta, G)$.

Theorem 7. *If $\mathcal{B}$ is a Boolean $\mathfrak{m}$-additive algebra, $\mathcal{I}$ is a principal ideal of $\mathcal{B}$, and G is a group of cardinality $\mathfrak{m}$, then the subgroup $S(\mathcal{I})$ is a direct summand of $S(\mathcal{B})$.*

Proof. Let $\mathcal{I} = \mathcal{I}(a)$ be the ideal of all elements of the form $a \cap u$ for $u \in \mathcal{B}$. Then the common part of the subgroups $S\big(\mathcal{I}(a)\big)$ and $S\big(\mathcal{I}(a')\big)$ consists of 0 only. For an arbitrary element x of $S(\mathcal{B}, G)$ we define elements y and z by the following relations:

$$y(g) = x(g) \cap a\,, \quad z(g) = x(g) \cap a' \quad \text{for} \quad g \in G\,,\ g \neq 0\,,$$
$$y(0) = x(0) \cap a \cup a'\,, \quad z(0) = x(0) \cap a' \cup a\,.$$

Obviously $y \in S\big(\mathcal{I}(a)\big)$, $z \in S\big(\mathcal{I}(a')\big)$ and it is easy to verify the relation $x+y = z$.

§ 8. Homomorphisms of groups $S(\mathcal{B}, G)$. We shall prove a few theorems on homomorphisms of groups $S(\mathcal{B}, G)$ and set of some connections between those homomorphisms and σ-measures defined on $\mathcal{B}$. The first papers concerning this problem are those of Specker [12] and Ehrenfeucht and Łoś [4].

The methods used in this section are similar to that used in the proofs of the theorems contained in [4].

J. Łoś has defined slender groups:

Definition. The torsion free group H is *slender* if and only if for every homomorphism h of the group $\sum_{n=1}^{*} C^{(n)}$ ($C^{(n)}$ infinite cyclic group) in the group H we have $h(C^{(n)}) = \{0\}$ for almost all n.

Let $\{G_t\}_{t\in T}$ be a family of torsion free groups and $\bar{\bar{T}} < \aleph_\iota$ (4) and let us write $S^* = \sum_{t\in T}^{*} G_t$, $S = \sum_{t\in T} G_t$. The following theorem was proved by J. Łoś and published in the book of L. Fuchs [5]:

(16) *If H is a slender group, and h a homomorphism of S^* in H such that $h(S) = \{0\}$, then $h(S^*) = \{0\}$.*

E. Sąsiada has proved in [9] that every torsion free, countable and reduced group is slender.

We shall consider measures μ defined on a σ-additive Boolean algebra $\mathcal{B}$ with values in a torsion free group H. The notion of a finitely additive measure with values in H is analogous to that of a real-valued measure. The σ-additivity in the case of a measure taking values in H cannot be defined in general and we restrict ourselves to measures μ that satisfy the following conditions.

(F_1) *If elements $a_1, a_2, \ldots$ of $\mathcal{B}$ are disjoint, then $\mu(a_n) = 0$ for almost all n,*

(F_2) *If elements $a_1, a_2, \ldots$ of $\mathcal{B}$ are disjoint and $\mu(a_n) = 0$ for $n = 1, 2, \ldots$ then $\mu(\bigcup_{n=1}^{\infty} a_n) = 0$,*

and we shall call them F-*measures*. Let us denote the set of all F-measures on $\mathcal{B}$ with values in a group H by $M(\mathcal{B}, H)$. This set can be considered as an abelian group with respect to the usual addition defined by the equality $(\mu_1+\mu_2)(a) = \mu_1(a)+\mu_2(a)$ for all $a \in \mathcal{B}$.

We are able to describe all measures belonging to $M(\mathcal{B}, H)$ and every such measure is uniquely determined by a finite number of measures taking values 0 and 1 defined on $\mathcal{B}$ and by the finite set of elements of H.

