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Semigroups on trees *

by

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Introduction. We consider here a special case of the test question: Given a continuum S, does it support the structure of a topological semigroup with zero and unit? In the case that S is one-dimensional it is known [1] that a necessary condition on S is that it be a generalized tree, i.e.: arc-wise connected, hereditarily unicoherent, and satisfy a certain arc-convergence property, namely, that for some point $0 \in S$ (necessarily a point of local connectivity) and for any net $\{p_a\}$ with $\{p_a\} \to p$, that $[0, p_a] \to [0, p]$. (See also [2].) It is conjectural that any one-dimensional generalized tree supports the desired structure. As a step in this direction we show here that any metric tree S whose endpoints I form a compact set admits such a structure. In fact we establish a stronger conclusion:

THEOREM. *I can be ordered so that* $\min(x, y)$ *is continuous for* $x, y \in I$, *and multiplication in* S *can be introduced so that* S *is realized as the continuous homomorphic image of the "fan" over* I, *i.e. the semigroup formed from* $I \times \{01\}$ *by shrinking* $I \times \{0\}$ *to a point* (here $\{01\}$ denotes the unit interval of real numbers provided with any continuous associative multiplication in which 0 acts as a zero and 1 acts as a unit).

Set-theoretic preliminaries. Throughout the paper S will denote a metric tree, or acyclic locally connected compactum. For equivalent formulations see [4], p. 88. The set of endpoints of S will be noted by I, and we assume that I is compact. The unique arc from p to q will be written $[p, q]$. We denote the boundary of A by $F(A)$, the complement of B in A by $A \setminus B$, the closure of A by A^*, and the empty set by $\square$.

We will make use of the fact ([4], p. 99) that a metric tree is a regular curve, i.e. about each point there is a small neighborhood with finite boundary. Also ([4], p. 89) the set of branch points of S, i.e. cutpoints of order > 2, is countable. We fix an element 0 of $S \setminus I$.

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LEMMA 1. *Let $x \in I$ and let U be an open set about x with $F(U)$ finite: then there is a non-branch point p in $[0, x]$ such that $C_x(S\backslash p)$, the component of $S\backslash p$ which contains x, is contained in U.*

Proof. Denote the boundary points of U by $b_1, \ldots, b_k$. Let V be an open connected set with $x \in V \subset U$ and $0 \notin V$. Let p_1 be a branch point in $[0, x] \cap V$; note that if there is no such p_1, then for any $p \in [0, x] \cap V$, $C_x(S\backslash p) \subset U$. There is a boundary point, say b_1, in $[0, p_1]$. If $C_x(S\backslash p_1) \not\subset U$, then we may assume there is abranch point p_2 on $(p_1, x]$. Let $b_2 \in F(U) \cap C_x(S\backslash p_1)$ and note that $b_2 \neq b_1$. After k steps, the boundary points have been exhausted, and we let $p \in (p_k, x]$ be a non-branch point. We then have $C_x(S\backslash p) \subset U$ and the proof is complete.

LEMMA 2. *Let U be an open set about I; then there is an open set N with $I \subset N \subset N^* \subset U$, $F(N)$ finite, $F(N)$ contains no branch points, $S\backslash N$ is connected, and for $b_1, b_2 \in F(N)$, $b_1 \notin [0, b_2]$.*

Proof. Let $x \in I$. Since S is a regular curve, there is an open set V about x with $V^* \subset U$ and $F(V)$ finite. By Lemma 1 there is a non-branch point p in $[0, x]$ with $C_x(S\backslash p) \subset V$. Let $N(x) = C_x(S\backslash p)$, and cover I by the sets $N(x)$. Then $I \subset \bigcup_{i=1}^{n} N(x_i)$, a finite union which is irredundant in the sense that no $N(x_i)$ is contained in the union of the others. Note that $N(x_i) \cap N(x_j) = \square$ if $i \neq j$. Let $N = \bigcup_{i=1}^{n} N(x_i)$; then $I \subset N \subset N^* \subset U$, and $F(N)$ is finite and consists of non-branch points. Also $S\backslash N = \bigcap_{i=1}^{n} [S\backslash N(x_i)]$ is a continuum. From the irredundancy it follows that for $b_1, b_2 \in F(N)$, $b_1 \notin [0, b_2]$.

