Developments in topological analysis *

by

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1. Introduction. The basic topological nature of many of the classical results of the theory of functions of a complex variable has long been recognized. In recent years, as the concepts and tools of topology have sharpened into more definitive form and become usable by a wider range of mathematical scholars, considerable effort has been made in the direction of separating out those phases of classical analysis which are topological in character and developing these in as purely a topological way as is poss'ble and easy.

It was recognized early by Stoïlow [1] that lightness and openness of the mapping generated by a non-constant analytic function represent the fundamental topological properties of this class of functions, in the sense that all other topological properties of the whole class of necessity are consequences of these two. This has led to a most intensive interest in and development of light open mappings and in their relation and applications to the results of classical analytic function theory. In a paper published in 1950 the present author [2] called attention to the desirability of proving the lightness and openness properties in a simple and purely topological way, based only on the assumed existence (and not continuity) of the derivative. Such a proof was given by Ursell and Eggleston [3] two years later, by making essential use of a topological index equivalent to the classical winding number.

Meanwhile a similar type of index had been studied and used effectively by Eilenberg [4] and by Kuratowski [5] in connection with the development of plane topology and the application of results to obtain properties of functions of analytic type. Using the terminology and methods of Kuratowski and Eilenberg the present author [2] was able to simplify the Ursell-Eggelston proof and exploit the method of argument to carry considerably further the development of analysis type results by topological means. During the preparation of the author's book on "Topological

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Analysis" (Princeton Press, 1958) and after its publication, important new steps were completed by R. L. Plunkett [6]. He was able to prove continuity of the derivative by these mappings and, a little later, succeeded in establishing also the openness of the derivative mapping. Lightness of the derivative mapping had already emerged earlier as a consequence of the identification of the set of zeros of the derivative with the set at which the mapping given by the original function is not locally topological.

This work was clearly pointed in the direction of showing by topological methods that existence of the first derivative implies the existence of the second derivative and thus also of all higher ordered ones. This goal was actually attained early in 1960 in some remarkable work of E. H. Connell [7] who effected the proof by making essential use of the openness of the original mapping along with the extensibility of openness of a mapping to the whole region R when opennes is assumed to hold on $R-p$, where $p \in R$. This latter result gives the maximum modulus conclusion under a similar assumption; and by applying this ingeniously to the differential quotient, Connell was able to obtain existence of the second derivative, termwise differentiably of uniformly convergent sequences and other classical results in remarkably simple fashion. Just recently Connell and Porcelli [8] have succeeded in establishing, in this same chain of results, the power series development of a function in a neighborhood of each point of a region in which the function is assumed only to have a first derivative. This has taken us very far indeed in the development of classical analysis from a topological base and it is confidently believed that the end has not yet been reached.

Largely as a result of a study of Connell's and Porcelli's papers to which he had access in manuscript form, the present author has been able to fit their conclusions into what is believed to be a simpler and still more natural sequence of deductions of classical theorems of complex variable analysis from a topological base. The essential features of this treatment will be given in the sections that follow in the present paper.

The differential quotient function

$$h(x, y) = \frac{f(x)-f(y)}{x-y}$$

for a function $f(x)$ defined and continuous in a region R of the complex plane will play a key rôle. It will be considered as a function of the two complex variables x and y in the cartesian product space $R\times R$. By developing and exploiting its uniform continuity properties on $R\times R-\Delta$, where Δ is the diagonal of $R\times R$, and employing standard extension results for uniformly continuous functions, the main results are obtained in a surprising easy and simple way. Further, it is not necessary to make direct use of opennes or lightness of the mapping generated by a differentiable function. Instead we are able to use directly a form of maximum modulus results already available through the circulation index which previously had been proven in the course of proving lightness and openness. As a result we are able thus to obtain new proofs for these two basic topological properties as by-products. This accomplishes a substantial shortening and simplification of the path leading to these results, rendering them even more accessible by topological methods.

2. Background material. We take an arbitrary mapping $\varphi(x)$ of an interval or simple arc ab into the complex plane Z and let p be a point of $Z-E$, where $E = \varphi(ab)$ is the image of ab. (Note: all *mappings* are assumed continuous.) Any continuous function $u(x)$, $x \in ab$, satisfying

$$e^u = \varphi(x)-p\,, \qquad x \in ab \tag{1}$$

is called a continuous branch of the logarithm of $\varphi(x)-p$ and (1) is referred to as an exponential representation of $\varphi(x)-p$. It is readily shown that every φ has such representations and, further, that the $u(x)$ is uniquely detrmined up to an additive constant. Thus when we define the *circulation index*

$$\mu_{ab}(\varphi, p) = u(b)-u(a)\,, \tag{2}$$

it follows that μ is independent of the particular $u(x)$ entering into the representation (1). Also, of course,

$$\mu(\varphi, p) = \mu(\varphi-p, 0) \tag{3}$$

and for any factorization $\varphi(x)-p = \varphi_1(x)\cdot\varphi_2(x)$, we have

$$\mu(\varphi, p) = \mu(\varphi_1, 0)+\mu(\varphi_2, 0)\,. \tag{4}$$

We note here that when no confusion is likely to result some or all of the symbols ab, φ, and p in the expression $\mu_{ab}(\varphi, p)$ may and will be omitted.

