Lambda-definable functionals of finite types *

by

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In RF §3 we introduced a notion of partial (or general) recursiveness of functions of variables of finite types RF 1.2. The formulation we chose was a new one rather than an extension to the higher types of variables of one of the previously known equivalent formulations for variables of type 0. (Summary in the introduction to [16].)

In RF §8, [16] and [17] we began the investigation of the relations between that new formulation and such extensions.

We continue this investigation in the present paper. Here we extend the Church-Kleene notation of λ-definability from number-theoretic functions to functions with variables of any finite types, and we prove the resulting notion to be equivalent to partial (or general) recursiveness as defined in RF §3.

For these limited objectives, we can give a treatment of λ-definability which is nearly self-contained. This will spare readers unfamiliar with λ-definability the necessity of finding their way around in a rather voluminous literature (1). From that literature we shall incorporate into our treatment only some six pages of Church-Rosser [7] and four of Kleene [11], which the reader will have to consult (2). In the proof of the equivalence to partial (or general) recursiveness, we shall presuppose also parts of RF and [17].

For historical orientation, we recall that three (equivalent) notions, λ-definability, Herbrand-Gödel general recursiveness, and Turing-Post computability, arose nearly simultaneously in the 1930's, and gave rise

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We cite [12] as "IM", and [13] and [15] as "RF".

(1) The development of the notion is in Church [1], [2] (especially § 9), [3], Kleene [9], [10], Rosser [19], and Church-Rosser [7]. Expositions, with full bibliographies, are in Church [5] and Curry-Feys [8].

(2) Hence we stick to the notation of those papers, using "$\{F\}(A_1, \ldots, A_n)$" or "$F(A_1, \ldots, A_n)$" to indicate the result of applying a function F to $A_1, \ldots, A_n$ as arguments. In Rosser [19], Church [5], etc. the Schönfinkel-Curry notation "$(FA_1 \ldots A_n)$" or "$FA_1 \ldots A_n$" is used.

to the Church-Turing thesis (Church [4], Turing [20]; cf. IM §62). Of these three, λ-definability was the first (so far as we are aware) to be studied intensively, and the only one the study of which began without the idea that the notion might encompass all "effectively calculable" number-theoretic functions. At the outset it was not apparent that even the predecessor function $x \dot{-} 1$ is λ-definable. The power of the sub-formalism of Church's formalism of [1] and [2] consisting of the three rules of λ-conversion (as modified in [9]), when applied to the definition of number-theoretic functions, revealed itself as the study progressed following the proof of the λ-definability of $x \dot{-} 1$ in February 1932 (published in [10]). Thus it was the results of the study, rather than the conception, of λ-definability which gave rise to the question whether all "effectively calculable" functions are λ-definable, which Church's thesis [4] answers affirmatively.

λ-definability is readily applied to other objects than number-theoretic functions, and thus is a natural choice to be tried when there is a question of how to formulate "effectiveness" for a new type of objects. An early example is the ordinal numbers 1935 (published in Church-Kleene [6]). The direction followed in some of the analysis in RF §5 was found in January 1955 using λ-definability, before we saw the convenience of the new formulation of partial and general recursiveness.

1. Before proceeding in §3 to the extension of λ-definability to higher types of variables, we introduce "λ-conversion with constants".

Consider a formal symbolism whose symbols are λ, infinitely many *variables* $a_0, a_1, a_2, \ldots$, finitely many *constants* $\alpha_1, \ldots, \alpha_g$ (g fixed $\geqslant 0$), parentheses ( ), braces { }, and brackets [ ]. The variables and constants we call *proper symbols*. (In Kleene [11] the only proper symbols were variables.)

The notion of 'formula', and that of 'free' occurence of a variable in a formula, are defined inductively, thus. 1. A variable is a *formula*, and occurs *free* in itself. 2. A constant is a *formula*. 3. If $\mathbf{M}$ and $\mathbf{N}$ are *formulas*, so is $\{\mathbf{M}\}(\mathbf{N})$, and the *free* occurences of variables in $\mathbf{M}$ and in $\mathbf{N}$ are *free* occurrences in $\{\mathbf{M}\}(\mathbf{N})$. 4. If $\mathbf{x}$ is a variable, and $\mathbf{M}$ is a *formula* containing at least one *free* occurrence of $\mathbf{x}$, then $\lambda\mathbf{x}[\mathbf{M}]$ is a *formula*, and the *free* occurrences in $\mathbf{M}$ of variables other than $\mathbf{x}$ are *free* occurrences in $\lambda\mathbf{x}[\mathbf{M}]$. 5. Formal expressions are *formulas*, and occurrences of variables in *formulas* are *free*, only as required by 1-4.

We may abbreviate the writing of formulas by omitting pairs of braces, and replacing pairs of brackets by dots ($\lambda\mathbf{x}[\mathbf{M}]$ becoming "$\lambda\mathbf{x}\cdot\mathbf{M}$") each having the longest reasonable scope. Also we abbreviate $\{\ldots\{\{\mathbf{M}\}(\mathbf{N}_1)\}(\mathbf{N}_2)\ldots\}(\mathbf{N}_n)$ to "$\{\mathbf{M}\}(\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n)$" or $\mathbf{M}(\mathbf{N}_1, \mathbf{N}_2, \ldots, \mathbf{N}_n)$, and $\lambda\mathbf{x}_1[\lambda\mathbf{x}_2\ldots\lambda\mathbf{x}_n[\mathbf{M}]\ldots]$ to "$\lambda\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_n[\mathbf{M}]$" or "$\lambda\mathbf{x}_1\mathbf{x}_2\ldots\mathbf{x}_n\cdot\mathbf{M}$". We denote formulas by bold-face letters, and use whatever letters are convenient as names for variables and constants. (Variables named by distinct letters in the designation of a formula shall be distinct from each other and from variables suppressed in the abbreviations for component formulas whenever it would make a difference for what we are doing whether they are distinct or not.)

Congruence of formulas is defined in analogy to IM p. 153, the λ-prefixes $\lambda\mathbf{x}$ playing the role here of the quantifiers $\forall\mathbf{x}$ and $\exists\mathbf{x}$ there. When $\mathbf{A}$ is congruent to $\mathbf{B}$, we write: $\mathbf{A}$ cong $\mathbf{B}$. That $\mathbf{N}$ is *free for* $\mathbf{x}$ *in* $\mathbf{M}$ is defined in analogy to IM p. 79. We write "$\mathrm{S}^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$" for the result of substituting $\mathbf{N}$ for the free occurrences of $\mathbf{x}$ in $\mathbf{M}$ (which substitution we will use only when it is *free* IM p. 80). A formula is *closed* if it contains no free (occurrence of a) variable.

A *λ-contraction* consists in passing from a formula (said to be *λ-contractible*) of the form $\{\lambda\mathbf{x}\cdot\mathbf{M}\}(\mathbf{N})$ where $\mathbf{x}$ is a variable, $\mathbf{M}$ is a formula (containing $\mathbf{x}$ free) and $\mathbf{N}$ is a formula to $\mathrm{S}^{\mathbf{x}}_{\mathbf{N}'}\mathbf{M}'|$ where $\mathbf{M}'$ is any congruent of $\mathbf{M}$ in which $\mathbf{N}$ is free for $\mathbf{x}$ and $\mathbf{N}'$ is any congruent of $\mathbf{N}$.

To each constant α there is to be correlated a natural number n_α.

A formula $\mathbf{A}$ is *normal*, if it has no (consecutive) part of the form $\{\lambda\mathbf{x}\cdot\mathbf{M}\}(\mathbf{N})$ where $\mathbf{x}$ is a variable and $\mathbf{M}$ and $\mathbf{N}$ are formulas or of the form $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$ where α is a constant and $\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha}$ are closed normal formulas. (This is a definition by recursion over the construction of $\mathbf{A}$.) Adapting a term from Curry-Feys [8], we call parts $\{\lambda\mathbf{x}\cdot\mathbf{M}\}(\mathbf{N})$ and $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$ as just described *λ-redexes* and *α-redexes*, respectively, or together *redexes*. (A normal formula is thus exactly one not containing redexes. But also a normal formula is exactly one not containing any λ-redex or part of the form $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$ where $\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha}$ are closed. For if a formula, containing no λ-redex, contains a part $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$ with $\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha}$ closed, then the innermost such part would be an α-redex.)

We now assume given an *evaluation* $\mathfrak{E}$ of the constants $\alpha_1, \ldots, \alpha_g$ which is an assignment to each constant α of a partial function a^* from n_α-tuples of congruence classes of closed normal formulas to a congruence class of closed normal formulas. Thus, if $\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha}$ are closed normal formulas, and $\mathbf{M}_1^*, \ldots, \mathbf{M}_{n_\alpha}^*$ are the classes of the formulas (closed and normal) congruent to $\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha}$, respectively, then $a^*(\mathbf{M}_1^*, \ldots, \mathbf{M}_{n_\alpha}^*)$ is either undefined, or consists of the class $\mathbf{N}^*$ of the formulas (closed and normal) congruent to some closed normal formula $\mathbf{N}_1$.

An *α-contraction* consists in passing from a formula (said to be *α-contractible*) of the form $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$ (α a constant, $\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha}$ closed normal formulas) for which $a^*(\mathbf{M}_1^*, \ldots, \mathbf{M}_{n_\alpha}^*)$ is defined to any member $\mathbf{N}$ of $\mathbf{N}^*$.

A *contraction* is a λ-contraction or an α-contraction. (The result of a contraction depends only on the congruence class of the formula contracted, and is determined only to within a congruence.) A *contractible* (i.e. λ-contractible or α-contractible) formula is necessarily a redex. But not every α-redex is contractible, unless $\mathfrak{E}$ consists of only total, i.e. completely defined, functions $\alpha_1^*, \ldots, \alpha_g^*$.

