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Independence in a certain class of abstract algebras

by

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1. In [5] E. Marczewski investigated a class of abstract algebras, called v-algebras, in which the notion of independence has the properties of linear independence. Later K. Urbanik [7] has given the representation theorem for v-algebras (called by him Marczewski's algebras). In this paper we shall investigate a wider class of algebras, namely v^*-algebras. Independence in this class of algebras has also the principal properties of linear independence. The investigation of v^*-algebras was suggested to me by Professor E. Marczewski.

For the terminology and notation used here see [4], [5], [6]. In particular we shall denote by $\boldsymbol{A}^{(n)}$ the set of all algebraic operations of n variables of a fixed algebra $\mathfrak{A}=(A,\boldsymbol{F})$ and by $\boldsymbol{A}^{(n,k)}$ the subset of $\boldsymbol{A}^{(n)}$ consisting of all operations depending on at most k variables. By $[a_1,\ldots,a_n]$ we shall denote the subalgebra of $\mathfrak{A}$ generated by the set $\{a_1,\ldots,a_n\}$.

2. We shall say that an algebra $\mathfrak{A}=(A,\boldsymbol{F})$ is a *v^*-algebra* if and only if it satisfies the following conditions:

(I) *If $a\in A$ and a is not an algebraic constant, then the set $\{a\}$ is a set of independent elements.*

(II) *If $\{a_1,\ldots,a_n\}$ is a set of independent elements, and $\{a_1,\ldots,a_n,a_{n+1}\}$ is not a set of independent elements, then $a_{n+1}\in[a_1,\ldots,a_n]$.*

Condition (I) may be treated as the degenerated case ($n=0$) of (II).

Evidently (see [5], p. 614) the v-algebras satisfy (I) and (II). On the other hand it is easy to construct a v^*-algebra which is not a v-algebra. Let G be a group of transformations of a set A. Let A_0 be the set of all fixed points of the transformations from G and let us assume that $g(A_0)\subset A_0$ for $g\in G$. Suppose, moreover, that there exists an $h\in G$, $h\neq e$ with at least two fixed points. Then the algebra $(A,\boldsymbol{F})$, where $\boldsymbol{F}$ is the family of all functions $g(x)$, $g\in G$, and all functions $f(x)=a$, $a\in A_0$, is a v^*-algebra and not a v-algebra.

From Theorem I below it follows that the simultaneous occurrence of conditions (I) and (II) is equivalent to the simultaneous occurrence of (I) and the following two conditions:

(III) *The set* $\{a_1, \ldots, a_n\}$ *in* A *is a set of independent elements if and only if none of the* a_i *belongs to the subalgebra generated by the other elements.*

(IV) *In every subalgebra with a basis consisting of* k *elements every* k *independent elements form a basis.*

The above observation is due to S. Świerczkowski.

(i) *If* $\mathfrak{A}$ *is a* v^**-algebra,* $\mathfrak{B}$ *is a subalgebra of* $\mathfrak{A}$, *then every set of independent elements in* $\mathfrak{B}$ *is also a set of independent elements in* $\mathfrak{A}$.

It is sufficient to prove this statement for finite sets only. Suppose that $\{a_1, \ldots, a_n\}$ is a set of independent elements in $\mathfrak{B}$; a_1 is independent in $\mathfrak{A}$, for otherwise, by (I), a_1 would be an algebraic constant in $\mathfrak{A}$, and *a fortiori* in $\mathfrak{B}$. Let k be the greatest integer $\leqslant n$ such that the set $\{a_1, \ldots, a_k\}$ is a set of independent elements in $\mathfrak{A}$. If $k \neq n$ then, by (II), $a_{k+1} \in [a_1, \ldots, a_k]$ and thus $\{a_1, \ldots, a_{k+1}\}$ would not be a set of independent elements in $\mathfrak{B}$. Thus $k = n$ and (i) is proved.

As an obvious consequence of (i) we infer that every subalgebra of a v^*-algebra is also a v^*-algebra.