Now in case $\varphi(b) = \varphi(a)$ so that our "curve" or image is closed, $\mu(\varphi, p)$ has the form $2k\pi i$, k an integer, so that $\mu(\varphi, p)/2\pi i = w(\varphi, p)$ is integer valued and is called the *winding number* of φ about p; for $\mu(x)$ is a logarithm of $\varphi(x)-p$ and thus of the form $\log|\varphi(x)-p|+i\operatorname{amp}[\varphi(x)-p]$. Further, as a function of p, $\mu(\varphi, p)$ is continuous in p and thus is constant in each component of $Z-E$. From this it follows that $\mu(\varphi, p) = 0$ for all points p in the unbounded component of $Z-E$. Indeed, for some such p sufficiently remote from E, the angle subtended by E at p is $< 2\pi$; hence the variation of the imaginary part of $u(x)$ on E is $< 2\pi$ in modulus, so that $k = 0$ and $\mu = 0$.

Let C be a simple closed curve and let f be a continuous function from C to the complex plane. We understand by a *traversal* of C a mapping ζ of an interval or simple arc ab onto C with $\zeta(a) = \zeta(b)$ but with $\zeta^{-1}(y)$ unique for $y \in C-\zeta(a)$. It is readily shown that $\mu_{ab}(f\zeta, p)$ depends only on the sense in which ζ traverses C. Thus for two traversals ζ and ζ_1 of C we have $\mu(f\zeta, p) = \pm\mu(f\zeta_1, p)$, the sign being $+$ or $-$ according as ζ and ζ_1 agree or disagree in sense. By comparing with the case when C is a circle centered at p, it is easily shown that for any traversal ζ of an arbitrary C in the complex plane and any point p within C, we have (where f is the indentity mapping)

$$(5)\qquad \frac{1}{2\pi i}\mu_{ab}(\zeta, p) = \pm 1 = w_c(z, p)\,.$$

Hence we define a *positive traversal* of C as a ζ for which we get $+1$ in (5).

It is clear that for any continuous complex-valued function $w = f(z)$ defined on a simple closed curve C in Z, all positive traversals of C give the same value of $\mu_{ab}(f\zeta, p)$ for $p \in W - f(C)$. Since we thus can compute μ directly from f on C, we write $\mu(f, p)$ instead of $\mu_{ab}(f\zeta, p)$, when the traversal is positive. Similarly we write $w_c(f, p)$ for the winding number of f about p when the traversal is positive.

Now let $f(z)$ be any mapping (continuous) from a simple closed curve C in a complex plane Z to a complex plane W and let R be the interior of C. The following additional conclusions are readily established.

(6) If $p \in W - f(C)$ and if f admits an extension (continuous) to R into $W - p$, [or if f is given on $C + R$ and $f(C+R)$ does not contain p], then

$$w_c(f, p) = 0\,.$$

(7) If f is continuous on $C + R$ and if there exists a dense open subset E_0 of $E - f(C)$, where $E = f(C+R)$, such that f is differentiable on $f^{-1}(E_0)$, then for any $p \in E - f(C)$, $w_c(f, p) > 0$.

The conclusions already stated lead naturally and simply to the

THEOREM. *If $w = f(z)$ is continuous on a simple closed curve C and on its interior R and is differentiable on the inverse of a dense open subset of $f(R+C) - f(C)$, then $f(R+C)$ consists of $f(C)$ together with certain bounded components of $W - f(C)$.*

COROLLARY 1. *If $|f(z)| \leqslant M$ on C, then $|f(z)| \leqslant M$ on $C + R$.*

COROLLARY 2. *If $f(z)$ is continuous on $R + C$ and differentiable on $R - F$ where F is a finite set of points, then $|f(z)| \leqslant M$ on C implies $|f(z)| \leqslant M$ on $R + C$.*

For a complete treatment of the above topics and results the reader is referred to the author's book on "Topological Analysis", Chapters V and VI. References will be found there to closely related work by Eilenberg, Kuratowski, Ursell and Eggleston and others. A similar sketch may be also found in earlier papers by the present author.