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(4) $\aleph_\iota$ is the least aleph (if such one exists) with the following property: there exists a σ-measure μ taking values 0 and 1 defined on all subsets of a set M with $\bar{\bar{M}} = \aleph_\iota$ and such that $\mu((m)) = 0$ for all $m \in M$ and $\mu(M) = 1$.

THEOREM 8. *If μ is a F-measure defined on $\mathcal{B}$ with values in a torsion free group H, then it is of the form $\mu(a) = \sum_{i=1}^{n} \mu_i(a)\cdot h_i$ $(a \in \mathcal{B})$, $\mu_1, \ldots, \mu_n$ being measures taking values 0 and 1 on $\mathcal{B}$, $h_1, \ldots, h_n$ elements of H. The measures $\mu_1, \ldots, \mu_n$ (if all different) and elements $h_1, \ldots, h_n$ (if different from 0) are uniquely determined (to the order) by the measure μ.*

The first to prove this theorem was probably A. Birula-Białynicki; the proof just presented is due to J. Łoś.

Proof. Let $\mathcal{J}$ be a σ-ideal in $\mathcal{B}$ consisting of all elements a, $a \in \mathcal{B}$, such that $\mu(b) = 0$ for all b contained in a, $b \subset a$. In the algebra $\overline{\mathcal{B}} = \mathcal{B}/\mathcal{J}$ we define the measure $\bar{\mu}$ by an equality $\bar{\mu}(\bar{a}) = \mu(a)$ ($\bar{a}$ is the coset of a). Obviously if $\bar{a} \in \overline{\mathcal{B}}$ and $\bar{a} \neq 0$, then there exists a $\bar{b} \subset \bar{a}$ such that $\bar{\mu}(\bar{b}) \neq 0$. The algebra $\overline{\mathcal{B}}$ is finite; in the opposite case there exists an infinite sequence $\bar{a}_1, \bar{a}_2, \ldots$ of disjoint elements of $\overline{\mathcal{B}}$ such that $\bar{\mu}(\bar{a}_n) \neq 0$ and consequently in the algebra $\mathcal{B}$ there exists an infinite sequence of disjoint elements $a_1, a_2, \ldots$ such that $\mu(a_n) \neq 0$—this is impossible since μ is a F-measure. The finite algebra $\overline{\mathcal{B}}$ is the algebra generated by its atoms $\bar{b}_1, \ldots, \bar{b}_n$ and $\bar{\mu}(\bar{b}_i) = h_i \neq 0$ for $i = 1, \ldots, n$. We define σ-measures $\mu_1, \ldots, \mu_n$, taking values 0 and 1, by the equalities

$$\mu_i(a) = \begin{cases} 1 & \text{if} \quad \bar{a} \supset \bar{b}_i\,, \\ 0 & \text{if} \quad \bar{a} \cap \bar{b}_i = 0\,. \end{cases}$$

By the definition of $\mu_1, \ldots, \mu_n$ it follows that $\bar{\mu}(\bar{a} \cap \bar{b}_i) = \mu_i(a)\cdot h_i$, and consequently for $a \in \mathcal{B}$ we have

$$\mu(a) = \sum_{i=1}^{n} \bar{\mu}(\bar{a} \cap \bar{b}_i) = \sum_{i=1}^{n} \mu_i(a)\cdot h_i\,.$$

The representation of μ as a linear form is unique since for every finite set of σ-measures taking values 0 and 1 defined on $\mathcal{B}$ there exist such elements $c_1, \ldots, c_n$ of $\mathcal{B}$ that $\mu_i(c_i) = 1$ and $\mu_j(c_i) = 0$ for $i \neq j$.