From (II) it follows by an argument familiar in the theory of linear spaces (see e.g. [1], pp. 178-179, theorem 7) that:

(ii) *If there exists a set of* n *generators of a* v^**-algebra, then every set of independent elements contains at most* n *elements.*

The proofs of the following three statements are identical with the proofs of the corresponding statements for v-algebras (see [5], pp. 614-616), and so we omit them.

(iii) *If* N *is a set of* n *independent elements in a* v^**-algebra and* M *is a set of* $n+1$ *independent elements, then there is an* a *in* $M \setminus N$ *such that* $N \cup \{a\}$ *is a set of independent elements.*

We thus see that v^*-algebras satisfy the axioms of independence formulated by H. Whitney [8]. Consequently, v^*-algebras are "Abhängigkeitsräume" as defined in [3].

(iv) *If* $A \neq A^{(0)}$ *then the following statements are equivalent in a* v^**-algebra*: (a) B *is a basis*, (b) B *is a minimal set of generators*, (c) B *is a maximal set of independent elements.*

(v) *Any two bases in a* v^**-algebra have the same cardinal number.*

We can now define the *dimension* of any v^*-algebra $\mathfrak{A}$ ($\dim\mathfrak{A}$) as the cardinal number of any basis in $\mathfrak{A}$. When we treat v^*-algebras as sets with independence in the sense of [8], then the dimension coincides with the rank, and when we treat them as "Abhängigkeitsräume" then the dimension coincides with the "Rang".

As we have seen above, most of the known principal theorems of linear independence in vector spaces remain true in v^*-algebras with one exception—it is no longer true that the algebra of all algebraic operations of k variables of a v^*-algebra is also a v^*-algebra for all $k \geqslant 1$. This theorem remains true for infinite-dimensional v^*-algebras. For finite-dimensional v^*-algebras the algebra of all algebraic operations of k variables is also a v^*-algebra for $1 \leqslant k \leqslant \dim\mathfrak{A}$; this follows from the remark that the algebra of all algebraic operations of n variables (where n is the dimension of $\mathfrak{A}$) of the algebra $\mathfrak{A}$ is isomorphic to $\mathfrak{A}$ (see [2], theorem 1) and from (i).

3. We shall now give another characterization of v^*-algebras. Let $\mathfrak{A}$ be an algebra. By Δ_k we shall denote the set of ordered k-tuples of algebraic operations of k variables such that if $(f_1, \ldots, f_k) \in \Delta_k$ then no f_i is generated in the algebra of all algebraic operations of k variables of $\mathfrak{A}$ by the other operations f_j ($k > 1$). By Δ_1 we shall denote the set of non-constant algebraic operations of one variable in $\mathfrak{A}$. By ∇_k we shall denote the set of ordered k-tuples of elements of $\mathfrak{A}$ such that if $(a_1, \ldots, a_k) \in \nabla_k$ then no a_i is generated by the other elements ($k > 1$). By ∇_1 we shall denote the set of elements of $\mathfrak{A}$ which are not algebraic constants. Now we prove:

THEOREM I. *If the algebra* $\mathfrak{A}$ *is a* v^**-algebra, then:*

(*) Δ_k *is a group of transformations* $\nabla_k \to \nabla_k$ *whenever* ∇_k *is non-void, and* Δ_1 *is a group without fixed points.*

Conversely if an algebra $\mathfrak{A}$ *satisfies* (*), *then* ∇_k *is the set of all independent* k*-tuples of elements of* $\mathfrak{A}$, *and moreover,* $\mathfrak{A}$ *is a* v^**-algebra.*