Although the concepts and results just sketched are valid and were proven for completely general simple closed curves and regions in the plane, a treatment limited to the very simplest cases would be entirely adequate for the applications which are to follow. Indeed, circles and squares and their interiors along with the region between an outer square and a finite number of inner squares provide all the generality needed. Thus all difficulties of subdivision of regions and approximation to general curves can be avoided.

3. The differential quotient function. Let $f(x)$ be a complex valued function defined and continuous in a region R of the complex plane. We define the function

$$h(x, y) = \frac{f(x) - f(y)}{x - y}, \quad \text{for} \quad x, y \in R\,,\ x \neq y;$$

$$h(x, x) = f'(x)\,, \quad \text{for} \quad x \in R\,, \text{ when } f'(x) \text{ exists at } x\,.$$

Thus $h(x, y)$ is defined at all points of the cartesian product space $R \times R$ except at points of the diagonal Δ of this space where f fails to be differentiable. Further, $h(x, y)$ is continuous in (x, y) at all points of $R \times R - \Delta$ and is continuous in x (and in y) separately at points (x, x) of Δ where it is defined, i.e. such that $f'(x)$ exists. We note also that $h(x, y)$ is symmetric in x and y.

(3.1) THEOREM. *Let $w = f(z)$ be continuous inside and on a simple closed curve C and differentiable at all points of the interior R of C except at a finite set F of points. Then if K is any compact set in R and $K_0 = K - K \cdot F$, the function $h(x, y)$ is uniformly continuous on $K_0 \times K_0$.*

Proof. Let $\varepsilon > 0$ be given. By uniform continuity of $h(x, y)$ on $C \times K$, there exists a $\delta > 0$ such that

$$|h(t, y) - h(t, y')| < \varepsilon/2 \quad \text{for all} \quad t \in C \text{ and all } y, y' \in K \text{ with } |y - y'| < \delta\,.$$

Now let $(x_1, y_1), (x_2, y_2) \in K_0 \times K_0$ with $|x_1 - x_2| < \delta$ and $|y_1 - y_2| < \delta$. We next define, for $z \in R$,

$$h(z) = h(z, y_1) - h(z, y_2)\,, \qquad g(z) = h(x_2, z) - h(x_1, z)\,.$$

Then $h(z)$ and $g(z)$ are continuous in $C + R$ and differentiable at all points $R - F - x_1 - x_2 - y_1 - y_2$ Whence, by § 2, Corollary 2,

$$|h(z)| \leqslant \max_{t \in C} |h(t, y_1) - h(t, y_2)| < \varepsilon/2\,, \quad \text{for all} \quad z \in R\,,$$

$$|g(z)| \leqslant \max_{t \in C} |h(x_2, t) - h(x_1, t)| < \varepsilon/2\,, \quad \text{for all} \quad z \in R\,.$$

Whence, substituting x_1 in the first and y_2 in the second of these relations, and adding,

$$\varepsilon > |h(x_1)| + |g(y_2)| \geqslant |h(x_1) - g(y_2)|$$
$$= |h(x_1, y_1) - h(x_1, y_2) - h(x_2, y_2) + h(x_1, y_2)|$$
$$= |h(x_1, y_1) - h(x_2, y_2)|\,.$$

(3.2) THEOREM. *Let $w = f(z)$ be continuous in a region S and differentiable at all points of $S-F$ where F is a finite set of points. Then $h(x, y)$ is defined and continuous at all points (x, y) in $S \times S$. Thus $f'(x)$ exists and is continuous at all points of S.*

Let z_0 be any point in S, let C be a circle with center z_0 and lying together with its interior R in the region S. Let K be a circular disk centered at z_0 and lying in R and let $K_0 = K - K \cdot F$. Now since by (3.1), $h(x, y)$ is uniformly continuous on $K_0 \times K_0$, by a standard extension result (see II, (2.3) of the author's "Topological Analysis", for example) it admits a unique continuous extension to the closure of $K_0 \times K_0$ and thus to all of $K \times K$. In particular this extension is valid at the point (z_0, z_0); and since (z_0, z_0) is interior to $K \times K$, we have that

$$\lim_{z\to z_0} h(z, z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$$

exists. Thus since f is differentiable at z_0, $h(z_0, z_0)$ was already defined there; and $h(x, y)$ must be continuous at (z_0, z_0), since it is identical with its extension to $K \times K$ and (z_0, z_0) is interior to $K \times K$. As z_0 was an arbitrary point of S, our first conclusion follows. Also, $f'(x)$ exists at all points of S as just shown and it is continuous at all points of S because it is identical with $h(x, y)$ on the diagonal Δ of $S \times S$.