A *reduction* consists in a contraction performed on a (consecutive) part **C** of a formula **A** (said to be *reducible*); i.e. **A** goes to **B** by a *reduction*, if from **A** we get **B** by replacing a contractible part **C** of **A** by a result of the contraction of **C**. We say then that **A** is *immediately reducible* to **B**, and write: **A** imr **B**. A normal formula is necessarily irreducible. But an irreducible formula is not necessarily normal, unless $\mathfrak{E}$ consists of only total functions. We distinguish *λ-reductions* and *α-reductions*. The inverse of a reduction is an *expansion*.

We say **A** is *reducible* to **B** (and write: **A** red **B**), if it is possible to pass from **A** to **B** by one or more reductions; **A** is *convertible* to **B** (**A** conv **B**), if it is possible to pass from **A** to **B** by zero or more reductions, expansions, and congruence transformations. If **A** is convertible to **B** without using expansions, we write: **A** cwe **B**. ('**A** red **B**' differs from '**A** cwe **B**' just in that for '**A** red **B**' at least one reduction must be used, with which the congruence transformations can then be lumped. Thus: **A** cwe **B** $\equiv$ (**A** red **B**) $\vee$ (**A** cong **B**).)

In Church-Rosser [7] §1 (following Church [1]), congruence transformations were obtained by repeated applications of a rule of inference I in each of which the **x**'s bound by a single λ-prefix λ**x** are changed (the analog of a single replacement in the predicate calculus using IM *73 or *74 p. 153). A second rule II gave λ-reduction separately from the auxiliary changes of bound variables, and a third rule III gave λ-expansion similarly. In the second form of conversion of Church-Rosser [7], given at the beginning of §2, a rule IV (due to Church [3]) was added giving α-reduction (separately from the changes of bound variables) in the case of a certain total evaluation $\mathfrak{E}$ of one constant δ with $n_\delta = 2$, and also a rule V giving the inverse operation (α-expansion) similarly. Generalizing those rules IV and V now to any specified list of constants $\alpha_1, \ldots, \alpha_g$ and evaluation $\mathfrak{E}$ of them, the above-mentioned relations between **A** and **B** become describable as follows in terms of the applications of Rules I-V by which, the relations assert, one can proceed from **A** to **B**.

A cong **B**: zero or more appls. of I (Church-Rosser: **A** conv-I **B**).
A imr **B**: one appl. of II or IV, with zero or more of I.
A red **B**: one or more appls. of II or IV, with zero or more of I.
A conv **B**: zero or more appls. of I, II, III, IV, V.
A cwe **B**: zero or more appls. of I, II, IV (Ch.-R: **A** conv-I-II-IV **B**).

If **A** conv **B**, and **B** is normal (irreducible), we call **B** a *normal* (*irreducible*) *form* of **A**.

2. Now we need the Church-Rosser theorems (Theorems 1 and 2 and Corollaries) for the present λ-conversion with constants. As we have remarked, this is a slight generalization of their second form of conversion ([7] p. 480). A forty-three page chapter of Curry-Feys [8] is devoted to the Church-Rosser theorems with side-results, extensions and applications. However to get what we need now and just that, we find it easier to go to the original paper of Church and Rosser [7], where it can be found in a brief compass. In taking over their proof, we shall annotate it ([3]).

We presuppose familiarity with the fact that in the present formulas (just as in those of IM) there is a unique proper pairing of parentheses, braces and brackets, which entails such results as the following (cf. IM Lemmas 1-4, pp. 24, 73, 74). If **A** is a formula, and **B** and **C** are (consecutive) parts of **A** which are formulas, then either **B** is **C**, or **B** is a proper part of **C**, or **C** is a proper part of **B**, or **B** and **C** are disjoint. Hence: If **y** is a variable, **P** is a formula containing **y** free, **Q** is a formula, and **R** is a formula-part of $\{\lambda\mathbf{y}\cdot\mathbf{P}\}(\mathbf{Q})$, then either **R** is the whole $\{\lambda\mathbf{y}\cdot\mathbf{P}\}(\mathbf{Q})$, or is the shown part $\lambda\mathbf{y}\cdot\mathbf{P}$, or is the shown **y**, or is a part of the shown **P**, or is a part of the shown **Q**. (Church and Rosser refer to Kleene [9] for results of this sort.)

Suppose **A** imr **B**, say by contracting the part **C**; and let **R** be a specified redex in **A**. (By a "redex" we shall always mean a given part, of one of the two above-described forms, of the formula **A** under consideration. Thus two different occurrences in **A** of the same formula $\{\lambda\mathbf{x}\cdot\mathbf{M}\}(\mathbf{N})$ constitute different λ-redexes.) We define certain parts of **B** to be the 'residuals of **R** in **B**' as follows. Let us tag the initial λ or α of **R** with a distinguishing mark, say a prefixed subscript $_1$. Thus if **R** is a part $\{\lambda\mathbf{x}\cdot\mathbf{M}\}(\mathbf{N})$, we rewrite **A** so that the part **R** becomes $\{{}_1\lambda\mathbf{x}\cdot\mathbf{M}\}(\mathbf{N})$; if **R** is a part $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$, so that it becomes ${}_1\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$. Now we perform the contraction of **C** with this tag (a subscript $_1$) adhering to the λ or α, respectively. The parts in the result **B** which are then similarly tagged with $_1$ are the *residuals of* **R** *in* **B**. One can follow out what this means in each of the 9 cases when **C** is a λ-redex (5 when **C** is an α-redex) to which one is led by considering whether **R** is a λ-redex or an α-redex and the possible mutual relations between **R** and **C** (remembering that

([3]) Curry and Feys [8] p. 149 express doubts about the proof of Church and Rosser, without locating a fallacy in it. These doubts seem to be connected with some failures by later writers to extract from it a list of properties of conversion to serve as the hypotheses of a more general topological theorem which would include the Church-Rosser theorem as a special case. We find the Church-Rosser proof correct.

in an α-redex $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$ the parts $\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha}$ are normal). Then one can, if he cares to take the trouble, formulate the definition of 'residuals' without resort to the tags. There are 0, $s \geqslant 1$, or 1, residuals, according as **R** is **C**, **C** is $\{\lambda \mathbf{y}\cdot\mathbf{P}\}(\mathbf{Q})$ with s occurrences of **y** in **P** and **R** is a part of **Q**, or otherwise. Each residual of a λ-redex is a λ-redex. Each residual of an α-redex $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$ is congruent to $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$, since $\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha}$ are closed and normal.

Similarly, if **A** cong **B**, a redex **R** in **A** has a single *residual in* **B** (congruent to **R**, if **R** is an α-redex).

More generally, if **A** cwe **B** through a specified sequence of zero or more reductions and congruence transformations, and if $\mathbf{R}_1, \ldots, \mathbf{R}_k$ ($k \geqslant 0$) is a list of distinct redexes in **A**, we can identify the *residuals* of $\mathbf{R}_1, \ldots, \mathbf{R}_k$, respectively, *in* **B** by tagging the redexes $\mathbf{R}_1, \ldots, \mathbf{R}_k$ in **A** with the respective subscripts $_1, \ldots, _k$, and observing where these subscripts come out after the specified sequence of reductions and congruence transformations. Since distinct redexes $\mathbf{R}_i$ and $\mathbf{R}_j$ ($i \neq j$) cannot have their initial λ or α in common, no λ or α will receive two subscripts in the tagging in **A**. Hence in the collection of all the residuals in **B** of $\mathbf{R}_1, \ldots, \mathbf{R}_k$, each residual will be a residual of just one of $\mathbf{R}_1, \ldots, \mathbf{R}_k$. A residual in **B** of a λ-redex $\mathbf{R}_i$ in **A** is a λ-redex; of an α-redex $\mathbf{R}_i$ is a congruent of $\mathbf{R}_i$. A redex $\mathbf{R}_i$ in **A** must possess (a non-empty class of) residuals in **B**, unless contractions are performed on residuals of $\mathbf{R}_i$ in **A** or intermediate formulas. In particular, an uncontractible α-redex in **A** will necessarily possess residuals in **B**.

We say that a sequence of zero or more reductions and congruence transformations **A** cong $\mathbf{A}_1$ imr $\mathbf{A}_2$ imr $\mathbf{A}_3$ ... imr $\mathbf{A}_n$ is a *sequence of contractions on* the redexes $\mathbf{R}_1, \ldots, \mathbf{R}_k$ in **A**, if each reduction in the sequence is by a contraction of a residual of one of $\mathbf{R}_1, \ldots, \mathbf{R}_k$. If in $\mathbf{A}_n$ there is no contractible residual of any of $\mathbf{R}_1, \ldots, \mathbf{R}_k$, we say the sequence *terminates*, with $\mathbf{A}_n$ as the *result*.

Now we take over Lemma 1 of Church-Rosser [7] p. 475, substituting for their parts $\{\lambda \mathbf{x}_j\cdot\mathbf{M}_j\}(\mathbf{N}_j)$ of **A** (all of them λ-redexes) a list of redexes $\mathbf{R}_1, \ldots, \mathbf{R}_k$ in **A** (which may include ones of each kind). In the proof pp. 475-478, the basis and Case 1 go as on p. 475. Before considering Cases 2 (a) and (b), we dispose of the new Case 2 (c): $\{\mathbf{F}\}(\mathbf{X})$ is an α-redex $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$ in the list $\mathbf{R}_1, \ldots, \mathbf{R}_k$. Then, since $\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha}$ are normal, $\{\mathbf{F}\}(\mathbf{X})$ is the only redex in **A** ($k = 1$), and the lemma is true with $m = 1$ or 0 according as $\{\mathbf{F}\}(\mathbf{X})$ is contractible or not.