Let g_0 be an arbitrary elements, different from 0, of the torsion free group G. For an arbitrary element a of $\mathcal{B}$ we denote by x_a the element ("characteristic function") of $S(\mathcal{B}, B)$ defined as follows:

$$x_a(g_0) = a\,, \quad x_a(0) = a'\,, \quad x_a(g) = 0 \quad \text{for} \quad g \neq 0\,,\ g \neq g_0\,.$$

LEMMA 9. *If $a_1, a_2, \ldots$ are disjoint elements of $\mathcal{B}$, and $S^*(a_1, a_2, \ldots)$ is the subgroup of $S(\mathcal{B}, G)$ that consists of elements x having the properties*

(i) *$x(g) = 0$ for $g \neq mg_0$, $m = 0, \pm 1, \pm 2, \ldots$,*

(ii) *$x(mg_0) \cap a_k = a_k$ or 0 for all k, m,*

then there exists an isomorphism ψ of the group $S^*(a_1, a_2, \ldots)$ onto $\sum_{n=1}^{\infty}{}^{*} C^{(n)}$ such that $\psi(x_{a_n}) = e_n$, e_n being a generator of the group $C^{(n)}$.

Proof. Isomorphism ψ is defined by the relation $\psi(x) = \langle k_n e_n \rangle$ and $k_n = m$ for such n that $a_n \subset x(mg_0)$; it is easy to see that ψ has all above properties.

THEOREM 9. *If H is a slender group and $h \in \mathrm{Hom}\big(S(\mathcal{B}, G), H\big)$, G being torsion free, then the function μ defined on $\mathcal{B}$ by an equality $\mu(a) = h(x_a)$ is a F-measure on $\mathcal{B}$ and the mapping $h \to \mu$ is a homomorphism of $\mathrm{Hom}\big(S(\mathcal{B}, G), H\big)$ into the group $M(\mathcal{B}, H)$.*

Proof. Since $x_{a_1 \cup \ldots \cup a_n} = x_{a_1} + \ldots + x_{a_n}$ for disjoint $a_1, \ldots, a_n$ holds, μ is obviously a finitely additive measure. Let $a_1, a_2, \ldots$ be a sequence of disjoint elements of $\mathcal{B}$ and consider the subgroup $S^*(a_1, a_2, \ldots)$ of $S(\mathcal{B}, G)$ and the isomorphism defined in Lemma 9. The function $h\psi^{-1}$ is a homomorphism of S^* in H and since H is slender, $h\psi^{-1}(e_n) = 0$ for almost all n and consequently $\mu(a_n) = h(x_{a_n}) = 0$ for almost all n.

If $\mu(a_n) = 0$ for $n = 1, 2, \ldots$, $a = \bigcup_{n=1}^{\infty} a_n$, and elements a_n are disjoint, then $h\psi^{-1}(e_n) = 0$ and by (16) $h\psi^{-1} = 0$. Consequently $\mu(a) = h(x_a) = 0$.

THEOREM 10. *If G is a torsion free group with $\overline{\overline{G}} < \aleph_s$, $\mathcal{B}$ is a $\overline{\overline{G}}$-additive Boolean algebra and μ is a F-measure defined on $\mathcal{B}$ with integer values, then defining the function h on $S(\mathcal{B}, G)$ by the equality $h(x) = \sum_{g \in G} \mu\big(x(g)\big)\cdot g$ we get the homomorphism of $S(\mathcal{B}, G)$ into G. The mapping $\mu \to h$ is a homomorphic one and maps $M(\mathcal{B}, C)$ into $\mathrm{Hom}\big(S(\mathcal{B}, G), G\big)$.*

Proof. Using standard arguments (see [3]) one can easily prove that for every set of disjoint elements $a_g \in \mathcal{B}$, $g \in G$, we have $\mu\big(\bigcup_{g \in G} a_g\big) = \mu(a_{g_1}) + \ldots + \mu(a_{g_n})$ for some $g_1, \ldots, g_n \in G$.