(3.21) COROLLARY. *If $w = f(z)$ is continuous in a region S and differentiable in S except possibly at the points of a closed, totally imperfect subset F of S, then $f(z)$ is differentiable and $f'(z)$ is continuous at all points of S.*

(3.3) REMOVABLE SINGULARITY THEOREM. *If $f(z)$ is bounded and differentiable on $R - z_0$, where R is a region containing the point z_0, then $a = \lim_{z\to z_0} f(z)$ exists and if $f(z_0)$ is defined to be a, then $f(z)$ is differentiable at z_0.*

For if we define $g(z) = (z-z_0)f(z)$ for $z \neq z_0$ and $g(z_0) = 0$, then $g(z)$ is continuous throughout R and differentiable on $R - z_0$. Hence by (3.2), $g(z)$ is differentiable also at z_0 so that

$$\lim_{z\to z_0} \frac{g(z)}{z-z_0} = \lim_{z\to z_0} f(z) = a, \quad \text{exists}.$$

Hence if $f(z_0)$ is defined to be a, $f(z)$ is continuous throughout R and, by (3.2), it must also be differentiable in R.

(3.4) LIOUVILLE'S THEOREM. *If $f(z)$ is bounded and differentiable in the whole plane Z, then $f(z)$ is constant in Z.*

For let z_0 be any point whatever in Z. Then if for any $r > 0$, C is the circle $|z - z_0| = r$, since $h(z, z_0)$ is differentiable in Z, we have

$$|h(z_0, z_0)| \leqslant \max_{t\in C} |h(t, z_0)| = \max_{t\in C} \left| \frac{f(t)-f(z_0)}{t-z_0} \right| \leqslant \frac{2M}{r},$$
$$\text{where} \quad |f(z)| \leqslant M \quad \text{in} \quad Z.$$

Hence $f'(z_0) = h(z_0, z_0) = 0$ for all z_0 in Z, as r can be arbitrarily large. Accordingly $f(z)$ must be constant.

4. The second derivative.

(4.1) THEOREM. *If $f(z)$ is differentiable in a region R so also is $f'(z)$.*

Proof. Let y_0 be any point of R. Then since the function $h(x, y_0)$ is continuous (in x) everywhere in R and differentiable (in x) at all points of $R - y_0$, by (3.2) it is also differentiable at y_0. Whence

$$(*) \qquad h'_x(y_0, y_0) = \lim_{x\to y_0} \frac{h(x, y_0)-h(y_0, y_0)}{x-y_0} = \lim_{x\to y_0} \frac{\dfrac{f(x)-f(y_0)}{x-y_0} - f'(y_0)}{x-y_0}.$$

Thus since for $x \neq y_0$,

$$h'_x(x, y_0) = \frac{f'(x)(x-y_0)-[f(x)-f(y_0)]}{(x-y_0)^2}$$

and since

$$\frac{f'(x)-f'(y_0)}{x-y_0} = \frac{f'(x)-\dfrac{f(x)-f(y_0)}{x-y_0}}{x-y_0} + \frac{\dfrac{f(x)-f(y_0)}{x-y_0}-f'(y_0)}{x-y_0},$$

we have

$$\lim_{x\to y_0} \frac{f'(x)-f'(y_0)}{x-y_0} = \lim_{x\to y_0} h'_x(x, y_0) + \lim_{x\to y_0} \frac{\dfrac{f(x)-f(y_0)}{x-y_0}-f'(y_0)}{x-y_0}$$
$$= h'_x(y_0, y_0) + h'_x(y_0, y_0) = 2h'_x(y_0, y_0)$$

by continuity of $h'(x, y_0)$ as given in (3.2) and by $(*)$ above. Accordingly $f''(y_0)$ exists and equals $2h'_x(y_0, y_0)$.

Thus we have the classical result that the existence of the first derivative in a region implies the existence and continuity of derivatives of all orders for a function of a complex variable.

5. Higher derivatives-order at a point.

(5.1) LEMMA. *Let $f(z)$ be differentiable in a region R containing the origin. If for $n > 0$ we have $f(0) = f'(0) = f''(0) \ldots = f^{(n-1)}(0) = 0$, then $\lim_{z\to 0} \dfrac{f(z)}{z^n}$ exists and equals $\dfrac{f^{(n)}(0)}{n!}$.*

Proof. By induction on n. For $n=1$ this follows from $f'(0) = \lim_{z\to 0} (f(z)/z)$. Suppose it holds for $n \leqslant k$ and that $f(0) = f'(0) \ldots = f^{(k)}(0) = 0$. Then

$$\lim_{z\to 0} \frac{f(z)}{z^k} = \frac{f^{(k)}(0)}{k!} = 0 .$$

Now define

$$h(z) = \frac{f(z)}{z^k} \quad \text{for} \quad z \neq 0 ,$$
$$= 0 \quad \text{for} \quad z = 0 .$$

Then $h(z)$ is continuous in R (since $\lim_{z\to 0} h(z) = h(0)$) and differentiable for $z \neq 0$. Thus it is differentiable also at $z = 0$. Whence