The next two lemmas are easy consequences of Lemma 2 and the fact that S is a regular curve; the proofs are omitted.

LEMMA 3. *Let U be an open set about I; then $S\backslash U$ has only finitely many branch points.*

LEMMA 4. *Each branch point of S has finite order.*

LEMMA 5. *Let $f\colon S \times S \to S$ be defined by: $p, q \in S \to [0, p] \cap [0, q] = [0, f(p, q)]$; then f is continuous.*

Proof. Let $p, q \in S$ and let W be an open set about $f(p, q)$. Suppose that $f(p, q)$ separates p from q. Let U be the component of $S\backslash f(p, q)$ which contains p, and let V be the component of $S\backslash f(p, q)$ which contains q. Then U and V are open and arc-wise connected. Let $p' \in U$, $q' \in V$; then $[p, p'] \subset U$, $[q, q'] \subset V$, and it follows that $[0, p'] \cap [0, q'] \supset [0, f(p, q)]$. But if $t \in [0, p'] \cap [0, q']$, it follows that $t \in [0, f(p, q)]$; we conclude that $f(p', q') = f(p, q)$, so that $f(U \times V) = f(p, q)$. Hence we may suppose that $f(p, q)$ does not separate p from q, i.e. we may assume that $f(p, q) = q$, so that either $q = 0$ or q separates 0 from p. Let U and V be disjoint open connected sets containing p and q respectively, with $V \subset W$, and let $p' \in U$, $q' \in V$. It follows that $f(p', q') = f(q, q') \in V$, so that $f(U \times V) \subset V$ and f is continuous.

Remetrization of S.

LEMMA 6. *Let ϱ be the given metric on S; then there is an equivalent metric d satisfying*

(i) *if p separates 0 from q then $d(0, p) < d(0, q)$ and*

(ii) *$d(0, x) = 1$ for each $x \in I$.*

Proof. Let V_1 be an open set containing I satisfying the conditions of Lemma 2, with $0 \notin V^*$. Let $F(V) = y_1, \ldots, y_n$. We map $[0, y_i]$ by a homeomorphism h_i onto the real interval $[0, 1-2^{-1}]$ with $h_i(0) = 0$, $i = 1, 2, \ldots, n$. We may further modify the h_i's so that (keeping the original notation) h_i agrees with h_j on $[0, y_i] \cap [0, y_j]$, $1 \leqslant i, j \leqslant n$. Let $m = \max[\varrho(y_1), I]$ and cover I by ϱ-spheres of radius $< m/2$ each of which is contained in V_1. Inside the resulting open set about I there is an open set V_2 satisfying the conditions of Lemma 2. Let $F(V_2) = z_1, \ldots, z_r$. We can find homeomorphisms k_i of $[0, f_i]$ onto the interval $[0, 1-2^{-2}]$ satisfying (i) k_i agrees with h_l on $[0, y_l] \cap [0, f_i]$, (ii) k_i agrees with k_j on $[0, z_i] \cap [0, z_j]$, $1 \leqslant i, j, k, l \leqslant r$. We iterate this procedure to obtain open sets V_i containing I, $F(V_i) = B_i$, $\bigcap_{i=1}^{\infty} V_i = I$, and corresponding finite sets of homeomorphisms H_i, where $H_i = \{a_b\colon a_b$ is a homeomorphism of $[0, b]$ onto $[0, 1-2^{-i}]$, $b \in B_i\}$ satisfying

(1) $b, b' \in B_i \to a_b$ agrees with a_b on $[0, b] \cap [0, b']$,

(2) $c \in B_{i-1} \to a_b$ agrees with a_c on $[0, b] \cap [0, c]$.

Let $D_i = \bigcup_{b \in B_i} [0, b]$ and let $D = \bigcup_{i=1} D_i$. It is easy to see that $D = S\backslash I$. Now fix $x \in I$ and define $a_x\colon [0, x] \to [0, 1]$ as follows:

Let $t \in [0, x]$; if $t = x$ define $a_x(t) = 1$. If $t \neq x$, then $t \in D$ so there is an integer i and $b \in B_i$ with $t \in [0, b]$. Define $a_x(t) = a_b(t)$. It follows from (i) and (ii) above that a_x is well defined and one to one. It is easy to see that a_x is continuous, hence a_x is a homeomorphism of $[0, x]$ onto $[0, 1]$. It is also clear that for $x, y \in I$, a_x agrees with a_y on $[0, x] \cap [0, y]$.