Proof. Suppose that $\mathfrak{A}$ is a v^*-algebra and ∇_k is non-void. Then $k \leqslant \dim\mathfrak{A}$. Δ_k is now the set of independent ordered k-tuples of operations from the algebra $A^{(k)}$ and ∇_k is the set of independent ordered k-tuples of elements of $\mathfrak{A}$. At first we prove that Δ_k is a set of transformations $\nabla_k \to \nabla_k$. It suffices to prove that if $a \in \nabla_k$, $f \in \Delta_k$ then $f(a) \in \nabla_k$. Let $a = (a_1, \ldots, a_k)$, $f = (f_1, \ldots, f_k)$. If $f(a) \notin \nabla_k$ then for some j and $F \in A^{(k-1)}$:

$$\begin{aligned} & f_j(a_1, \ldots, a_k) \\ & = F\big(f_1(a_1, \ldots, a_k), \ldots, f_{j-1}(a_1, \ldots, a_k), f_{j+1}(a_1, \ldots, a_k), \ldots, f_k(a_1, \ldots, a_k)\big). \end{aligned}$$

From the independence of $\{a_1, \ldots, a_k\}$ it easily follows that $\{f_1, \ldots, f_k\}$ is a dependent system, contrary to our assumption. Now we prove that Δ_k is a group. Let $e = (x_1, \ldots, x_k)$. Evidently $e \in \Delta_k$, and, for every $f \in \Delta_k$, $ef = fe = f$.

(a) *For every* $f \in \Delta_k$ *there exists an* $f^{-1} \in \Delta_k$ *such that* $f^{-1}f = e$.

As we have seen in § 2, the algebra $A^{(k)}$ is a v^*-algebra (obviously k-dimensional) and so every set of k independent operations of k variables of $\mathfrak{A}$ generates $A^{(k)}$. Let $f = (f_1, \ldots, f_k)$. Then there exist operations $g_1, \ldots, g_k$ such that for $i = 1, \ldots, k$ we have $g_i(f_1, \ldots, f_k) = x_i$. If $(g_1, \ldots, g_k) \notin \Delta_k$ then, for some j and $F \in A^{(k-1)}$, $g_j = F(g_1, \ldots, g_{j-1}, g_{j+1}, \ldots, g_k)$. By putting here $x_i \to f_i$ we obtain $x_j = F(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_k)$, which is a contradiction. Thus $f^{-1} = (g_1, \ldots, g_k) \in \Delta_k$ and $f^{-1}f = e$.

(b) *For every* $f, g \in \Delta_k$, $fg \in \Delta_k$.

Let $a \in V_k$. Then $g(a) \in V_k$ and $fg(a) \in V_k$. Hence the set $\{f_i(g_1(a_1, \ldots, a_k), \ldots, g_k(a_1, \ldots, a_k))\}_{i=1}^{k}$ is a set of independent elements and this can happen only if $fg \in \Delta_k$.

From (a), (b) and the obvious remark that associativity holds in Δ_k we infer that Δ_k is a group. If for an $f \in \Delta_1$ and an $a \in V_1$ we have $f(a) = a$ then from (I) follows $f(x) = x$ for every x. The first part of the theorem is thus proved.

Suppose now that Δ_k are groups of transformations $V_k \to V_k$ whenever V_k is non-void and Δ_1 is without fixed points. It suffices to prove that $\mathfrak{A}$ is a v^*-algebra.

LEMMA. *If* $(a_1, \ldots, a_{k-1}, b) \in V_k$, $\{a_1, \ldots, a_k\}$ *is a set of independent elements*, $(a_1, \ldots, a_k, b) \notin V_{k+1}$ *then* $b \in [a_1, \ldots, a_k]$.

Proof. At first let $k = 1$. Then $a_1 \in V_1$, $(a_1, b) \notin V_2$. There exists an $f \in A^{(1)}$ such that $b = f(a_1)$; thus $b \in [a_1]$, or $a_1 = f(b)$. In the latter case if $f \notin \Delta_1$ then f is a constant, and so a_1 is an algebraic constant, contrary to our assumption. Thus $f \in \Delta_1$, and $b = f^{-1}(a_1)$.