(i) $$h'(0) = \lim_{z\to 0} \frac{f(z)/z^k}{z} = \lim_{z\to 0} \frac{f(z)}{z^{k+1}} .$$

Also since $0 = [f'(z)]'_{z=0} = [f'(z)]''_{z=0} = \ldots = [f'(z)]^{(k-1)}_{z=0}$, we have

(ii) $$\lim_{z\to 0} \frac{f'(z)}{z^k} = \frac{[f'(z)]^{(k)}_{z=0}}{k!} = \frac{f^{(k+1)}(0)}{k!} .$$

Further, by continuity of $h'(z)$ at $z = 0$, we get

(iii) $$h'(0) = \lim_{z\to 0} h'(z) = \lim_{z\to 0} \frac{z^k f'(z) - kz^{k-1} f(z)}{z^{2k}}$$
$$= \lim_{z\to 0} \frac{f'(z)}{z^k} - \lim_{z\to 0} \frac{kf(z)}{z^{k+1}} .$$

Thus by (i), (ii) and (iii) we get

$$(k+1) \lim_{z\to 0} \frac{f(z)}{z^{k+1}} = \lim_{z\to 0} \frac{f'(z)}{z^k} = \frac{f^{(k+1)}(0)}{k!}$$

or

$$\lim_{z\to 0} \frac{f(z)}{z^{k+1}} = \frac{f^{(k+1)}(0)}{(k+1)!} ,$$

which is our conclusion for $n = k+1$.

(5.2) Theorem. *Let $f(z)$ be differentiable in a region R. If for some point $a \in R$, $0 = f'(a) = f''(a) = \ldots$, (all derivatives of f vanish at a) then f is constant in R.*

Proof. Suppose first that $a = 0 = f(a)$. Let the circle C: $|z| = r$ be chosen so that it lies together with its interior in R. By (5.1), $\lim_{z\to 0} (f(z)/z^n) = 0$ for all n and the function $h(z) = f(z)/z^n$ for $z \neq 0$, $h(0) = 0$, is differentiable in R. Thus for any n and all z inside C,

$$|h(z)| \leqslant \max_{t\in C} |h(t)| \leqslant \max_{t\in C} \left|\frac{f(t)}{t^n}\right| = \frac{M}{r^n}$$

where $M = \max_{t\in C} |f(t)|$. Thus $|f(z)| \leqslant M|z/r|^n$ so that $f(z) \equiv 0$ for $|z| < r$.

Now in general, applying the case just handled to the function $g(z) = f(a+z) - f(a)$, we conclude that $f(a+z) \equiv f(a)$ for $|z| < r$. Thus $f(z)$ is constant in a circular neighborhood of $z = a$. Since R is connected, f must be constant throughout R. For if R_a is the set of all points $x \in R$ such that $f(z) \equiv f(a)$ in some circular neighborhood of x, R_a is open in R by definition; and if b is any limit point of R_a in R, all derivatives of f likewise vanish at b so that $f(x) \equiv f(a)$ in a neighborhood of b. Hence R_a is also closed in R.

(5.3) *Let $f(z)$ be non-constant and differentiable in a region R, let $a \in R$ and let n be the least positive integer such that $f^{(n)}(a) \neq 0$. Then*

$$f(z) - f(a) = (z-a)^n \varphi(z) ,$$

where $\varphi(z)$ is differentiable in R and $\varphi(a) \neq 0$.

For by (5.1), if $\varphi(z) = (f(z)-f(a))/(z-a)^n$ for $z \neq a$ and $\varphi(a) = f^n(a)/n!$, $\lim_{z\to a} \varphi(z) = \varphi(a)$ so that our conclusion follows.

Note. The integer n is usually called the *order* or *local degree* of f at the point a.

(5.4) *If $f(z)$ is non-constant and differentiable in a region R, for any $a \in R$ we have $f(z) \neq f(a)$ for all $z \neq a$ in a sufficiently small neighborhood of a.*

6. Lightness and openness.

(6.1) Theorem. *If $f(z)$ is non-constant and differentiable in a region R, f is light and strongly open in R.*

By (5.4) each point of $f^{-1}(y)$ is an isolated point of $f^{-1}(y)$, for any $y \in f(R)$. Hence f is surely light. Now to prove openness, let U be any open set in R and let $w_0 \in f(U)$ and $z_0 \in U \cdot f^{-1}(w_0)$. Then if C is a circle with center z_0 lying together with its interior I in U and such that $f(z) \neq w_0$ on C, by the theorem in § 2, $f(C+I)$, and therefore also $f(U)$, contains the complementary domain of $f(C)$ to which w_0 belongs. Accordingly $f(U)$ is open in the w-plane so that f is strongly open on R.