In handling Case 2 (a), and Case 2 (b) (that $\{\mathbf{F}\}(\mathbf{X})$ is a λ-redex $\{\lambda \mathbf{x}_p\cdot\mathbf{M}_p\}(\mathbf{N}_p)$ among $\mathbf{R}_1, \ldots, \mathbf{R}_k$), we tag $\mathbf{R}_1, \ldots, \mathbf{R}_k$ with subscripts $_1, \ldots, _k$ in **A** initially, and the tags ride on the tagged symbols through the reductions and congruence transformations. We must verify that each replacement of a sequence of contractions which is used in the proof

leaves unaltered not only the endformula itself but also the tags which adhere to some of its symbols. This is covered by p. 476 first footnote. The replacements come under three types or "processes", appearing first in p. 476 top lines, p. 476 lines 24-27, and p. 477 first (full) paragraph, respectively. For illustration, we elaborate the third (and most complicated) one.

In amplifying this paragraph, we leave to the reader the easy new case that instead of $\{\lambda \mathbf{z}\cdot\mathbf{R}\}(\mathbf{S})$ we have an α-redex. Now we can stick with the notation of Church-Rosser. Also for simplicity we show the formulas as they appear if the bound variables of $\lambda \mathbf{y}\cdot\mathbf{P}$ undergo no changes during ζ (in the contrary case there are congruence transformations which don't change the argument). The residual $\{\lambda \mathbf{z}\cdot\mathbf{R}\}(\mathbf{S})$ of a part of $\mathbf{M}_p$ is the residual of a part of the **P** of $\{\lambda \mathbf{y}\cdot\mathbf{P}\}(\mathbf{N}_p)$, which part we write $\{\lambda \mathbf{z}\cdot\mathbf{Q}\}(\mathbf{T})$. Say for illustration **y** is distinct from **z**, **Q** and **T** each contain two free occurrences of **y**, **Q** contains two of **z**, and **P** contains one of **y** outside the part $\{\lambda \mathbf{z}\cdot\mathbf{Q}\}(\mathbf{T})$. To show this, we write $\{\lambda \mathbf{y}\cdot\mathbf{P}\}(\mathbf{N}_p)$ thus (cf. IM pp. 78, 79):

$$\text{(a)} \qquad \{\lambda \mathbf{y}\cdot\mathbf{P}\big(\mathbf{y}, \{\lambda \mathbf{z}\cdot\mathbf{Q}(\mathbf{z},\mathbf{z},\mathbf{y},\mathbf{y})\}\big(\mathbf{T}(\mathbf{y},\mathbf{y})\big)\big)\}(\mathbf{N}_p).$$

The contraction β' reduces this to

$$\text{(b)} \qquad \mathbf{P}\big(\mathbf{N}_p, \{\lambda \mathbf{z}\cdot\mathbf{Q}(\mathbf{z},\mathbf{z},\mathbf{N}_p,\mathbf{N}_p)\}\big(\mathbf{T}(\mathbf{N}_p,\mathbf{N}_p)\big)\big),$$

where by the requirement of freedom in the substitution performed in a λ-contraction $\mathbf{N}_p$ does not contain **z** free (otherwise the bound variable **z** in the $\lambda \mathbf{z}\cdot\mathbf{Q}$ of **P** would have had first to be changed). The contractions of ζ which are on residuals of parts of $\mathbf{N}_p$ reduce this to

$$\text{(c)} \qquad \mathbf{P}\big(\mathbf{N}_{p1}, \{\lambda \mathbf{z}\cdot\mathbf{Q}(\mathbf{z},\mathbf{z},\mathbf{N}_{p2},\mathbf{N}_{p3})\}\big(\mathbf{T}(\mathbf{N}_{p4},\mathbf{N}_{p5})\big)\big).$$

Finally, the contraction of $\{\lambda \mathbf{z}\cdot\mathbf{R}\}(\mathbf{S})$, which is

$$\{\lambda \mathbf{z}\cdot\mathbf{Q}(\mathbf{z},\mathbf{z},\mathbf{N}_{p2},\mathbf{N}_{p3})\}\big(\mathbf{T}(\mathbf{N}_{p4},\mathbf{N}_{p5})\big),$$

reduces this to

$$\text{(d)} \qquad \mathbf{P}\Big(\mathbf{N}_{p1}, \mathbf{Q}\big(\mathbf{T}(\mathbf{N}_{p4},\mathbf{N}_{p5}), \mathbf{T}(\mathbf{N}_{p4},\mathbf{N}_{p5}), \mathbf{N}_{p2}, \mathbf{N}_{p3}\big)\Big),$$

since as noted **z** is not free in $\mathbf{N}_p$, and hence not in $\mathbf{N}_{p2}, \mathbf{N}_{p3}$. Under Church and Rosser's process, (a) is reduced to (d) instead via

$$\text{(b')} \qquad \{\lambda \mathbf{y}\cdot\mathbf{P}\Big(\mathbf{y}, \mathbf{Q}\big(\mathbf{T}(\mathbf{y},\mathbf{y}), \mathbf{T}(\mathbf{y},\mathbf{y}), \mathbf{y}, \mathbf{y}\big)\Big)\}(\mathbf{N}_p),$$

$$\text{(c')} \qquad \mathbf{P}\Big(\mathbf{N}_p, \mathbf{Q}\big(\mathbf{T}(\mathbf{N}_p,\mathbf{N}_p), \mathbf{T}(\mathbf{N}_p,\mathbf{N}_p), \mathbf{N}_p, \mathbf{N}_p\big)\Big).$$

Not only is the same formula (d) obtained, but clearly the tags on symbols in (a) (derived from tags introduced into **A**) ride through the new reductions to arrive at the same positions in (d) as before ([4]).

Now we can take over Church-Rosser [7] Lemma 2 and proof pp. 478-479, substituting for the part $\{\lambda \mathbf{x}\cdot\mathbf{M}\}(\mathbf{N})$ a redex of either kind (in I, $\mathbf{B}_1$ cong **B**; in II, $\mathbf{B}_k$ cwe $\mathbf{B}_{k+1}$). Finally we have the following versions of their Theorems 1 and 2 and Corollaries pp. 479-480. (The proof of Theorem 1 from Lemma 2 is on p. 473.)

THEOREM 1. *If* **A** *conv* **B**, *there is a conversion of* **A** *to* **B** *in which no expansion precedes any reduction.*

COROLLARY 1. *If* **B** *is a normal* (*or irreducible*) *form of* **A**, *then* **A** cwe **B**.

COROLLARY 2. *If* **A** *has a normal* (*or irreducible*) *form, that form is unique to within a congruence.*

THEOREM 2. *If* **B** *is a normal* (*or irreducible*) *form of* **A**, *then there is a number m such that any sequence of reductions starting from* **A** *will lead to* **B** (*to within a congruence*) *after at most m reductions.*

COROLLARY. *If a formula has a normal* (*irreducible*) *form, every part of it which is a formula has a normal* (*irreducible*) *form.*

Throughout §§ 1 and 2 the notions have been relative to a fixed choice of the list of constants $\alpha_1, \ldots, \alpha_g$, the numbers $n_{\alpha_1}, \ldots, n_{\alpha_g}$, and the evaluation $\mathfrak{E}$.

3. Now we introduce "λ-conversion with symbols for functions of finite types". The formation rules are as in § 1, except that now moreover each of the constants $\alpha_1, \ldots, \alpha_g$ shall be a symbol for a function of a given positive integral type (with each $n_\alpha = 1$); say the constants are $\alpha_1^1, \ldots, \alpha_{n_1}^1, \ldots, \alpha_1^r, \ldots, \alpha_{n_r}^r$. Conversion will be defined relative to an *interpretation* $\mathfrak{a}_1^1, \ldots, \mathfrak{a}_{n_1}^1, \ldots, \mathfrak{a}_1^r, \ldots, \mathfrak{a}_{n_r}^r$ (briefly, $\mathfrak{a}$) of these constants, each by a respective object of its type (RF 1.2).

The list of constants and interpretation will be varied now. Specifically, we shall define by a transfinite inductive definition the ternary relation **A** $\mathfrak{a}$-conv **B**, where $\mathfrak{a}$ includes an interpretation of at least each constant in **A** or **B**. But first we introduce some auxiliary terminology.

([4]) This is the case not only when just $\mathbf{R}_1, \ldots, \mathbf{R}_k$ are tagged in **A**, but likewise when distinct numerals are used to tag all the redexes in **A**; and the same is true for the other two types of replacement. So the proof establishes, in addition to the congruence of **A′** and **A″** (stated in the lemma), that the residuals of each redex in **A** are correspondingly located in **A′** and **A″** (are the same, if **A′** is **A″**). This enters into the induction in a rather obvious way indicated in the second footnote p. 476; thus, if a terminating sequence of contractions on a list of redexes in **F** with result **F′** is carried out in the obvious way on $\{\mathbf{F}\}(\mathbf{M})$ to produce $\{\mathbf{F}'\}(\mathbf{M})$, the residuals in **M** are unchanged.

The formulas $\lambda fx\cdot f(x)$, $\lambda fx\cdot f(f(x))$, $\lambda fx\cdot f(f(f(x)))$, ..., and any congruents of them, we call *numerals* (*for* the natural numbers $0, 1, 2, \ldots$), and we abbreviate them by $\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots$, respectively. When "$x$", "$y$", ... denote natural numbers, "$\mathbf{x}$", "$\mathbf{y}$", ... shall denote the corresponding numerals (determined to within a congruence).