Define $d\colon S \times S \to [0, 2]$ as follows: Let $p, q \in S$; there exist $x, y \in I$ with $p \in [0, x]$, $q \in [0, y]$. Define

$$d(p, q) = \begin{cases} |a_x(p) - a_x f(p, q)| + |a_y(q) - a_y f(p, q)| & \text{if } f(p, q) \text{ separates } p \text{ from } q, \\ |a_x(p) - a_x(q)| & \text{if } f(p, q) \text{ does not separate } p \text{ from } q. \end{cases}$$

It can be readily shown that d is a metric, and we note the property that for $p, q \in S$, $d(p, q) = d(p, f(p, q)) + d(f(p, q), q)$. To show that d is equivalent to ϱ we will show that $\varrho(p_n, p)\to 0$ implies $d(p_n, p)\to 0$. If $p \in I$, then using Lemma 3 there is a subsequence p_{n_k} of p_n, and $x \in I$, such that (p_{n_k}) and p lie on $[0, x]$. It is then immediate that $d(p_n, p)\to 0$. Hence we may assume that $p \notin I$. Given $\varepsilon > 0$ there is an integer k with $2^{-k} < \frac{1}{2}\varepsilon$, and an integer n_0 such that $n \geqslant n_0$ implies $p_n \in V_k$ and $f(p_n, p) \in V$ (the latter by the continuity of f). Then $d(p_n, p) = d(p_n, f(p_n, p)) + d(f(p_n, p), p) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$, so that $d(p_n, p)\to 0$. From the compactness of S it follows that d is equivalent to ϱ. The remaining properties claimed for d are immediate from the construction.

The ordering on I. Let $B = \{0\}$ together with the set of branch points of S, and let $b \in B$: By Lemma 4, $S\backslash b$ has finitely many components $C_i(b)$ which do not contain 0. Let $I_i(b) = I \cap C_i(b)^*$ (the set of endpoints of the tree $C_i(b)^*$ excluding b). Define a relation $R(b)$ on $\bigcup_i I_i(b)$ by: $(x, y) \in R(b)$ iff [1] $x \in I_i(b)$, $y \in I_j(b)$ and $i < j$. We note that $R(b)$ is transitive and satisfies the property $(x, y) \in R(b) \to (y, x) \notin R(b)$. Now let $R = \bigcup_{b\in B} R(b) \cup \Delta$ where Δ is the diagonal of $I\times I$. To see that R is an ordering, let $x, y \in I$ with $x \neq y$ and we show either $(x, y) \in R$ or $(y, x) \in R$. Let $f(x, y) = b$ where f is the function of Lemma 5; then $x \in I_i(b)$, $y \in I_j(b)$ with $i \neq j$. Hence either $(x, y) \in R(b) \subset R$ or $(y, x) \in R(b) \subset R$. Further, it is easily seen that $(x, y) \in R \to (y, x) \notin R$. To show R is transitive, suppose $x, y, z \in I$ with $x \leqslant y \leqslant z$. We may assume $x < y < z$, and we consider $[0, x] \cup [0, y]$. If $f(x, y)$ separates 0 from $f(y, z)$, then y and z belong to the same $C_i f(x, y)$, so that $(x, z) \in R(f(x, y)) \subset R$. If $f(y, z)$ separates x from $f(x, y)$, then x and z belong to the same $C_i(f(x, y))$, so that $(z, y) \in R(f(x, y)) \subset R$, hence $(y, z) \notin R$, a contradiction. If $f(y, z)$ separates 0 from y, and $f(y, z) = f(x, y)$, then x, y, and z belong to distinct $C_i f(x, y)$'s, and hence x, y, z are ordered transitively. Finally, if $f(y, z) \in [0, f(x, y))$, then x and y belong to the same $C_i f(y, z)$ so since $(y, z) \in R(f(y, z))$ it follows that $(x, z) \in R(f(y, z)) \subset R$.