(6.11) Corollary (Maximum modulus theorem). *If $|f(z)| \leqslant M$ on the boundary* Fr(O) *of a bounded open set O, then $|f(z)| < M$ for all $z \in O$.*

(For various applications see, for example, p. 77 of "Topological Analysis".)

7. Power series expansion. The validity of the usual power series development in a neighborhood of any point of a region in which $f(z)$ is differentiable results readily with the aid of (5.1). The reader is reminded that, especially here, we are following proofs of Connell and Porcelli to be found in papers referred to in § 1.

(7.1) THEOREM. *Let $w = f(z)$ be differentiable in the region R: $|z| < 1$ and be continuous on $\bar{R}$ and satisfy $|f(z)| \leqslant 1$ on $\bar{R}$. Then $\left|\frac{f^{n}(0)}{n!}\right| \leqslant 2^n$ for all n and the MacLaurin's series for f converges to f for $|z| < \frac{1}{2}$.*

Proof. By induction on n. For $n = 1$ we have

$$\left|\frac{f'(0)}{1}\right| \leqslant \max_{t\in C}\left|\frac{f(t)-f(0)}{t}\right| \leqslant \max_{t\in C}|f(t)-f(0)| \leqslant 2, \quad \text{where} \quad C = \bar{R}-R.$$

Assume

$$\left|\frac{f^{(j)}(0)}{j!}\right| \leqslant 2^j \quad \text{for all} \quad j < k.$$

Define

$$h(z) = f(z)-f(0)-f'(0)z-\frac{f''(0)z^2}{2!}-\ldots-\frac{f^{(k-1)}(0)z^{k-1}}{(k-1)!}.$$

Then

$$h(0) = 0, \quad h'(0) = f'(0)-f'(0) = 0, \ \ldots,$$
$$h^{(k-1)}(0) = f^{(k-1)}(0)-\frac{(k-1)!f^{(k-1)}(0)}{(k-1)!} = 0.$$

Hence by (5.1), $\lim_{z\to 0}\frac{h(z)}{z^k}$ exists and equals $\frac{h^{(k)}(0)}{k!}$ which is the same as $\frac{f^{(k)}(0)}{k!}$. Thus

$$\left|\frac{f^{(k)}(0)}{k!}\right| \leqslant \max_{t\in C}\left|\frac{h(t)}{t^k}\right| = \max_{t\in C}|h(t)| \leqslant 1+1+2+\ldots+2^{k-1} = 2^k,$$

by the induction hypothesis. Hence the inequality $\left|\frac{f^{(n)}(0)}{n!}\right| \leqslant 2^n$ holds for all n. This gives

$$\left|\frac{h(z)}{z^k}\right| \leqslant \max_{t\in C}\left|\frac{h(t)}{t^k}\right| = \max_{t\in C}|h(t)| \leqslant 2^k \quad \text{for all} \quad k.$$

Thus $|h(z)| \leqslant |2z|^k$ for all k so that $h(z)\to 0$ as $k\to\infty$ for $|z| < \frac{1}{2}$.

(7.11) COROLLARY. *If $f(z)$ is differentiable for $|z-a| < r$ and continuous for $|z-a| \leqslant r$, the Taylor's series for $f(z)$ converges to $f(z)$ for $|z-a| < \frac{1}{2}r$.*

8. Local topological analysis. Degree. Let $f(z)$ be non-constant and differentiable in a region R. As noted in (5.3) for each $a \in R$ the least positive integer k_a such that $f^{(k_a)}(a) \neq 0$ is the *local degree* of f at a. In particular $k_a = 1$ if $f'(a) \neq 0$.

(8.1) THEOREM. *For any sufficiently small simple closed curve C lying in R and enclosing a, we have*

$$w_c[f, f(a)] = k_a.$$

Proof. By (5.3) we have the representation

$$f(z)-f(a) = (z-a)^{k_a}\varphi(z), \quad z \in R,$$

where $\varphi(z)$ is differentiable in R and $\varphi(a) \neq 0$. Now let C be taken small enough so that if I is its interior, then $C+I$ lies in R and the origin of the w plane W lies in the unbounded component of $W-\varphi(C+I)$. Then by § 2, $w_c(\varphi, 0) = 0$ so that by (3) and (4) of § 2, we get

$$\begin{aligned} w_c[f, f(a)] = w_c[f(z)-f(a), 0] &= w_c[(z-a)^{k_a}, 0]+w_c(\varphi, 0) \\ &= w_c[(z-a)^{k_a}, 0] \\ &= k_a w_c(z-a, 0) \\ &= k_a w_c(z, a) = k_a \end{aligned}$$

since $w_c(z, a) = 1$ by (5) of § 2.