If x, y are elements of $S(\mathcal{B}, G)$ and $z = x+y$, then

$$\begin{aligned} h(x+y) = h(z) &= \sum_{g} \mu\big(z(g)\big)\cdot g \\ &= \sum_{g} \mu\big[\bigcup_{g'} x(g') \cap y(g-g')\big]\cdot g \\ &= \sum_{g}\sum_{g'} \mu[x(g') \cap y(g-g')]\cdot g' + \sum_{g}\sum_{g'} \mu[x(g') \cap y(g-g')]\cdot(g-g') \\ &= \sum_{g'}\sum_{g} \mu[x(g') \cap y(g-g')]\cdot g' + \sum_{g''}\sum_{g'} \mu[x(g') \cap y(g'')]\cdot g'' \\ &= \sum_{g'} \mu\big(x(g')\big)\cdot g' + \sum_{g''} \mu\big(y(g'')\big)\cdot g'' \\ &= h(x) + h(y)\,. \end{aligned}$$

All the sums are finite and the operations that have been performed are admissible.

THEOREM 11. *If $\mathfrak{B}$ is σ-additive Boolean algebra and μ is a F-measure defined on $\mathfrak{B}$ with values in the group H, then defining the function h on $S(\mathfrak{B}, C)$ by the equality $h(x) = \sum_{n=-\infty}^{+\infty} n\cdot\mu(x(n))$ we get the homomorphism of $S(\mathfrak{B}, C)$ into H. The mapping $\mu\to h$ is homomorphic one and maps $M(\mathfrak{B}, H)$ into* $\mathrm{Hom}(S(\mathfrak{B}, C), H)$.

The proof of Theorem 11 is similar to that of Theorem 10.

The only essential property of the group of rational integers used in Theorems 10 and 11 is that there exists a bilinear mapping of $C\times G$ into G (resp. $C\times H$ into H); hence both theorems may be appropriately generalized.

The homomorphisms that we have got in Theorems 10 and 11 starting with a given F-measure are, in fact, integrals (with respect to this measure) in $\mathfrak{B}$ in the sense of Sikorski's paper [10]. Sikorski has considered σ-measures with real values and has defined the integral for a homomorphism of the σ-algebra of Borel sets on the real line into $\mathfrak{B}$, but this definition may be adopted to in the situation presented. The generalization of the notion of real function on $\mathfrak{B}$ which is used in [10] is similar to that discussed at the beginning of our paper.

By Theorem 9 we get the natural homomorphic mapping of the group $\mathrm{Hom}(S(\mathfrak{B}, G), H)$ in the group $M(\mathfrak{B}, H)$; this mapping is, in general, not isomorphic. In order to have more precise information about that mapping we need some lemma concerning the uniqueness of extending the homomorphism defined on the subgroup $S(\mathcal{J})$ to the homomorphism defined on $S(\mathcal{J}_\beta)$.

LEMMA 10. *If $\mathcal{J}$ is a finitely additive ideal in a σ-additive Boolean algebra $\mathfrak{B}$, an h is a homomorphism of $S(\mathfrak{B}, C)$ into a slender group H and $h(x_a) = 0$ for $a\in\mathcal{J}$, then $h(x) = 0$ for $x\in S(\mathcal{J}_\beta)$.*

Proof. If x is an arbitrary element of $S(\mathcal{J}_\beta)$, then $\nu(x) = \bigcup_{n\neq 0} x(n)\in\mathcal{J}_\beta$ and $\nu(x)$ may be represented as a join of disjoint elements $a_m\in\mathcal{J}$: $\nu(x) = \bigcup_{m=1}^{\infty} a_m$ in such a manner that each a_m is contained in one of the $x(n)$. By Lemma 9, there exists an isomorphism that maps $S^*(a_1, a_2, \ldots)$ onto $S^* = \sum_{n=1}^{\infty}{}^*\{e_n\}$, $\psi(x_{a_n}) = e_n$. By an equality $\bar h\psi(x) = h(x)$ we define a homomorphism $\bar h$ of S^* into H such that $\bar h(e_n) = 0$ for $n = 1, 2, \ldots$ By (16) we have $\bar h = 0$ and then $h(x) = 0$.