To show that R is closed in $I\times I$, let $(x, y) \in I\times I\backslash R$, and let U and V be disjoint connected open sets about x and y respectively. Then either $x' \in U \cap I, y' \in V \cap I \to (x', y') \in R$ (as determined by the branch point $f(x, y)$) or vice versa. In the first case $(U \cap I \times V \cap I) \cap R = \square$ and the other case is similar.

It now follows easily that the multiplication $m\colon I\times I\to I$ defined by $m(x, y) = \min(x, y)$ is continuous.

The multiplication in S. To each $t \in S$ we assign two coordinates $x(t)$ and $a(t)$, where $x(t) = \operatorname{Sup}\{y \in I\colon t \in [0, y]\}$ and $a(t) = d(0, t)$. For $p, q \in S$ we define $st = [\min(x(p), x(q)), a(p)\cdot a(q)]$, where Sup and min

[1] iff = if and only if.

refer to the ordering just introduced on I, and multiplication in $[0, 1]$ is any continuous associative multiplication in which 0 acts as a zero and 1 acts as a unit. The details of showing that this multiplication is continuous can be carried through, but it is more revealing to take a different point of view.

We define $g\colon I\times I\to[0 1]$ by $g(x, y) = d(0, f(x, y))$. Note that since f is continuous, so is g. Define $\varphi\colon I\times[0, 1]\to S$ by: $\varphi(x, a)$ is the element of S at d-distance a from 0 along the arc $[0, x]$, and let R be the equivalence relation determined by φ; i.e. $[(x, y), (y, b)] \in R$ iff $\varphi(x, a) = \varphi(y, b)$. We will write $(x, a)\sim(y, b)$ to denote $[(x, a), (y, b)] \in R$. It is easy to see that $(x, a)\sim(y, b)$ iff $a = b \leqslant g(x, y)$. We will show that R is a closed congruence, i.e. that R is closed in $(I\times[0, 1])\times(I\times[0, 1])$ and that $(x, a)\sim(y, b), (x', a')\sim(y', b')\to(xx', aa')\sim(yy', bb')$. If $(x, a)\sim(y, b)$ and $(x', a')\sim(y', b')$, then $a = b \leqslant g(x, y)$, $a' = b' \leqslant g(x', y')$, so $aa' = bb' \leqslant g(x, y)\cdot g(x', y') \leqslant \min[g(x, y), g(x', y')]$, the last inequality holding for *any* continuous associative multiplication on $[0, 1]$ in which 0 acts as a zero and 1 acts as a unit ([3], p. 128).

Hence it suffices to show that $\min[g(x, y), g(x', y')] \leqslant g(xx', yy')$. There are four cases.

Case 1. $x \leqslant x'$, $y \leqslant y'$.

Case 2. $x' \leqslant x$, $y' \leqslant y$.

Case 3. $x \leqslant x'$, $y' \leqslant y$.

Case 4. $x' \leqslant x$, $y \leqslant y'$.

The first two of these cases are clear, and Cases 3 and 4 are dual. We will establish Case 3, i.e. $\min[g(x, y), g(x', y')] \leqslant g(x, y')$. It will be convenient to consider the subspace $[0, x] \cup [0, y']$. We have several possibilities:

(1) $f(x, x') \in [0, f(x, y')]$.

Then $f(x, x') = f(x', y')$; now $f(x', y')$ and $f(x, y')$ lie on $[0, y']$, so $g(x', y') \leqslant g(x, y')$.

(2) $f(x, x') \in [f(x, y'), x]$; then $f(x', y') = f(x, y')$, so $g(x', y') = g(x, y')$. At this point we may assume that $f(x', y') \in [f(x, y'), y']$.

(3) $f(x, y) \in [0, f(x, y')]$; then as in (1) we conclude that $g(x, y) \leqslant g(x, y')$.

(4) $f(x, y) \in (f(x, y), x]$; then by the construction of the ordering, x and y compare in the same way with x' and y'. Since $x < x'$, it follows that $y < y'$, a contradiction.

(5) $f(x, y) \in (f(x, y'), y']$; then $f(x, y') = f(x, y)$, so $g(x, y') = g(x, y)$. This completes the argument that R is a congruence.