We say that a formula **F** *λ-defines* [*λ-defines from* $\beta_1^1, \ldots, \beta_{l_1}^1, \ldots, \beta_1^s, \ldots, \beta_{l_s}^s$] a partial function $\varphi(a_1, \ldots, a_{n_0}, \alpha_1^1, \ldots, \alpha_{n_1}^1, \ldots, \alpha_1^r, \ldots, \alpha_{n_r}^r)$, if (i) **F** is closed and contains no constants [only constants $\boldsymbol{\beta}_1^1, \ldots, \boldsymbol{\beta}_{l_s}^s$ distinct from $\alpha_1^1, \ldots, \alpha_{n_r}^r$ and interpreted by $\beta_1^1, \ldots, \beta_{l_s}^s$], (ii) for each $a_1, \ldots, a_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r$ for which $\varphi(a_1, \ldots, a_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ is defined, $\mathbf{F}(\mathbf{a}_1, \ldots, \mathbf{a}_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ $\alpha_1^1, \ldots, \alpha_{n_r}^r$-conv $\mathbf{y}$ [$\beta_1^1, \ldots, \beta_{l_s}^s, \alpha_1^1, \ldots, \alpha_{n_r}^r$-conv $\mathbf{y}$] where $y = \varphi(a_1, \ldots, a_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$, (iii) for each $a_1, \ldots, a_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r$ for which $\varphi(a_1, \ldots, a_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ is undefined, $\mathbf{F}(\mathbf{a}_1, \ldots, \mathbf{a}_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ $\alpha_1^1, \ldots, \alpha_{n_r}^r$-conv $\mathbf{y}$ [$\beta_1^1, \ldots, \beta_{l_s}^s, \alpha_1^1, \ldots, \alpha_{n_r}^r$-conv $\mathbf{y}$] for no numeral **y**. (If the function φ is total, as in the definition of **A** $\mathfrak{a}$-conv **B** below, (iii) does not come into play.)

If there is such an **F**, we say that φ is *partial λ-definable* [*partial λ-definable from* $\beta_1^1, \ldots, \beta_{l_s}^s$], and further, in case φ is total, that φ is *λ-definable* [*λ-definable from* $\beta_1^1, \ldots, \beta_{l_s}^s$].

If $\varphi_1(\mathfrak{a}_1)$ is a function of variables $\mathfrak{a}_1$ not in an order of nondecreasing type, then in applying the foregoing definitions we identify $\varphi_1(\mathfrak{a}_1)$ with the function $\varphi(\mathfrak{a}) = \varphi_1(\mathfrak{a}_1)$ where $\mathfrak{a}$ is the result of permuting the variables $\mathfrak{a}_1$ to an order of nondecreasing type preserving the given order within each type (cf. RF 1.3).

Normal formula is defined as before (§ 1), with parts of the form $\alpha^j(\mathbf{M})$ playing the role of the $\alpha(\mathbf{M}_1, \ldots, \mathbf{M}_{n_\alpha})$ there. We shall take over other needed terminology from §§ 1, 2 when the sense should be clear.

Now we give the seven direct clauses of the inductive definition of **A** $\mathfrak{a}$-conv **B**. The last two use the phrase "λ-defines from $\mathfrak{a}$", which stands there as abbreviation for its definiens given above (which contains the symbol for the relation now being defined).

1. If **A** cong **B**, then **A** $\mathfrak{a}$-conv **B**. 2-3. If **B** comes from **A** by a λ-reduction, and **B** $\mathfrak{a}$-conv **C** (**A** $\mathfrak{a}$-conv **C**), then **A** $\mathfrak{a}$-conv **C** (**B** $\mathfrak{a}$-conv **C**). 4-5. If **A** contains as a (consecutive) part $\alpha^1(\mathbf{m})$ where α^1 is one of $\alpha_1^1, \ldots, \alpha_{n_1}^1$ and **m** is a numeral, **B** comes from **A** by replacing this part by **n** where $n = \mathfrak{a}^1(m)$, and **B** $\mathfrak{a}$-conv **C** (**A** $\mathfrak{a}$-conv **C**), then **A** $\mathfrak{a}$-conv **C** (**B** $\mathfrak{a}$-conv **C**). 6-7. If **A** contains as a part $\alpha^j(\mathbf{M})$ where α^j is one of $\alpha_1^j, \ldots, \alpha_{n_j}^j$ for some j $(2 \leqslant j \leqslant r)$ and **M** is a normal formula, **M** λ-defines from $\mathfrak{a}$ a function β^{j-1}, **B** comes from **A** by replacing this part by **n** where $n = \mathfrak{a}^j(\beta^{j-1})$, and **B** $\mathfrak{a}$-conv **C** (**A** $\mathfrak{a}$-conv **C**), then **A** $\mathfrak{a}$-conv **C** (**B** $\mathfrak{a}$-conv **C**).

Remark 1. To obtain partial λ-definability, and λ-definability, from assumed partial or total functions $\psi_1, \ldots, \psi_l$ of $m_1, \ldots, m_l$ variables respectively (cf. RF 3.14), we generalize the $\beta_1^1, \ldots, \beta_{l_s}^s$ above to include $\psi_1, \ldots, \psi_l$, and supply direct clauses 8-9 allowing a reduction or expansion when **A** contains a part $\psi_i(\mathbf{M}_1, \ldots, \mathbf{M}_{m_i})$ where $\mathbf{M}_1, \ldots, \mathbf{M}_{m_i}$ are numerals for, or normal formulas which λ-define from $\psi_1, \ldots, \psi_l$, $\mathfrak{a}$, the respective members of an m_i-tuple of objects for which ψ_i is defined.

4. By induction, in the form corresponding to the inductive definition of '**A** $\mathfrak{a}$-conv **B**', we see that the steps in a given verification that $\mathbf{A}_0$ $\mathfrak{a}$-conv **C** can be arranged in a (*definite*) $\mathfrak{a}$-*conversion tree* (*of* $\mathbf{A}_0$ *to* **C**) of the following sort (cf. RF beginning 5.3). $\mathbf{A}_0$ is at the 0-position. After any n-position occupied say by **A** (by **B**) under the circumstances of Clause 1, 2, 4 or 6 (Clause 3, 5, or 7), there may be an $n+1$-position occupied by **B** (by **A**). In the case of Clause 6 or 7 with $j=2$ [with $j>2$], if that $n+1$-position exists, we call it the *upper* $n+1$-position and call the n-position a *node*, and there are infinitely many *lower* $n+1$-positions occupied by the formulas $\mathbf{M}(\mathbf{a})$ where $a = 0, 1, 2, \ldots$ [occupied all by the same formula $\mathbf{M}(\alpha^{j-2})$, where α^{j-2} is the type-$j-2$ constant next in a given list after those interpreted by $\mathfrak{a}$, but with all the type-$j-2$ functions a^{j-2} as the interpretations of α^{j-2}]. Every branch ends. In particular, the *top* branch, which contains at each node the upper $n+1$-position, ends with **C**. All other branches end with numerals.

The top branch, or the sequence $\mathbf{A}_0, \ldots, \mathbf{A}_k$ of its formulas, of a definite $\mathfrak{a}$-conversion tree we call an ($\mathfrak{a}$-)*conversion* of $\mathbf{A}_0$ to $\mathbf{A}_k$.

We may also speak of an $\mathfrak{a}$-*conversion tree* not necessarily definite, which results by arranging in the described manner the stages in an attempted verification that $\mathbf{A}_0$ $\mathfrak{a}$-conv **C** for a specified or unspecified **C** (cf. RF 9.1). The differences from a definite $\mathfrak{a}$-conversion tree, such as we have just inferred to exist whenever we already have $\mathbf{A}_0$, **C** and a particular demonstration by the inductive definition that $\mathbf{A}_0$ $\mathfrak{a}$-conv **C**, are two. First, branches may continue ad infinitum. Second, at a node the top branches of the subtrees beginning with the lower $n+1$-positions may not all of them end in numerals, in which case the upper $n+1$-position cannot be filled.

In the case of conversion in which we are chiefly interested, the initial formula $\mathbf{A}_0$ is $\mathbf{F}(\mathbf{a}_1, \ldots, \mathbf{a}_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ where **F** λ-defines a function φ which is defined for the arguments $a_1, \ldots, a_{n_0}, a_1^1, \ldots, a_{n_r}^r$, and the last formula in the top branch is the numeral **y** for the number $y = \varphi(a_1, \ldots, a_{n_0}, a_1^1, \ldots, a_{n_r}^r)$. A definite $\mathfrak{a}$-conversion tree in this case, and in an elliptical sense its top branch (i.e. a conversion), constitutes a kind of "computation" of the function value $\varphi(a_1, \ldots, a_{n_0}, a_1^1, \ldots, a_{n_r}^r)$. When we do not know whether $\varphi(a_1, \ldots, a_{n_0}, a_1^1, \ldots, a_{n_r}^r)$ is defined, a not necessarily definite $\mathfrak{a}$-conversion tree with $\mathbf{F}(\mathbf{a}_0, \ldots, \mathbf{a}_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ at the 0-position and with each next position filled when possible constitutes an "attempted computation" of $\varphi(a_1, \ldots, a_{n_0}, a_1^1, \ldots, a_{n_r}^r)$.

However, the consistency of this kind of computation is not trivial, as it was of the kind in RF 3.9 and beginning 5.3 or of that in RF 10.2.

There, given the object at the 0-position, the entire tree so far as it was constructible was determined. This is not the case here, given the formula $\mathbf{A}_0$ at the 0-position together with the interpretation $\mathfrak{a}$ of the constants in use.

The consistency now will come out of the (first) Church-Rosser theorem extended to the present λ-conversion with symbols for functions of finite types.

If **A** $\mathfrak{a}$-conv **C** where **C** is normal, **C** is an $\mathfrak{a}$-*normal form of* **A**.