THEOREM 12. *If $\mathfrak{B}$ is a σ-additive Boolean algebra and H is a slender group and for arbitrary a of $\mathfrak{B}$ x_a is an element of $S(\mathfrak{B}, C)$ defined by the equalities*

$$x_a(1) = a\,,\quad x_a(0) = a'\,,\quad x_a(n) = 0\quad \text{for}\quad n\neq 0, 1\,,$$

then the mapping of the group $\mathrm{Hom}(S(\mathfrak{B}, C), H)$ *onto* $M(\mathfrak{B}, H)$: $h\to\mu$ *defined as follows*

$$\mu(a) = h(x_a)$$

is isomorphic, and the reciprocal mapping: $\mu\to h$ is the following one

$$h(x) = \sum_{n=-\infty}^{+\infty} n\cdot\mu(x(n))\,.$$

Proof. At first we shall prove the mapping $h\to\mu$ to be isomorphic. It is obviously homomorphic; let us suppose that $h(x_a) = 0$ for all $a\in\mathfrak{B}$. Then by Lemma 10 (taking $\mathcal{J} = \mathcal{J}_\beta = \mathfrak{B}$), we have $h(x) = 0$ for all $x\in S(\mathfrak{B}, C)$ and $h = 0$. Let μ be an arbitrary F-measure of $M(\mathfrak{B}, H)$ and consider the homomorphism h defined above; then evidently $h(x_a) = \mu(a)$ and μ is the image of h.

Let $\mathcal{J}$ be an arbitrary finitely additive ideal in $\mathfrak{B}$, H a slender group and let us consider the subgroup of $\mathrm{Hom}(S(\mathfrak{B}, C), H)$ consisting of all homomorphisms h such that $h(x_a) = 0$ for all $a\in\mathcal{J}$. By Lemma 10 all these homomorphisms are 0 on the group $S(\mathcal{J}_\beta)$. By Theorems 12 and 5 the following theorem holds:

THEOREM 13. *If $\mathfrak{B}$ is a σ-additive Boolean algebra, $\mathcal{J}$ is a finitely additive ideal in $\mathfrak{B}$ and H is a slender group, then the subgroup consisting of all homomorphisms $h\in\mathrm{Hom}(S(\mathfrak{B}, C), H)$ such that $h(x_a) = 0$ for all $a\in\mathcal{J}$ is isomorphic with the group $M(\mathfrak{B}/\mathcal{J}_\beta, H)$.*

§ 9. Groups $S(\mathfrak{B}, G)$ with distributive $\mathfrak{B}$. For complete direct sums the following "commutativity" holds:

$$\Big(\sum_{t\in T}{}^*G_t\Big)^{\mathfrak{m}}\approx\sum_{t\in T}{}^*(G_t)^{\mathfrak{m}}$$

or, which is equivalent,

$$S\Big(2^V, \sum_{t\in T}{}^*G_t\Big)\approx\sum_{t\in T}{}^*S(2^V, G_t)$$

for an arbitrary set V. This last relation can be generalized to groups $S(\mathfrak{B}, G)$, but we must impose some additional assumptions on the algebra $\mathfrak{B}$.

DEFINITION. The $\mathfrak{m}$-additive Boolean algebra $\mathfrak{B}$ is said to be $\mathfrak{m}$-*distributive* if the relation

$$\bigcap_{t\in T}\bigcup_{u\in U_t} a_{t,u} = \bigcup_{f\in F}\bigcap_{t\in T} a_{t,f(t)}$$

(F being the set of all functions f defined on T with values $f(t)$ in U_t) holds for arbitrary non-empty sets T, U_t ($t\in T$) satisfying $\sum_{t\in T}\overline{\overline{U}}_t\leqslant\mathfrak{m}$, and for arbitrary elements $a_{t,u}\in\mathfrak{B}$.