Suppose that we have proved the lemma for every $k < N$ and V_N is non-void. Let our assumptions be satisfied. There exists an $f \in A^{(N)}$ such that $b = f(a_1, \ldots, a_N)$ and so $b \in [a_1, \ldots, a_N]$, or there exists a j such that $a_j = f(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_N)$. In the latter case let us define:

$$z = (x_1, \ldots, x_{j-1}, f(x_1, \ldots, x_N), x_{j+1}, \ldots, x_N)\,.$$

If $z \notin \Delta_N$ then there exist a j_0 and a $g \in A^{(N-1)}$ such that

$$x_{j_0} = g(x_1, \ldots, x_{j_0-1}, x_{j_0+1}, \ldots, x_{j-1}, f(x_1, \ldots, x_N), x_{j+1}, \ldots, x_N)$$

(for f depends on x_j and so the equality $f = h(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$ is impossible). If we now put $x_j = b$, $x_i = a_i$ $(i \neq j)$ we obtain $a_j = g(a_1, \ldots, a_{j_0-1}, a_{j_0+1}, \ldots, a_N)$ and this contradicts the independence of $(a_1, \ldots, a_N)$. Thus $z \in \Delta_N$. Let us define:

$$c = (a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_N)$$

If $c \in V_N$, then $z(c) = (a_1, \ldots, a_N)$; thus $z^{-1}(a_1, \ldots, a_N) = c$ and $b \in [a_1, \ldots, a_N]$. If $c \notin V_N$, then by applying the inductional hypothesis we easily obtain $b \in [a_1, \ldots, a_N]$. The lemma is thus proved.

Now we prove that (I) and (II) are satisfied. (I) follows from the fact that Δ_1 has no fixed points. (II) we prove by induction. Suppose at first that $n = 1$, i.e. a_1 is an independent element and $\{a_1, a_2\}$ is a pair of dependent elements. If $a_2 \notin V_1$, then obviously $a_2 \in [a_1]$. If $a_2 \in V_1$, but $(a_1, a_2) \notin V_2$, then $a_2 \in [a_1]$ or $a_1 \in [a_2]$. In the latter case $a_1 = f(a_2)$, where $f \in \Delta_1$, and so $a_2 = f^{-1}(a_1)$. It remains to consider the case $(a_1, a_2) \in \Delta_2$. There exist two different algebraic operations, $f(x, y)$ and $g(x, y)$, such that $f(a_1, a_2) = g(a_1, a_2) = b$. Let $z = (f, g)$. If we had $z \in \Delta_2$, then $z(a_1, a_2) = (b, b) \in V_2$, which is impossible. Thus $z \notin \Delta_2$ and we obtain $f(x, y) = G(g(x, y))$ (by interchanging if necessary the operations f, g). By putting here $x = a_1$, $y = a_2$ we obtain $b = G(b)$, whence b is an algebraic constant. Now let us observe that the pairs $(h(x, y), x)$ and $(h(x, y), y)$ do not belong to Δ_2 for any operation $h(x, y)$ such that $h(a_1, a_2) = b$ because $(h(x, y), x)(a_1, a_2) = (b, a_1) \notin V_2$, and analogously in the second case. So we see that $h(x, y)$ must be a constant. Consequently the operations $f(x, y)$ and $g(x, y)$ considered above are constant and so they are equal, contrary to our assumption.

Assume now that we have proved (II) for $n \leqslant N-1$ and let $\{a_1, \ldots, a_N\}$ be a set of independent elements and let $\{a_1, \ldots, a_{N+1}\}$ be a set of not independent elements. We shall prove that $a_{N+1} \in [a_1, \ldots, a_N]$. If $(a_1, \ldots, a_{N+1})$ does not belog to V_{N+1}, then this follows easily from the lemma. Suppose therefore that $(a_1, \ldots, a_{N+1}) \in V_{N+1}$. There exist two different algebraic operations $f(x_1, \ldots, x_{N+1})$, $g(x_1, \ldots, x_{N+1})$ such that $f(a_1, \ldots, a_{N+1}) = g(a_1, \ldots, a_{N+1}) = b$. Then $z = (f(x_1, \ldots, x_{N+1}), g(x_1, \ldots, x_{N+1}), x_{i_1}, \ldots, x_{i_{N+1}})$ does not belong to Δ_{N+1} for any set of indices $1 \leqslant i_1, \ldots, i_{N+1} \leqslant N+1$, for otherwise $z(a_1, \ldots, a_{N+1}) = (b, b, a_3, \ldots)$ would belong to V_{N+1} and this is evidently false.