(8.11) *$f'(a) \neq 0$ if and only if $k_a = 1$. Also $f(z)$ generates a local homeomorphism at a if $f'(a) \neq 0$, (or $k_a = 1$).*

To see the latter, we have only to note that if $f'(a) = h(a, a) \neq 0$, then $h(x, y) \neq 0$ in a neighborhood $U \times U$ of (a, a) where U is a neighborhood of a in R. Thus $\frac{f(x)-f(y)}{x-y} \neq 0$ for all $x, y \in U$ with $x \neq y$.

(8.2) THEOREM. *$f(z)$ is locally topologically equivalent at a to the power mapping $w = z^{k_a}$. Thus for a suitably chosen topological disk D enclosing a we have $f(z)-f(a) = [s(z)]^{k_a}$ for $z \in D$, where $s(z)$ is a homeomorphism on D.*

Proof. Let $f(z)-f(a) = (z-a)^{k_a}\varphi(z)$, where $\varphi(a) \neq 0$, and φ is differentiable in R. Consider the function

$$s(z) = (z-a)\varphi(z)^{1/k_a}.$$

Since $s'(z) = \varphi(z)^{1/k_a}+(1/k_a)\varphi(z)^{1/k_a-1}\varphi'(z)(z-a)$, so that $s'(a) = \varphi(a)^{1/k_a} \neq 0$, s generates a local homeomorphism of Z into the complex plane S in the neighborhood of a. Now the function

$$w = g(s) = s^{k_a} = (z-a)^{k_a}\varphi(z) = f(z)-f(a),$$

is a power function in the neighborhood of $s = 0$. Thus $f(z)-f(a)$ factors into the form

$$f(z)-f(a) = g[s(z)] = [(z-a)\varphi(z)^{1/k_a}]^{k_a},$$

where $s(z)$ is a homeomorphism in a neighborhood of a and g is a power mapping of degree k_a.

(8.21) COROLLARY. *For any $z \in D-a$, there are exactly k_a distinct points of $f^{-1}f(z)$ on D. In particular, f is a local homeomorphism at a if and only if $k_a = 1$ (or $f'(a) \neq 0$).*

(8.22) COROLLARY. *At any point a in R, local degree = local multiplicity = winding number at a.*

DEFINITION. For any $y \in f(R)$, the sum $k(y)$ (finite or infinite) of the local degrees of f at all points of $R\cdot f^{-1}(y)$ will be called the *degree* of f at y.

(8.3) THEOREM. *Let $f(z)$ be continuous on a simple closed curve C and differentiable on the interior R of C. For any component Q of $f(R+C)-f(C)$, $k(y)$ is finite and constant on Q. Indeed we have*

$$k(y) = w_c(f, y) = k, \quad \text{for} \quad \text{all } y \in Q.$$

This is an immediate consequence of (8.22) together with the results stated in § 2.

(8.31) COROLLARY. *On $R\cdot f^{-1}(Q)$, f is compact and of constant degree. For any $p \in Q$, the number of p-places (each counted with multiplicity) of f in R is constant and $= w_c(f, p)$.*

(8.23) COROLLARY. *If $f(z) \neq 0$ on C, there are exactly $w_c(f, 0)$ zeros of f within C.*

As would be expected, Rouché's theorem and other results on zeros and poles are also immediate consequences of this sequence of results. For details, see "Topological Analysis", Ch. VIII, for example.

9. Sequences. Termwise differentiability. Using only the results in §§ 1-3 above we now establish the standard result.

(9.1) THEOREM. *If the sequence of differentiable functions $[f_n(z)]$ in a region R converges almost uniformly to $f(z)$ in R, then $f(z)$ is differentiable in R and the sequence $[f_n'(z)]$ converges almost uniformly to $f'(z)$ in R.*

Proof. Let M be any compact set in R. We shall now show that the sequence of corresponding differential quotient functions $h_n(x, y)$ converge uniformly on $M\times M$. Let O be a bounded open set such that $M \subset O \subset \bar{O} \subset R$ and let r be the distance from M to the boundary $\mathrm{Fr}(O)$ of O. Then if $\varepsilon > 0$ there exists an N such that

(i) $$|f_n(t)-f_m(t)| < r\varepsilon/2, \quad \text{for all} \quad t \in \bar{O} \text{ and } m, n > N.$$