THEOREM 14. *Let* $\{G_t\}$, $t\in T$, *be a family of groups*, $S^*=\sum_{t\in T}^{*}G_t$, $\mathfrak{n}=\sum_{t\in T}\overline{\overline{G}}_t$ *and let* $\mathcal{B}$ *be a* $\overline{\overline{S}}{}^*$*-additive and* $\mathfrak{n}$*-distributive Boolean algebra. Then*

$$S\Big(\mathcal{B},\sum_{t\in T}^{*}G_t\Big)\approx\sum_{t\in T}^{*}S(\mathcal{B},G_t)\,.$$

Proof. We define an isomorphic mapping from left to right as follows: let us write

$$S_{t_0}^{*}=\sum_{t\in T\setminus(t_0)}^{*}G_t$$

and for $x\in S(\mathcal{B},S^*)$

$$\varphi(x)=\langle x_t\rangle\quad\text{ and }\quad x_t(g_t)=\bigcup_{g\in S_t^*}x(g_t+g)\,.$$

We shall prove four properties of the mapping φ just defined.

1. $x_t\in S(\mathcal{B},G_t)$. In fact, for each pair of different elements $g_t,g_t'\in G$ we have $x(g_t+g)\frown x(g_t'+g')=0$ for $g,g'\in S_t^*$ and moreover

$$\bigcup_{g_t\in G_t}x_t(g_t)=\bigcup_{g\in S^*}x(g)=e\,.$$

2. $\varphi(x+y)=\varphi(x)+\varphi(y)$ *for each pair* $x,y\in S(\mathcal{B},S^*)$. Let us write $z=x+y$, $\varphi(z)=\langle z_t\rangle$, $\varphi(x)=\langle x_t\rangle$, $\varphi(y)=\langle y_t\rangle$; then

$$\begin{aligned}z_t(g_t)&=\bigcup_{g\in S_t^*}z(g_t+g)=\bigcup_{g\in S_t^*}\ \bigcup_{g'\in S^*}x(g')\frown y(g_t+g-g')\\&=\bigcup_{g'\in S^*}x(g')\frown\bigcup_{g\in S_t^*}y(g_t+g-g')=\bigcup_{g'\in S^*}x(g')\frown y_t(g_t-g_t')\\&=\bigcup_{g_t'\in G_t}\ \bigcup_{g'\in S_t^*}x(g_t'+g')\frown y_t(g_t-g_t')\\&=\bigcup_{g_t'\in G_t}x_t(g_t')\frown y_t(g_t-g_t')=(x_t+y_t)(g_t);\end{aligned}$$

thus $z_t=x_t+y_t$ and $\varphi(z)=\varphi(x)+\varphi(y)$.

3. $x\neq 0$ *implies* $\varphi(x)\neq 0$. In fact, if $x\neq 0$ then there exists such a $g\in S^*$, $g\neq 0$ that $x(g)\neq 0$; moreover, for some $t_0\in T$ we have $g_{t_0}\neq 0$ and then $x_{t_0}(g_{t_0})\supset x(g)\neq 0$ and $\varphi(x)=\langle x_t\rangle\neq 0$.

4. Let $x_t\in S(\mathcal{B},G_t)$ for all $t\in T$ and consider an element $x\in S(\mathcal{B},S^*)$ defined by

$$x(g)=\bigcap_{t\in T}x_t(g_t)\quad\text{ for each }\quad g=\langle g_t\rangle\in S^*\,.$$