THEOREM 3. *If a formula* **A** *has an* $\mathfrak{a}$-*normal form, its* $\mathfrak{a}$-*normal form is unique to within a congruence.*

Proof. The theorem is equivalent to: If **A** $\mathfrak{a}$-conv **C**, **B** $\mathfrak{a}$-conv **D**, **A** cong **B**, and **C** and **D** are normal, then **C** cong **D**. Suppose to the contrary that we have two (definite) $\mathfrak{a}$-conversion trees T and U, whose top branches are conversions of **A** to **C** and of **B** to **D**, respectively, where **A** cong **B**, and **C** and **D** are normal, but not **C** cong **D**. Now either (Case 1) T and U contain a pair of subtrees T_1 and U_1 (both in T, or both in U, or one in T and one in U), each of which begins with a lower next position to a node in the top branch of the tree T or U to which it belongs, and whose top branches are $\mathfrak{a}_1$-conversions of $\mathbf{A}_1$ to $\mathbf{C}_1$ and of $\mathbf{B}_1$ to $\mathbf{D}_1$, respectively, for a common interpretation $\mathfrak{a}_1$ of the constants (extending $\mathfrak{a}$ to the additional constant α^{j-2}, if $j>2$), where $\mathbf{A}_1$ cong $\mathbf{B}_1$, and $\mathbf{C}_1$ and $\mathbf{D}_1$ are normal, but not $\mathbf{C}_1$ cong $\mathbf{D}_1$; or this is not so (Case 2). In Case 1, the same alternative applies to T_1 and U_1; etc. If Case 1 applied each time ad infinitum, then by König's lemma [18] we would have an infinite sequence of positions along a branch of T or U. But each branch of a definite $\mathfrak{a}$-conversion tree is finite. Therefore Case 2 applies eventually. To simplify notation, say it applies initially.

We define functions a^* which will constitute an evaluation $\mathfrak{E}$ in the sense of § 1 of the constants interpreted by $\mathfrak{a}$. For each such type-1 constant α^1 and numeral **m**, take $a^{1*}(\mathbf{m})$ to be **n** where $n = a^1(m)$. For each such type-j constant α^j ($j>1$) and congruence class $\mathbf{M}^*$ of closed normal formulas, take $a^{j*}(\mathbf{M}^*)$ to be **n** where $n = a^j(\beta^{j-1})$, when in the top branch of one of the given $\mathfrak{a}$-conversion trees T and U a node (i.e. an application of Clause 6 or 7) occurs with $\alpha^j(\mathbf{M})$ for an $\mathbf{M} \in \mathbf{M}^*$ as the part contracted and β^{j-1} as the function whose values are "computed" by the subtrees beginning with the lower next positions to the node (the result of the contraction being **n**); this function β^{j-1} is unique for the given $\mathbf{M}^*$, or the Case 2 hypothesis would be contradicted. In all remaining cases, $a^*(\mathbf{M}^*)$ shall be undefined. Now we have an evaluation $\mathfrak{E}$; and furthermore the given conversions, of **A** to **C** and of **B** to **D**, are conversions in the sense of § 1 for this $\mathfrak{E}$. Hence by § 2 Theorem 1 Corollary 2, **C** cong **D**, contradicting our supposition.

THEOREM 4. (i) *For each list of variables, each closed formula* **F** *containing only constants interpreted by* $\mathfrak{a}$ *λ-defines from* $\mathfrak{a}$ *a unique partial function* φ *of those variables.* Hence: (ii) *For a given interpretation* $\mathfrak{a}$, $\mathfrak{a}$-*conversion as defined here is conversion in the sense of* § 1 *for the following evaluation* $\mathfrak{E}_0$: $\alpha^{1*}(\mathbf{m})$ *is* **n** *where* $n = \alpha^1(m)$; *for* $j > 1$, $\alpha^{j*}(\mathbf{M}^*)$ *is* **n** *where* $n = \alpha^j(\beta^{j-1})$, *when* $\mathbf{M} \in \mathbf{M}^*$ *and* **M** *λ-defines* β^{j-1} *from* $\mathfrak{a}$; $\alpha^*(\mathbf{M}^*)$ *is undefined otherwise.* Hence: (iii) *The Church-Rosser Theorems* 1 *and* 2 *and Corollaries in* § 2 *apply.*

Proof. (i) By Theorem 3, since a numeral **y** is normal.

In an $\mathfrak{a}$-conversion tree (not necessarily definite), by a *completed step* we mean the passage from a formula in an n-position to the formula in the $n+1$-position if the n-position is not a node (the upper $n+1$-position if the n-position is a node). An $\mathfrak{a}$-conversion tree in which each completed step is a reduction (which may have assimilated into it a congruence transformation) we call an $\mathfrak{a}$-*reduction tree.*

THEOREM 5. *If* $\mathbf{A}_0$ *has an* $\mathfrak{a}$-*normal form, then each* $\mathfrak{a}$-*reduction tree with* $\mathbf{A}_0$ *at the* 0-*position has only finite branches.*

Proof. We restate the theorem as: If $\mathbf{A}_0$ has an $\mathfrak{a}$-normal form **C**, and **A** cong $\mathbf{A}_0$, then each $\mathfrak{a}$-reduction tree **T** with **A** at the 0-position has only finite branches. This we prove by induction. Consider a given $\mathfrak{a}$-conversion tree **U** of $\mathbf{A}_0$ to **C**; the hypothesis of the induction will apply to any proper subtree of **U**.

Simply by Theorem 4 (iii) with Theorem 2, the top branch of **T** ends (not necessarily with a congruent of **C**).

To deal with the other branches, consider the evaluation $\mathfrak{E}$ which differs from $\mathfrak{E}_0$ of Theorem 4 (ii) in that for $j > 1$ $\alpha^{j*}(\mathbf{M}^*)$ is defined only in case a contraction or expansion of $\alpha^j(\mathbf{M}_1)$, for some $\mathbf{M}_1 \in \mathbf{M}^*$, occurs in the top branch of **U**.

We show that, in any sequence of reductions starting with **A**, no formula contains an α-redex not contractible under the evaluation $\mathfrak{E}$. For otherwise the sequence of reductions up to the first formula inclusive in which such a redex occurs would be a sequence of reductions in the sense of § 1 for $\mathfrak{E}$ as the evaluation. No further reductions in that sense could eliminate the residuals of this uncontractible α-redex. But by Theorem 2 for $\mathfrak{E}$ as the evaluation, continuing the reductions must lead to a normal formula (congruent to **C**).

Hence in **T** the top branch is a sequence of reductions in the sense of § 1 for $\mathfrak{E}$ as the evaluation. Therefore in this branch each α^j-reduction for $j > 1$ is by contracting a redex $\alpha^j(\mathbf{M})$ congruent to a redex $\alpha^j(\mathbf{M}_1)$ contracted in the top branch of **U**. So each lower next position to a node in the top branch of **T** is occupied by a formula $\mathbf{M}(\mathbf{a})$ for $j = 2$ ($\mathbf{M}(\alpha^{j-2})$ for $j > 2$) congruent to one at a lower next position to a node in the top branch of **U**, which has a numeral as $\mathfrak{a}$-normal ($\mathfrak{a}, \alpha^{j-2}$-normal) form. Hence, by the hypothesis of the induction, each non-top branch of **T** also ends.

Given the formula **A** and interpretation $\mathfrak{a}$, the result of a reduction of **A** in which the first innermost redex (i.e. the leftmost redex not containing a redex as proper part) is contracted is unique to within a congruence; and it becomes completely unique when the choice of the bound variables is specified by a suitable convention. (Such a convention will be introduced implicitly in 5.1.) We then call the reduction *standard.* We call an $\mathfrak{a}$-reduction tree *standard* when each reduction in it is standard and the choice of the bound variables in the numerals **a** in the formulas $\mathbf{M}(\mathbf{a})$ at the lower next positions to α^2-nodes is specified by a convention (to be introduced in 5.1).

THEOREM 6. *If* $\mathbf{A}_0$ *has an* $\mathfrak{a}$-*normal form* **C**, *there is a standard* $\mathfrak{a}$-*reduction tree of* $\mathbf{A}_0$ *to* **C** (*to within a congruence*).

Proof. We restate the theorem as: If $\mathbf{A}_0$ has an $\mathfrak{a}$-normal form **C**, and **A** cong $\mathbf{A}_0$, then there is a standard $\mathfrak{a}$-reduction tree of **A** to **C** (to within a congruence). This we prove by induction. Consider a given $\mathfrak{a}$-conversion tree **U** of $\mathbf{A}_0$ to **C**.

Simply by Theorem 4 (iii) with Theorem 2, there is an $\mathfrak{a}$-conversion tree **T** of **A** to a congruent of **C** whose top branch is a sequence of standard $\mathfrak{a}$-reductions with numerals as specified at the lower next positions to α^2-nodes.

Furthermore, as in the proof of Theorem 5, each lower next position to a node in the top branch of **T** is occupied by a formula congruent to one at a lower next position to a node in the top branch of **U**, which has an $\mathfrak{a}$-normal ($\mathfrak{a}, \alpha^{j-2}$-normal) form. So by the hypothesis of the induction, the subtrees of **T** starting with these lower next positions can be replaced by standard $\mathfrak{a}$-reduction trees.

5. THEOREM 7. *Each partial λ-definable* (*Each λ-definable*) *function* $\varphi(a_1, \ldots, a_{n_0}, \alpha_1^1, \ldots, \alpha_{n_1}^1, \ldots, \alpha_1^r, \ldots, \alpha_{n_r}^r)$ *is partial* (*general*) *recursive.*

Proof. 5.1. We introduce a Gödel numbering of formulas. To the $i+1$-st variable (of *index* i) we assign the *Gödel number* $\langle 1, 0, i\rangle$ (cf. RF 2.1), to the $i+1$-st type-j constant α_{i+1}^j the *Gödel number* $\langle 1, j, i\rangle$, to $\{\mathbf{M}\}(\mathbf{N})$ the *Gödel number* $\langle 3, m, n\rangle$ where m and n are the *Gödel numbers* of **M** and **N**, and to $\lambda\mathbf{x}[\mathbf{M}]$ the *Gödel number* $\langle 5, x, m\rangle$ where x and m are the *Gödel numbers* of **x** and **M**.