Now for any fixed $y \in M$ we have, for all $x \in M$,

(ii) $$\begin{aligned}|h_n(x, y)-h_m(x, y)| &\leq \max_{t\in \mathrm{Fr}(O)} |h_n(t, y)-h_m(t, y)| \\ &= \max_{t\in \mathrm{Fr}(O)} \left|\frac{f_n(t)-f_n(y)}{t-y} - \frac{f_m(t)-f_m(y)}{t-y}\right| \\ &\leq \frac{1}{r}\max_{t\in \mathrm{Fr}(O)} |f_n(t)-f_m(t)+f_m(y)-f_n(y)| \\ &\leq \frac{1}{r}\max_{t\in \mathrm{Fr}(O)} [|f_n(t)-f_m(t)|+|f_n(y)-f_m(y)|] \\ &< \frac{1}{r}\cdot 2\frac{r\varepsilon}{2} = \varepsilon \quad \text{by (i)}.\end{aligned}$$

Thus for all $x, y \in M$, i.e. $(x, y) \in M\times M$,

$$|h_n(x, y)-h_m(x, y)| < \varepsilon \quad \text{for} \quad m, n > N,$$

so that $h_n(x, y)$ converges uniformly on $M\times M$.

Now in particular $h_n(x, y)$ converges uniformly on the set $\Delta\cdot M\times M$ and on this set $h_m(z, z) \equiv f_n'(x)$. Thus $f_n'(z)$ converges uniformly on M and hence converges almost uniformly on R.

Now let $g(x, y) = \lim_{n\to\infty} h_n(x, y)$ for $(x, y) \in R\times R$. Then $g(x, y)$ is continuous in $R\times R$; and for $x_0 \neq y_0$ we have

$$\begin{aligned} g(x_0, y_0) = \lim_{n\to\infty} h_n(x_0, y_0) &= \frac{1}{x_0-y_0}[\lim_{n\to\infty} f_n(x_0) - \lim_{n\to\infty} f_n(y_0)] \\ &= \frac{1}{x_0-y_0}[f(x_0)-f(y_0)] = h(x_0, y_0).\end{aligned}$$

Thus if $z_0 \in R$, continuity of $g(x, y)$ gives $\lim_{z\to z_0} g(z, z_0) = g(z_0, z_0)$ so that $\lim_{z\to z_0} h(z, z_0) = g(z_0, z_0)$, since $g(z, z_0) \equiv h(z, z_0)$ for $z \neq z_0$. Thus, $\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$ exists and equals $g(z_0, z_0)$. Hence $f'(z_0)$ exists and equals $g(z_0, z_0)$; and this concludes the proof of the theorem since we already have shown that $f_n'(z)$ converges almost uniformly to $g(z_0, z_0) = f'(z_0)$.

This result yields an interesting second proof for the

THEOREM. *If $f(z)$ is differentiable in a region R, so also is $f'(z)$.* (See § 4 above.)

For let $z_0 \in R$ and let C be a circle with center z_0 and lying with its interior I in R. Then if a_n is any sequence of non-zero numbers converging to 0 and with $z+a_n \subset R$ for $z \in I$, each of the functions $h(z+a_n, z)$ is differentiable in I. Also by uniform continuity of $h(x, y)$, the sequence $[h(z+a_n, z)]$ converges uniformly to $f'(z)$ on I. Thus by (9.1), $f'(z)$ is itself differentiable on I.

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Schoenflies problems

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Dedicated to the Fundamenta Mathematicae on the occasion of the publication of its 50th volume, with grateful appreciation of what this journal of mathematics has meant to the world of mathematics during the last fifty years.

§ 1. Introduction. The theorem that the union of a Jordan curve and its interior in a 2-plane is a closed 2-cell is commonly called a Schoenflies theorem. The problems which arise in attempting to generalize this theorem in euclidean spaces of higher dimension are called *Schoenflies problems.*

The generalization which suggests itself first is false. Let M be a topological $(n-1)$-sphere in an euclidean n-space E with $n>1$. Let $\mathring{J}M$ be the open interior of M and JM the closure of $\mathring{J}M$. It is not always true that JM is a closed n-cell for $n>2$. See Ref. [0].

A major advance in formulating a valid Schoenflies extension theorem when $n>2$ was made by Barry Mazur in Ref. [2]. Mazur concerned himself with a topological $(n-1)$-sphere on an euclidean n-sphere. We shall present a theorem which is essentially that of Mazur, but in which Mazur's n-sphere is replaced by the euclidean n-space E. This use of an euclidean n-space E in place of an eucliden n-sphere accords with subsequent developments which we shall present.

Mazur's theorem. Mazur made two assumptions, the second of which, as we shall see, is unnecessary. Let S be an $(n-1)$-sphere in E with center at the origin and radius 1. In our formulation of Mazur's theorem these hypotheses are as follows.

I. Let φ be a homeomorphism of an open neighborhood N of S into E under which points interior (exterior) to S go into points interior (exterior) to the $(n-1)$-manifold $\varphi(S)=M$.

II. Suppose that there exists in N a neighborhood of a point P of S in the form of a star of euclidean cells incident with P such that on each cell of the star φ is linear.