The elements $x(g)$ are obviously disjoint and

$$\bigcup_{g\in S^*}x(g)=\bigcup_{g\in S^*}\ \bigcap_{t\in T}x_t(g_t)=\bigcap_{t\in T}\ \bigcup_{g_t\in G_t}x_t(g_t)=e$$

by the $\mathfrak{n}$-distributivity of $\mathcal{B}$. Moreover, for $y=\varphi(x)$ we have

$$\begin{aligned}y_{t_0}(g_{t_0})&=\bigcup_{g\in S_{t_0}^*}x(g_{t_0}+g)\\&=\bigcup_{g\in S_{t_0}^*}\ \bigcup_{t\in T\setminus(t_0)}x_t(g_t)\frown x_{t_0}(g_{t_0})\\&=x_{t_0}(g_{t_0})\frown\bigcup_{g\in S_{t_0}^*}\ \bigcap_{t\in T\setminus(t_0)}x_t(g_t)\\&=x_{t_0}(g_{t_0})\frown\bigcup_{t\in T\setminus(t_0)}\ \bigcup_{g_t\in G_t}x_t(g_t)=x_{t_0}(g_{t_0});\end{aligned}$$

thus $y_{t_0}=x_{t_0}$ for an arbitrarily chosen $t_0\in T$.

All 1-4 imply φ to be an isomorphic mapping.

It is known that the complete direct sum of algebraically compact groups is also such a group. We generalize this theorem to groups $S(\mathcal{B},G)$. At first we shall prove a very special case of that theorem.

LEMMA 11. *Let* G *be an infinite algebraically compact group with* $p^\infty\overline{\overline{G}}=\{0\}$, $\overline{\overline{G}}=\aleph_\alpha$, $\operatorname{cf}\alpha>0$, *and let* $\mathcal{B}$ *be a* $\overline{\overline{G}}$*-additive and* $\overline{\overline{G}}$*-distributive Boolean algebra; then* $S(\mathcal{B},G)$ *is also algebraically compact group.*

Proof. Let us write $S=S(\mathcal{B},G)$; since $p^\infty G=\{0\}$, then by Lemma 1 it follows that $p^\infty S=\{0\}$ and the p-adic topology of S satisfies separation axioms.

Let $\{x_n\}$ be a sequence of elements of S that satisfy the Cauchy condition:

(17) *for each* k, *there exists an* $N(k)$ *such that for each* $n,m\geqslant N(k)$ $x_n-x_m\in p^kS$ *holds.*

If $\bigcap_{i=1}^{\infty}x_i(g_i)\neq 0$ then

(18) $$g_n-g_m\in p^kG\quad\text{ for all }\quad n,m\geqslant N(k)$$

and the sequence $\{g_i\}$ converges. In fact, we have the inclusions

$$\begin{aligned}\bigcap_{i=1}^{\infty}x_i(g_i)&\subset x_n(g_n)\frown x_m(g_m)\\&\subset\bigcup_{g'}x_n(g')\frown x_m(g_m-g_n+g')=(x_n-x_m)(g_n-g_m)\,,\end{aligned}$$

and by our assumptions, Lemma 1 and (17) we have $g_n-g_m\in p^kG$.

We shall prove that the sequence $\{x_n\}$ converges in the p-adic topolcgy of S to the element x defined by the equality

(19) $$x(g)=\bigcup\bigcap_{i=1}^{\infty}x_i(g_i)\,,$$

the join being taken ever all sequences $\{g_i\}$ ($g_i\in G$) convergent to g (by our assumption $\operatorname{cf}\alpha>0$ and we have $\overline{\overline{G}}{}^{\aleph_0}=\overline{\overline{G}}$ summands).

At first we prove $x \in S$; all $x(g)$ are obviously disjoint. Using the distributivity of $\mathcal{B}$ we get

$$\bigcup_g x(g) = \bigcup \bigcap_{i=1}^{\infty} x_i(g_i) = \bigcup_{\{g_i\}} \bigcap_{i=1}^{\infty} x_i(g_i) = \bigcap_{i=1}^{\infty} \bigcup_g x_i(g) = e ,$$

the join in the second term being taken over all convergent sequences $\{g_i\}$, that in the third one over all sequences $\{g_i\}$.