We define some primitive recursive functions and predicates, which will have stated properties.

$$\begin{aligned} \mathrm{sb}(m, n, x) &= n \quad \text{if} \quad m = x\,, \\ &= \langle 3, \mathrm{sb}((m)_1, n, x), \mathrm{sb}((m)_2, n, x)\rangle \quad \text{if} \quad (m)_0 = 3\,, \\ &= \langle 5, (m)_1, \mathrm{sb}((m)_2, n, x)\rangle \quad \text{if} \quad (m)_0 = 5\ \&\ (m)_1 \neq x\,, \\ &= m \quad \text{otherwise}\,. \end{aligned}$$

If m and n are the Gödel numbers of formulas **M** and **N**, and x is the Gödel number of a variable **x**, then $\mathrm{sb}(m, n, x)$ is the Gödel number of $\mathbf{S}^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}|$.

$$\mathrm{Cd}(m) \equiv (i)_{i<m}[\mathrm{sb}(m, \langle 1, 0, i+1\rangle, \langle 1, 0, i\rangle) = m].$$

If m is the Gödel number of a formula **M**, $\mathrm{Cd}(m) \equiv$ {**M** is closed}.

$$\begin{aligned}
\mathrm{rx}(k) &= k \quad \text{if} \quad (k)_0 = 3 \ \&\ \mathrm{rx}\big((k)_1\big) = 0 \ \&\ \mathrm{rx}\big((k)_2\big) = 0 \ \& \\
&\qquad\qquad [(k)_{1,0} = 5 \vee \{(k)_{1,0} = 1 \ \&\ (k)_{1,1} > 0 \ \&\ \mathrm{Cd}\big((k)_2\big)\}], \\
&= \mathrm{rx}\big((k)_1\big) \quad \text{if} \quad (k)_0 = 3 \ \&\ \mathrm{rx}\big((k)_1\big) > 0, \\
&= \mathrm{rx}\big((k)_2\big) \quad \text{if} \quad \big[\{(k)_0 = 3 \ \&\ \mathrm{rx}\big((k)_1\big) = 0\} \vee (k)_0 = 5\big] \ \&\ \mathrm{rx}\big((k)_2\big) > 0, \\
&= 0 \quad \text{otherwise}.
\end{aligned}$$

If k is the Gödel number of a formula **K**, then $\mathrm{rx}(k) = 0$ if **K** is normal, and $\mathrm{rx}(k)$ is the Gödel number of the first innermost redex **R** in **K** otherwise.

$$\begin{aligned}
\mathrm{repl}(k, b) &= b \quad \text{if} \quad \mathrm{rx}(k) = k, \\
&= \langle 3, \mathrm{repl}\big((k)_1, b\big), (k)_2\rangle \quad \text{if} \quad (k)_0 = 3 \ \&\ \mathrm{rx}\big((k)_1\big) > 0, \\
&= \langle 3, (k)_1, \mathrm{repl}\big((k)_1, b\big)\rangle \quad \text{if} \quad (k)_0 = 3 \ \&\ \mathrm{rx}\big((k)_1\big) = 0 \\
&\qquad\qquad\qquad\qquad \&\ \mathrm{rx}\big((k)_2\big) > 0, \\
&= \langle 5, (k)_1, \mathrm{repl}\big((k)_2, b\big)\rangle \quad \text{if} \quad (k)_0 = 5 \ \&\ \mathrm{rx}\big((k)_2\big) > 0, \\
&= k \quad \text{otherwise}.
\end{aligned}$$

If k and b are the Gödel numbers of formulas **K** and **B**, then $\mathrm{repl}(k, b) = k$ if **K** is normal, and $\mathrm{repl}(k, b)$ is the Gödel number of the result of replacing the first innermost redex in **K** by **B** otherwise.

$$\begin{aligned}
\mathrm{incr}(m, s) &= \langle 3, \mathrm{incr}\big((m)_1, s\big), \mathrm{incr}\big((m)_2, s\big)\rangle \quad \text{if} \quad (m)_0 = 3, \\
&= \langle 5, 5^s\cdot(m)_1, \mathrm{incr}\big(\mathrm{sb}\big((m)_2, 5^s\cdot(m)_1, (m)_1\big), s\big)\rangle \quad \text{if} \quad (m)_0 = 5, \\
&= m \quad \text{otherwise}.
\end{aligned}$$

If m is the Gödel number of a formula **M** containing no variable of index $\geqslant s$, then $\mathrm{incr}(m, s)$ is the Gödel number of the congruent formula resulting from **M** by increasing the index of each bound occurrence of a variable in **M** by s.

$$\begin{aligned}
&\mathrm{nu}_0(0, h, i) = \langle 3, \langle 1, 0, h\rangle, \langle 1, 0, i\rangle\rangle, \\
&\mathrm{nu}_0(n+1, h, i) = \langle 3, \langle 1, 0, h\rangle, \mathrm{nu}_0(n, h, i)\rangle. \\
&\mathrm{nu}_0(n, h, i) = \{\text{the Gödel number of } f\big(\ldots\big(f(x)\big)\ldots\big) \text{ with } n+1\ f\text{'s}, \\
&\qquad\qquad \text{where } f \text{ and } x \text{ are the } h+1\text{-st and } i+1\text{-st variables}\}.
\end{aligned}$$

$$\mathrm{nu}(n, h, i) = \langle 5, \langle 1, 0, h\rangle, \langle 5, \langle 1, 0, i\rangle, \mathrm{nu}_0(n, h, i)\rangle\rangle.$$

If $h \neq i$, $\mathrm{nu}(n, h, i) =$ {the Gödel number of the numeral for n having the $h+1$-st and $i+1$-st variables as its f and x}.

$$\begin{aligned}
&\mathrm{nu}(n) = \mathrm{nu}(n, 0, 1). \\
&\mathrm{nu}(n) = \{\text{the Gödel number of a specified numeral for } n\}. \\
&\mathrm{Nu}(m) \equiv (En)_{n<m}(Eh)_{h<m}(Ei)_{i<m}[h \neq i \ \&\ \mathrm{nu}(n, h, i) = m]. \\
&\mathrm{Nu}(m) \equiv \{m \text{ is the Gödel number of a numeral}\}.
\end{aligned}$$

$$\mathrm{nu}^{-1}(m) = \big(\mu y_{y<\langle m,m,m\rangle}[(y)_1 \neq (y)_2 \ \&\ \mathrm{nu}\big((y)_0, (y)_1, (y)_2\big) = m]\big)_0.$$

If m is the Gödel number of a numeral, that numeral is for the natural number $\mathrm{nu}^{-1}(m)$:

If m and n are the Gödel numbers of formulas **M** and **N**, with **M** containing **x** free, then $\mathrm{incr}(m, m+n)$ is the Gödel number of a formula **M′** congruent to **M** in which **N** is free for **x**. So if **P** is $\{\lambda\mathbf{x}\cdot\mathbf{M}\}(\mathbf{N})$ with Gödel number p, then $\mathrm{sb}\big(\mathrm{incr}\big((p)_{1,2}, (p)_{1,2}+(p)_2\big), (p)_2, (p)_{1,1}\big)$ is that of a formula $\mathbf{S}^{\mathbf{x}}_{\mathbf{N}}\mathbf{M}'|$ coming from **P** by a λ-contraction.

$$\begin{aligned}
\mathrm{red}_0(k) = \mathrm{repl}\Big(k, \mathrm{sb}\Big(&\mathrm{incr}\Big(\big(\mathrm{rx}(k)\big)_{1,2}, \big(\mathrm{rx}(k)\big)_{1,2}+\big(\mathrm{rx}(k)\big)_2\Big), \\
&\big(\mathrm{rx}(k)\big)_2, \big(\mathrm{rx}(k)\big)_{1,1}\Big)\Big).
\end{aligned}$$

If k is the Gödel number of a formula **K** whose first innermost redex is a λ-redex, $\mathrm{red}_0(k)$ is the Gödel number of the result of contracting that redex within **K** (i.e. of a single standard reduction of **K**).

In analyzing via Gödel numbering a reduction in which an α^j-redex ($j \geqslant 1$) is contracted, we shall suppose the $i+1$-st type-j constant α^j_{i+1} to be interpreted in $\mathfrak{a}$ by $(\alpha^j)_i$ where α^j is a specified type-j object (cf. **RF** 2.1).

$$\mathrm{red}_1(k, \alpha^1) = \mathrm{repl}\Big(k, \mathrm{nu}\Big(\Big(\alpha^1\Big(\mathrm{nu}^{-1}\Big(\big(\mathrm{rx}(k)\big)_2\Big)\Big)\Big)_{(\mathrm{rx}(k))_{1,2}}\Big)\Big).$$

If k is the Gödel number of a formula **K** whose first innermost redex is an α^1-redex, $\mathrm{red}_1(k, \alpha^1)$ is the result of a single standard reduction of **K**.

$$\mathrm{red}_j(k, \alpha^j, \beta^{j-1}) = \mathrm{repl}\Big(k, \mathrm{nu}\Big(\big(\alpha^j(\beta^{j-1})\big)_{(\mathrm{rx}(k))_{1,2}}\Big)\Big) \quad (j > 1).$$

If k is the Gödel number of a formula **K** whose first innermost redex is a contractible α^j-redex $\alpha^j(\mathbf{M})$, with β^{j-1} the function which **M** λ-defines from $\mathfrak{a}$, then $\mathrm{red}_j(k, \alpha^j, \beta^{j-1})$ is the Gödel number of the result of a single standard reduction of **K**.

In analyzing $\alpha^1_1, \ldots, \alpha^1_{n_1}, \ldots, \alpha^r_1, \ldots, \alpha^r_{n_r}$-conversion, we shall take the constants interpreted by $\alpha^j_1, \ldots, \alpha^j_{n_j}$ to be the first n_j type-j constants ($j = 1, \ldots, r$). Then $n = \langle 0, n_1, \ldots, n_r\rangle$ will be used to keep track of the numbers of these constants which have been admitted. At a node with $j > 2$, the α^{j-2} shall be the next unused type-$j-2$ constant, i.e. the $n_{j-2}+1$-st where $n_{j-2} = (n)_{j-2}$.