To show that R is closed, let $[(x, a), (y, b)] \in (I\times[0, 1])^2\backslash R$. It is then false that $a = b \leqslant g(x, y)$, so either (1) $a \neq b$ or (2) $a = b > g(x, y)$.

In case (1), there are open sets U about a and V about b with $U \cap V = \square$, so that $(I \times U) \times (I \times V)$ is open and misses R. In case (2), using the continuity of g, there are open sets U about a, N about x, and M about y with $g(x', y') < a'$ for each $x' \in N \cap I$, $y' \in M \cap I$, and $a' \in U$. Then $[(N \cap I) \times D] \times [(M \cap I) \times U]$ is open and misses R. Thus R is closed, and it follows that φ is continuous.

Since $I \times [0, 1]$ is a compact topological semigroup and R is a closed congruence, it follows that $I \times [0, 1]/R$ is again a compact topological semigroup. Denote the natural homomorphism by η. We have the diagram

$$I \times [0, 1] \xrightarrow{\eta} I \times [0, 1]/R \xrightarrow{\varphi^*} S, \qquad \underrightarrow{\quad\varphi\quad}$$

where φ^* is the function induced by φ. It follows from the continuity of φ that φ^* is continuous, and is 1-1 onto and hence a homeomorphism. We identify $I \times [0, 1]/R$ with S so that S is now a topological semigroup and is the continuous homomorphic image of $I \times [0, 1]$. Now let T be the relation on $I \times [0, 1]$ defined by $T = (I \times 0) \times (I \times 0) \cup \Delta$ where Δ is the diagonal of $(I \times [0, 1])^2$. Then T is a closed congruence, $I \times [0, 1]/T$ is a compact topological semigroup, and the natural mapping β is a continuous homorphism. Since $T \subset R$, there is induced a continuous homomorphism α as indicated:

$$I \times [0, 1] \xrightarrow{\beta} I \times [0, 1]/T \xrightarrow{\alpha} I \times [0, 1]/R = S, \qquad \underrightarrow{\quad\eta\quad}$$

Thus S has been realized as the continuous homomorphic image of $I \times [0, 1)/T$, which is the "fan" over I, i.e. the topological semigroup obtained from $I \times [0, 1]$ by shrinking $I \times \{0\}$ to a point.

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On groups of functions defined on Boolean algebras

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With an arbitrary set T and abelian group G one may connect the group of all functions x defined on T with values in G. These groups are known as complete direct sums and can be considered also as groups of functions defined on T with values in G that are measurable with respect to the complete Boolean algebra 2^T of all subsets of T. Let us write for an arbitrary function x: $T \to G$ and arbitrary $g \in G$

$$x(g) = \{t \in T;\ x(t) = g\}.$$

Then we obviously have

(1) $$x(g) \in 2^T \quad \text{for all} \quad g \in G,$$

(2) $$\bigcup_{g \in G} x(g) = T,$$

(3) $$x(g) \cap x(g') = 0 \quad \text{for} \quad g \neq g',$$

(4) $$(x+y)(g) = \bigcup_{g' \in G} x(g') \cap y(g-g').$$

Now we get a clear way for generalizations; let G be an arbitrary abelian group of cardinality $\overline{\overline{G}} = \mathfrak{m}$ and $\mathcal{B}$ a Boolean $\mathfrak{m}$-additive algebra with maximal element e. Elements of the group $S(\mathcal{B}, G)$ are functions x defined on G and satisfying the following conditions:

(5) $$x(g) \in \mathcal{B} \quad \text{for all} \quad g \in G,$$

(6) $$\bigcup_{g \in G} x(g) = e,$$

(7) $$x(g) \cap x(g') = 0 \quad \text{for} \quad g \neq g',$$

the sum $z = x+y$ of two such functions is defined by the equation

(8) $$z(g) = \bigcup_{g' \in G} x(g') \cap y(g-g') \quad \text{for} \quad g \in G.$$

Since $\mathcal{B}$ is $\mathfrak{m}$-additive, then the sum in (8) exists and elements $z(g)$ are well-defined and satisfy conditions (5)-(7).