Hence

$$f(x_1, \ldots, x_{N+1}) = H(g(x_1, \ldots, x_{N+1}), x_{i_1}, \ldots, x_{i_{N-1}})$$

or

$$g(x_1, \ldots, x_{N+1}) = H(f(x_1, \ldots, x_{N+1}), x_{i_1}, \ldots, x_{i_{N-1}})$$

or for some j

$$x_{i_j} = H(f(x_1, \ldots, x_{N+1}), g(x_1, \ldots, x_{N+1}), x_{i_1}, \ldots, x_{i_{j-1}}, x_{i_{j+1}}, \ldots, x_{i_{N-1}})$$

with some $H(x_1, \ldots, x_N)$. If for a set of indices $1 \leqslant i_1, \ldots, i_{N-1} \leqslant N$ the set $\{b, a_{i_1}, \ldots, a_{i_{N-1}}\}$ were a set of independent elements, then by putting $x_i = a_i$ in the equalities just established we would easily obtain a contradiction. Hence the set $\{b, a_{i_1}, \ldots, a_{i_{N-1}}\}$ is not a set of independent elements for any set of indices $1 \leqslant i_1, \ldots, i_{N-1} \leqslant N$. Remembering that the set $\{a_1, \ldots, a_N\}$ is a set of independent elements we infer from the inductive hypothesis that for every set of indices $1 \leqslant i_k \leqslant N$, $b \in [a_{i_1}, \ldots, a_{i_{N-1}}]$. Thus

$$b = f_1(a_2, \ldots, a_N) = f_2(a_1, a_3, \ldots, a_N) = \ldots = f_N(a_1, \ldots, a_{N-1})\,.$$

From the independence of $\{a_1, \ldots, a_N\}$ we easily infer that all the operations f_i should be constants and so b is an algebraic constant.

Now let h be any operation such that $h(a_1, \ldots, a_{N+1}) = b$. Then $(h, x_{i_1}, \ldots, x_{i_N})$ does not belong to Δ_{N+1} for any set of indices $1 \leqslant i_k \leqslant N+1$. Suppose that h depends on x_k. Then $(h, x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{N+1}) \notin \Delta_{N+1}$, consequently for some j and H $x_j = H\big(h(x_1, \ldots, x_{N+1}), x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{N+1}\big)$. By putting $x_i = a_i$ we obtain (considering that b is an algebraic constant): $a_j \in [a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{N+1}]$, contrary to the assumption that $(a_1, \ldots, a_{N+1}) \in V_{N+1}$. Hence h does not depend on x_k for any k and so h is a constant. But in the place of h we can put f and g, and so they are constant, whence they are equal and this contradicts the assumption that $\{a_1, \ldots, a_{N+1}\}$ is not a set of independent elements. This proves our theorem.

4. It is natural to ask for a representation theorem for v^*-algebras. It seems to be a difficult problem, for, as can easily be seen, if the classes of all n-ary algebraic operations are identical for $(A, \boldsymbol{F})$ and $(A, \boldsymbol{G})$ where $(A, \boldsymbol{F})$ is an n-dimensional v^*-algebra, then $(A, \boldsymbol{G})$ is also a v^*-algebra.