5.2. We say that a formula **A** (containing only constants interpreted by $\mathfrak{a}$) has the *$\mathfrak{a}$-value y*, if **A** $\mathfrak{a}$-conv **y**; and has no *$\mathfrak{a}$-value*, otherwise. In the first case, by Theorem 3 the y is unique and by Theorem 6 there is a standard $\mathfrak{a}$-reduction tree of **A** to **y**.

Consider the following recursion, for a given $r \geqslant 0$.

$$(*)\quad \mathrm{val}_r(n, k, \alpha^1, \ldots, \alpha^r)$$

$$\simeq \mathrm{val}_r\big(n, \mathrm{red}_0(k), \alpha^1, \ldots, \alpha^r\big) \quad \text{if} \quad \big(\mathrm{rx}(k)\big)_{1,0} = 5 \quad (\text{Case } 1),$$

$$\simeq \mathrm{val}_r\big(n, \mathrm{red}_1(k, \alpha^1), \alpha^1, \ldots, \alpha^r\big) \quad \text{if} \quad \big(\mathrm{rx}(k)\big)_{1,0} = 1 \ \& \ \big(\mathrm{rx}(k)\big)_{1,1} = 1 \ \& \ \mathrm{Nu}\Big(\big(\mathrm{rx}(k)\big)_2\Big) \quad (\text{Case } 2),$$

$$\simeq \mathrm{val}_r\Big(n, \mathrm{red}_2\Big(k, \alpha^2, \lambda x \mathrm{val}_r\big(n, \langle 3, (\mathrm{rx}(k))_2, \mathrm{nu}(x)\rangle, \alpha^1, \ldots, \alpha^r\big)\Big), \alpha^1, \ldots, \alpha^r\Big)$$
$$\text{if} \quad \big(\mathrm{rx}(k)\big)_{1,0} = 1 \ \& \ \big(\mathrm{rx}(k)\big)_{1,1} = 2 \quad (\text{Case } 3.2),$$

$$\simeq \mathrm{val}_r\Big(n, \mathrm{red}_j\Big(k, \alpha^j, \lambda\beta^{j-2}\mathrm{val}_r\Big(p_{j-2}\, n, \langle 3, (\mathrm{rx}(k))_2, \langle 1, j-2, (n)_{j-2}\rangle\rangle,$$
$$\alpha^1, \ldots, \alpha^{j-3}, \lambda\sigma^{j-3}\big(p_{(n)_{j-2}} \exp \beta^{j-2}(\sigma^{j-3})\big) \cdot \prod_{i<(n)_{j-2}} \big(p_i \exp\big(\alpha^{j-2}(\sigma^{j-3})\big)_i\big),$$
$$\alpha^{j-1}, \ldots, \alpha^r\Big)\Big), \alpha^1, \ldots, \alpha^r\Big) \quad \text{if} \quad \big(\mathrm{rx}(k)\big)_{1,0} = 1 \ \& \ \big(\mathrm{rx}(k)\big)_{1,1} = j$$
$$(\text{Case } 3.j \text{ for } j = 3, \ldots, r),$$

$$\simeq \mathrm{nu}^{-1}(k) \quad \text{if} \quad \mathrm{Nu}(k) \quad (\text{Case } 4),$$

$$\simeq \mathrm{val}_r(n, k, \alpha^1, \ldots, \alpha^r) \quad \text{otherwise} \quad (\text{Case } 5).$$

The function val_r computed by the recursion $(*)$ in the sense of RF 10.2 (but here we haven't first written the recursion with a function variable ζ) has the following property. *When $n = \langle 0, n_1, \ldots, n_r\rangle$, k is the Gödel number of a formula* **K** *containing only the constants $\alpha^1_1, \ldots, \alpha^1_{n_1}, \ldots, \alpha^r_1, \ldots, \alpha^r_{n_r}$, and α^j_{i+1} is interpreted in $\mathfrak{a}$ by $(\alpha^j)_i$, then $\mathrm{val}_r(n, k, \alpha^1, \ldots, \alpha^r)$ is defined if and only if* **K** *has an $\mathfrak{a}$-value, in which case $\mathrm{val}_r(n, k, \alpha^1, \ldots, \alpha^r, \ldots)$ is that $\mathfrak{a}$-value.* This is established by two inductions, one over the computation by $(*)$, the other over a standard $\mathfrak{a}$-reduction tree of **K** to the numeral **y** for its $\mathfrak{a}$-value.

5.3. The functional $\mathsf{F}(\zeta; n, k, \alpha^1, \ldots, \alpha^r)$ used in constructing the right side of $(*)$ is normal in the sense of RF 10.1, as we see thus. The function $\lambda\sigma^{j-3}\ldots$ substituted in Case 3.j is of the sort $\lambda\tau^{j-3}\ldots$ stipulated for S4$'.j-2$ in RF 9.8 ($j = 3, \ldots, r$). The $\lambda\beta^{j-2}\ldots$ in Case 3.j (the $\lambda x\ldots$ in Case 3.2) comes directly under the function variable α^j (the function variable α^2) via the construction of $\mathrm{red}_j(k, \alpha^j, \alpha^{j-1})$ (of $\mathrm{red}_1(k, \alpha^2, \alpha^1)$),

and so comes under S8.j (S8.2). The case definition can be handled by repeated applications of S5$'$ in RF 10.1. Otherwise only the primitive recursive schemata S1-S8 RF 1.3 are required, after introductions of ζ by S$'$0 RF 10.1.

Hence by LXIV in RF 10.3: $\mathrm{val}_r(n, k, \alpha^1, \ldots, \alpha^r)$ *is partial recursive.*

5.4. Suppose **F** with Gödel number f λ-defines $\varphi(a_1, \ldots, a_{n_0}, \alpha^1_1, \ldots, \alpha^1_{n_1}, \ldots, \alpha^r_1, \ldots, \alpha^r_{n_r})$ with $n_r > 0$ if $r > 0$. For the given n_0 and r, we easily define a primitive recursive function $\varkappa$ such that $\varkappa(f, a_1, \ldots, a_{n_0}, n_1, \ldots, n_r)$ is the Gödel number of $\mathbf{F}(\mathbf{a}_1, \ldots, \mathbf{a}_{n_0}, \alpha^1_1, \ldots, \alpha^1_{n_1}, \ldots, \alpha^r_1, \ldots, \alpha^r_{n_r})$ where $\alpha^j_1, \ldots, \alpha^j_{n_j}$ are the first n_j type-j constants ($j = 1, \ldots, r$). Then by 5.2,

$$(**)\quad \varphi(a_1, \ldots, a_{n_0}, \alpha^1_1, \ldots, \alpha^r_{n_r}) \simeq \mathrm{val}_r\big(\langle 0, n_1, \ldots, n_r\rangle,$$
$$\varkappa(f, a_1, \ldots, a_{n_0}, n_1, \ldots, n_r), \langle \alpha^1_1, \ldots, \alpha^1_{n_1}\rangle, \ldots, \langle \alpha^r_1, \ldots, \alpha^r_{n_r}\rangle\big).$$

The functions $\langle \alpha^j_1, \ldots, \alpha^j_{n_j}\rangle$ ($j = 1, \ldots, r$) on the right are of the form $\lambda\tau^{j-1}\ldots$ for LXI in RF 9.7, provided $n_2, \ldots, n_{r-1} > 0$, and by 5.3 val_r is partial recursive; so then φ is partial recursive.

To handle λ-definable functions with $n_2, \ldots, n_{r-1}$ not all > 0, we can proceed as in (iii) of the proof of RF LXVIII. We introduce successively sets of functions similar to $\mathrm{val}_r(n, k, \alpha^1, \ldots, \alpha^r)$ but lacking $1, 2, \ldots, r-2$ of the variables $\alpha^2, \ldots, \alpha^{r-1}$. The successive recursions come under (a*) of RF 10.4, so by RF LXVIII (i) these functions are partial recursive.

6. THEOREM 8. *Each partial (general) recursive function $\varphi(a_1, \ldots, a_{n_0}, \alpha^1_1, \ldots, \alpha^1_{n_1}, \ldots, \alpha^r_1, \ldots, \alpha^r_{n_r})$ is partial λ-definable (λ-definable).*

Proof. **6.1.** We now take over Kleene [11] § 2 pp. 344-347. Note the definitions of I and J p. 341. In [11] "**A** conv **B**" means that **A** is transformable to **B** by zero or more λ-reductions, λ-expansions, and congruence transformations; and in this section we further write "**A** red **B**" when at least one reduction, but no expansion, is included. In (15) p. 346, etc. "free symbols" means constants and free occurrences of variables. The set of the free symbols of a formula remains unchanged throughout a λ-conversion. We now show that (1)-(22) *of* [11] *hold also reading "red" in place of "conv" (provided the conclusion of* (11) *is first restated as* "min(**x**, **y**) *conv* **x** *and* min(**y**, **x**) *conv* **x**"). Of statements "**A** *conv* **B**" where **A** is not normal but **B** is, the strengthening to "**A** *red* **B**" is immediate by Church-Rosser [7] (or § 2) Theorem 1. Of the remaining statements, the strengthened form follows directly from the previous proof, except that in (13) and (14) we replace the given $\mathbf{C}^\circ$ by $I(\mathbf{C}^\circ)$ ($\mathbf{A}'$ by $I(\mathbf{A}')$) to exclude the possibility of no reduction.