We shall consider the difference $x_j - x$ for $j \geqslant N(k)$. By the definition of x we have the relation

$$(x_j - x)(g) = \bigcup_{g'} x_j(g+g') \cap x(g') = \bigcup \bigcap_{i=1}^{\infty} x_i(g_i) ,$$

the join in the last term being taken over all convergent sequences $\{g_i\}$ such that $g_j = g + \lim_i g_i$. Let us suppose that $(x_j - x)(g) \neq 0$; then for some sequence $\{g_i\}$ with the above properties we have $\bigcap_{i=1}^{\infty} x_i(g_i) \neq 0$, and by (18) for $m \geqslant N(k)$ we have $\lim_i g_i - g_m \in p^k G$ and $g_j - g_m = (g + \lim_i g_i) - g_m \in p^k G$; consequently $g \in p^k G$ and by Lemma 1, $x_j - x \in p^k S$.

THEOREM 15. *Let G be an algebraically compact group and $\mathcal{B}$ a $\overline{\overline{G}}$-additive and $\overline{\overline{G}}$-distributive Boolean algebra. Then $S(\mathcal{B}, G)$ is an algebraically compact group.*

Proof. By Lemma 1, if G_0 is a divisible part of G, then $S(\mathcal{B}, G_0)$ is divisible and Theorem 14 implies $S(\mathcal{B}, G) = S(\mathcal{B}, G_0) + \sum_{p \in P}^{*} S(\mathcal{B}, G_p)$, G_p being algebraically compact with $p^{\infty} G_p = \{0\}$. Consequently, it is sufficient to consider only the groups $S(\mathcal{B}, G_p)$.

The group G_p may be represented as

$$G_p = \operatorname{compl}(I_p^{\mathfrak{m}_0}) + \operatorname{compl} \sum_{n=1}^{\infty} (C_{p^n})^{\mathfrak{m}_n}$$

for some cardinals $\mathfrak{m}_0, \mathfrak{m}_1, \ldots$ Let us write $\mathfrak{s}_n = \sum_{k=n}^{\infty} \mathfrak{m}_k$; then we have $\mathfrak{s}_1 \geqslant \mathfrak{s}_2 \geqslant \ldots$ and there exist such a cardinal $\mathfrak{s}$ and an index N that $\mathfrak{s}_n = \mathfrak{s}$ for $n \geqslant N$. It is easy to see that the group $\sum_{n=N}^{\infty} (C_{p^n})^{\mathfrak{m}_n}$ contains the subgroup $G_p' = \sum_{n=N}^{\infty} (C_{p^n})^{\mathfrak{s}}$ and the cardinality of $\operatorname{compl} G_p'$ is $\mathfrak{s}^{\aleph_0}$ which is not confinal with $\aleph_0$ (by (14)). Finally the group G_p can be represented as

$$G_p = \sum_{n=1}^{N-1} (C_{p^n})^{\mathfrak{m}_n} + G_p''$$

and $\overline{\overline{G_p''}}$ is not confinal with $\aleph_0$ by Lemma 8 and our previous remarks.

The group $S(\mathcal{B}, \sum_{n=1}^{N-1} (C_{p^n})^{\mathfrak{m}_n})$ is of bounded order and thus it is algebraically compact and the group $S(\mathcal{B}, G_p'')$ is algebraically compact by Lemma 11, consequently the group $S(\mathcal{B}, G_p)$ is also an algebraically compact group.

Remark. It is easy to construct the group $S(\mathcal{B}, G)$ with finite G and $S(\mathcal{B}, G)$ admitting no compact topology. In fact let $\mathcal{B}$ be any field of sets of cardinality $\overline{\overline{\mathcal{B}}} = \aleph_0$ and let G be any non-trivial finite group. Then the group $S(\mathcal{B}, G)$ is of cardinality $\aleph_0$ and hence admits no compact topology.

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