In this direction we obtained only the following, very special result:

THEOREM II. *If $\mathfrak{A}$ is an n-dimensional v^*-algebra, and $A^{(n)} = A^{(n,1)}$, then there exist a group G of transformations of the set A and a subset $A_0 \subset A$ composed of all fixed points of the transformations from G and such that $G(A_0) \subset A_0$, and moreover every algebraic operation of n variables is of the form:*

$$f(x_1, \ldots, x_n) = g(x_i) \quad \textit{for} \quad g \in G \ \textit{and} \ 1 \leqslant i \leqslant n$$

or

$$f(x_1, \ldots, x_n) = a \quad \textit{for} \quad a \in A_0 .$$

Proof. It suffices to prove that Δ_1 is a group of transformations $A \to A$ (and not only $V_1 \to V_1$). The associativity and the existence of the unit element are obvious. Now let $g = f^{-1}$ in Δ_1. Then $fg(x) = gf(x) = x$ in V_1 and from the fact that no element in V_1 forms a set of dependent elements it follows that $x = fg(x) = gf(x)$ in A. If we now define $A_0 = A \setminus \Delta_1$ we at once obtain the theorem.

We shall say that an n-dimensional algebra is minimal if its fundamental operations depend on at most n variables. The combination of theorems I and II gives thus a representation for all minimal algebras satisfying $A^{(n)} = A^{(n,1)}$ (where n is the dimension of the algebra).

5. In [3] O. Haupt, G. Nöbeling and C. Pauc define the direct sum of subsets $B_1, \ldots, B_k$ of an "Abhängigkeitsraum" as follows: $B = B_1 + B_2 + \ldots + B_k$ if and only if $B = \bigcup_{i=1}^{k} B_i$, $\dim B_i \neq 0$, and $\dim B = \sum_{i=1}^{k} \dim B_i$.

We shall write $\mathfrak{A} = \sum_{k=1}^{m} \mathfrak{A}_k$ and say that the algebra $\mathfrak{A}$ is decomposable if $m \geqslant 2$, $\mathfrak{A}_k = (A_k, \boldsymbol{F})$ are subalgebras of $\mathfrak{A}$, $\bigcup_{k=1}^{m} A_k = A$ and $\sum_{k=1}^{m} \dim \mathfrak{A}_k = \dim \mathfrak{A}$. It is easy to see that a v^*-algebra is decomposable if and only if it is a direct sum of subsets in the sense of [3]. Following [3] we shall say that $\mathfrak{A}$ is totally decomposable if there exists such a decomposition $\mathfrak{A} = \sum_{k=1}^{m} \mathfrak{A}_k$ that $\dim \mathfrak{A}_i = 1$ for $i = 1, \ldots, m$. We shall now prove

THEOREM III. *If $\mathfrak{A}$ is a v^*-algebra, then the following conditions are equivalent:*

(a) *$\mathfrak{A}$ is decomposable,*

(b) *$\mathfrak{A}$ is totally decomposable,*

(c) *$A^{(k)} = A^{(k,1)}$ for $k \leqslant \dim \mathfrak{A}$.*

(We thus see that Theorem II gives a representation for minimal decomposable v^*-algebras.)

Proof. The implication (b)$\to$(a) is trivial.

(a)$\to$(c). Suppose $\mathfrak{A} = \sum_{k=1}^{m} \mathfrak{A}_k$, $\dim \mathfrak{A} = n$, $\max_k \dim \mathfrak{A}_k = \dim \mathfrak{A}_1 = r$. Let $f(x_1, \ldots, x_n)$ depend actually on $x_1, \ldots, x_s$ $(s \leqslant n)$. Let $\{a_1, \ldots, a_n\}$ form a basis of $\mathfrak{A}$. Evidently in each algebra $\mathfrak{A}_k$ there are exactly $\dim A_k$ elements from that basis. Let $f(a_1, \ldots, a_n) = b$, and let $b \in \mathfrak{A}_{k_0}$, $\dim \mathfrak{A}_{k_0} = p$, $a_{i_1}, \ldots, a_{i_p} \in A_{k_0}$. There exists an operation $g(x_1, \ldots, x_p)$ such that $g(a_{i_1}, \ldots, a_{i_p}) = b$, because $a_{i_1}, \ldots, a_{i_p}$ form a basis of $\mathfrak{A}_{k_0}$. From the independence of the a_i-s it follows that $f(x_1, \ldots, x_n) = g(x_{i_1}, \ldots, x_{i_p})$ and so f depends only on p variables, and thus $s \leqslant p \leqslant r$. We can write $f(x_1, \ldots, x_n) = h(x_1, \ldots, x_s)$. We can always suppose that $a_1, \ldots, a_r \in A_1$, $a_{r+1} \notin A_1$. Suppose now that $s \neq 0, 1$. Let $h(a_1, \ldots, a_{s-1}, a_{r+1}) = c \in \mathfrak{A}_{k_1}$, $\dim \mathfrak{A}_{k_1} = q$, and $a_{r_1}, \ldots, a_{r_q} \in A_{k_1}$. There exists an $F(x_1, \ldots, x_q)$ such that $F(a_{r_1}, \ldots, a_{r_q}) = c = h(a_1, \ldots, a_{s-1}, a_{r+1})$ and from the independence of the a_i-s we infer that the number of variables on which h depends is not greater than the number of numbers from the sequence $1, 2, \ldots, s-1, r+1$ which belong to $\{r_1, \ldots, r_q\}$. But this number is less than s, for if every number of the sequence $1, 2, \ldots, s-1, r+1$ belongs to $r_1, \ldots, r_q$ then $a_1, \ldots, a_{s-1}, a_{r+1} \in A_{k_1}$ and this is impossible. Hence h and consequently f does not depend on s variables, contrary to our assumption.

(c)$\to$(b). Let $\dim \mathfrak{A} = n$; $A^{(n)} = A^{(n,1)}$. Let $a_1, \ldots, a_n$ be a basis of $\mathfrak{A}$, and let us define $\mathfrak{A}_i = [a_i]$ for $i = 1, 2, \ldots, n$. It suffices to prove that $A = \bigcup_{i=1}^{n} A_i$. Suppose $b \in A \setminus \bigcup_{i=1}^{n} A_i$. There exists an $f(x_1, \ldots, x_n)$ such that $f(a_1, \ldots, a_n) = b$ but $f(x_1, \ldots, x_n) = g(x_{i_0})$ for suitable i_0 and so $b = g(a_{i_0}) \in A_{i_0}$ contrary to our assumption.

The theorem is thus proved.

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Semigroups on trees *

by

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Introduction. We consider here a special case of the test question: Given a continuum S, does it support the structure of a topological semigroup with zero and unit? In the case that S is one-dimensional it is known [1] that a necessary condition on S is that it be a generalized tree, i.e.: arc-wise connected, hereditarily unicoherent, and satisfy a certain arc-convergence property, namely, that for some point $0 \in S$ (necessarily a point of local connectivity) and for any net $\{p_a\}$ with $\{p_a\} \to p$, that $[0, p_a] \to [0, p]$. (See also [2].) It is conjectural that any one-dimensional generalized tree supports the desired structure. As a step in this direction we show here that any metric tree S whose endpoints I form a compact set admits such a structure. In fact we establish a stronger conclusion:

THEOREM. *I can be ordered so that* $\min(x, y)$ *is continuous for* $x, y \in I$, *and multiplication in* S *can be introduced so that* S *is realized as the continuous homomorphic image of the "fan" over* I, *i.e. the semigroup formed from* $I \times \{01\}$ *by shrinking* $I \times \{0\}$ *to a point* (here $\{01\}$ denotes the unit interval of real numbers provided with any continuous associative multiplication in which 0 acts as a zero and 1 acts as a unit).

Set-theoretic preliminaries. Throughout the paper S will denote a metric tree, or acyclic locally connected compactum. For equivalent formulations see [4], p. 88. The set of endpoints of S will be noted by I, and we assume that I is compact. The unique arc from p to q will be written $[p, q]$. We denote the boundary of A by $F(A)$, the complement of B in A by $A \setminus B$, the closure of A by A^*, and the empty set by $\square$.

We will make use of the fact ([4], p. 99) that a metric tree is a regular curve, i.e. about each point there is a small neighborhood with finite boundary. Also ([4], p. 89) the set of branch points of S, i.e. cutpoints of order > 2, is countable. We fix an element 0 of $S \setminus I$.

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