Let $L_0 \to 0$, $L_1 \to I$, $L_{j+2} \to \lambda b \cdot b(L_j)$ ($j = 0, 1, 2, \ldots$). Then L_0 is a numeral, and for $j > 0$ (by an induction with a double basis), L_j λ-defines a type-j function. So letting $H_j \to \lambda c \cdot c(L_{j-1}, I)$, and using (4):

$$(23)\qquad H_j(\alpha^j)\ \mathfrak{a}\text{-}red\ I \quad (j = 1, 2, 3, \ldots).$$

The $k=0$ case of the next proposition (24) is immediate, by taking $\mathbf{G}$ as $\mathbf{F}$. To infer it in general by induction, consider any $k>0$. By the hypothesis of the induction, there is a $\mathbf{G}'$ such that $\mathbf{G}'(\mathbf{n})$ for $n=0,1,2,\ldots$ red $\mathbf{A}_1,\ldots,\mathbf{A}_{k-1},\mathbf{F}(0),\mathbf{F}(1),\mathbf{F}(2),\ldots,$ respectively. Using (17), pick $\mathbf{B}$ so that $\mathbf{B}(0)\operatorname{red}\lambda n\cdot\mathbf{G}'(n\dot{-}1)$ and $\mathbf{B}(1)\operatorname{red}\lambda n\cdot n(I,\mathbf{A}_0)$. Now $\lambda n\cdot\mathbf{B}(1\dot{-}n,n)$ has the properties of $\mathbf{G}$ in:

(24) *If* $\mathbf{A}_0,\ldots,\mathbf{A}_{k-1},\mathbf{F}$ *have no free symbols, there is a formula* $\mathbf{G}$ *(without free symbols) such that* $\mathbf{G}(\mathbf{n})$ *for* $n=0,1,2,\ldots$ *red* $\mathbf{A}_0,\ldots,\mathbf{A}_{k-1},\mathbf{F}(0),\mathbf{F}(1),\mathbf{F}(2),\ldots,$ *respectively*.

Applying (24) with $k=1$, $\mathbf{A}_0\to\lambda bc\cdot b(c)$ and $\mathbf{F}\to\lambda n\cdot n\big(\lambda dbca\cdot d(\lambda t\cdot b(t,a),c(a)),\mathbf{A}_0\big)$, and using (3) and induction:

(25) *There is a formula* $\mathbf{G}$ *such that* $\mathbf{G}(\mathbf{n})$ *red* $\lambda bca_1\ldots a_n\cdot b(c(a_1,\ldots,a_n),a_1,\ldots,a_n)$ $(n=0,1,2,\ldots)$.

Similarly with $\mathbf{A}_0\to\lambda bt\cdot b(t)$, $\mathbf{F}\to\lambda n\cdot n\big(\lambda dba\cdot d(\lambda t\cdot b(t,a)),\mathbf{A}_0\big)$:

(26) *There is a formula* $\mathbf{G}$ *such that* $\mathbf{G}(\mathbf{n})$ *red* $\lambda ba_1\ldots a_nt\cdot b(t,a_1,\ldots,a_n)$ $(n=0,1,2,\ldots)$.

Similarly with $\mathbf{F}\to\lambda nm\cdot n\big(\lambda dba\cdot d(b(a)),\mathbf{A}_0(m)\big)$:

(27) *There is a formula* $\mathbf{G}$ *such that, if* $\mathbf{A}_0(\mathbf{m})$ *red* $\lambda b\mathfrak{c}\cdot b(\mathbf{C}_1,\ldots,\mathbf{C}_p)$ *where* $\mathbf{C}_1,\ldots,\mathbf{C}_p$ *contain only the variables* $\mathfrak{c}$ *and no constants, then* $\mathbf{G}(\mathbf{n},\mathbf{m})$ *red* $\lambda ba_1\ldots a_n\mathfrak{c}\cdot b(a_1,\ldots,a_n,\mathbf{C}_1,\ldots,\mathbf{C}_p)$ $(n=0,1,2,\ldots)$.

Similarly with $\mathbf{F}\to\lambda nm\cdot n\big(\lambda dbca\cdot d(b(\mathbf{H},a),c),\mathbf{A}_0(m)\big)$:

(28) *If* $\mathbf{H}$ *has no free symbols, there is a formula* $\mathbf{G}$ *such that, if* $\mathbf{A}_0(\mathbf{m})$ *red* $\lambda bc\mathfrak{c}\cdot b(\mathbf{C}_1,\ldots,\mathbf{C}_p)$ *where* $\mathbf{C}_1,\ldots,\mathbf{C}_p$ *contain only the variables* $c,\mathfrak{c}$ *and no constants, then* $\mathbf{G}(\mathbf{n},\mathbf{m})$ *red* $\lambda bca_1\ldots a_n\mathfrak{c}\cdot b(\mathbf{H},a_1,\ldots,\mathbf{H},a_n,\mathbf{C}_1,\ldots,\mathbf{C}_p)$ $(n=0,1,2,\ldots)$.

Remark 2. We can improve (15), (17), (19), (20), (24), (28) to allow constants, either α^j's or as in Remark 1 also ψ's, reduction then becoming $\mathfrak{a}$-reduction where $\mathfrak{a}$ includes an interpretation of these constants. (This would be used in proving the relativized form of Theorem 8.) To allow ψ's, we assign them type designations. For example, we can write $\psi(b,\beta_1^1,\beta_2^1,\beta^3)$ as $\psi^{\mathrm{t}}(b,\beta_1^1,\beta_2^1,\beta^3)$ where $\mathrm{t}=(0{:}0,1,1,3)$. Then letting $H_{\mathrm{t}}\to\lambda b\cdot b(L_0,L_1,L_1,L_3,I)$, and using (4) and the result preceding (23), $H_{\mathrm{t}}(\psi^{\mathrm{t}})$ $\mathfrak{a}$-red I. Thus (23) is extended from "pure" to "special" types ([14] bottom p. 95). For any function of a special type t, we use the unimproved (15) to pick a formula C_{t}' such that $C_{\mathrm{t}}'(I)\operatorname{red}H_{\mathrm{t}}$ and $C_{\mathrm{t}}'(H)\operatorname{red}I$, also a formula C' such that $C'(I)\operatorname{red}H$ and $C'(H)\operatorname{red}I$, and let $C_{\mathrm{t}}\to\lambda x\cdot C_{\mathrm{t}}'(C'(x))$. Then $C_{\mathrm{t}}(I)\operatorname{red}I$ and $C_{\mathrm{t}}(H)\operatorname{red}H_{\mathrm{t}}$. To prove (15) when $\mathbf{C}$ may contain constants, we now replace each term $\mathbf{T}$ not a constant (each constant $\mathbf{T}$ of type t) by $y(\mathbf{T})$ as before (by $C_{\mathrm{t}}(y,\mathbf{T})$). The improved (17), (19), (20), (24), (28) follow as before, using the improved (15) in place of the unimproved.

6.2. Next we undertake to translate the recursion (7) of [17] 3.9, for a fixed r but varying odd m, into an $\mathfrak{a}$-reducibility relationship. The role of φ will be played by a fixed formula $\mathbf{L}$, which will be picked presently using (19) (so $\mathbf{L}(-1)$ red I). Given values of m,z,q,x will be expressed by numerals $\mathbf{m},\mathbf{z},\mathbf{q},\mathbf{x}$. For given odd m $(=\langle 0,n_1,\ldots,n_r\rangle$ with $n_r>0$ if $r>0)$, the function arguments $\alpha_1^1,\ldots,\alpha_{n_1}^1,\ldots,\alpha_1^r,\ldots,\alpha_{n_r}^r$ (briefly, $\mathfrak{a}$) will be the interpretation of constants $\alpha_1^1,\ldots,\alpha_{n_1}^1,\ldots,\alpha_1^r,\ldots,\alpha_{n_r}^r$, say the first $n_1,\ldots,n_r$ of types $1,\ldots,r$, respectively. The left side of [17] (7) will thus be translated, for given values of m,z,q,x, by the formula $\mathbf{L}(\mathbf{m},\mathbf{z},\mathbf{q},\mathbf{x},\alpha_1^1,\ldots,\alpha_{n_1}^1,\ldots,\alpha_1^r,\ldots,\alpha_{n_r}^r)$.

The right side of [17] (7) is given by cases, the number of cases varying with m. But we shall consider Cases $7.1,\ldots,7.n_1$ (where $n_1=(m)_1$) as Subcases $1,\ldots,n_1$ of Case 7; the Case 7 hypothesis is $(z)_0=7$ & $1\leqslant(q)_{1,0}\leqslant(m)_1$. Similarly, we shall consider Cases $8\cdot j\cdot 1,\ldots,8\cdot j\cdot n_j$ as Subcases $1,\ldots,n_j$ of Case $8\cdot j$ $(j=2,\ldots,r)$. We now list the cases $1,2,3,4,5\mathrm{a},5\mathrm{b},6\mathrm{a},6\mathrm{b},7,8\cdot 2,\ldots,8\cdot r,9,10$ as the i-case for $i=0,\ldots,r+9$.

We shall wish the formula $\mathbf{L}(\mathbf{m},\mathbf{z},\mathbf{q},\mathbf{x},\alpha_1^1,\ldots,\alpha_{n_r}^r)$ expressing the left side of [17] (7) to be $\mathfrak{a}$-reducible to a formula $\mathbf{R}_{\mathbf{L},m,z,q,x}$ expressing the right side in the case and subcase which apply for the given values of m,z,q,x. First, we must see that there is such a formula. By (22), the standard primitive recursive number-theoretic functions are λ-definable, and we now employ formally the usual notations for them. For example, we write $(\mathbf{A})_{\mathbf{I}}$ as a abbreviation for $\mathbf{E}(\mathbf{A},\mathbf{I})$ where $\mathbf{E}$ is some formula λ-defining $\lambda ai\,(a)_i$, $\prod_{i<\mathbf{X}}p_i^{(\mathbf{X})_i+1}$ for $\mathbf{F}(\mathbf{X})$ where $\mathbf{F}$ is some formula λ-defining $\lambda x\prod_{i<x}p_i^{(x)_i+1}$, etc. In the 10-$,\ldots,r+7$-cases (Cases $8\cdot j$ for $j=3,\ldots,r$), σ^{j-2} shall be translated as a variable t just following the type-$j-2$ constants (so that when σ^{j-2} is substituted for t by a λ-reduction, $\mathbf{L}$ will be applied to its arguments in order of increasing type). In this manner we do indeed, given $\mathbf{L},m,z,q,x$, determine a formula $\mathbf{R}_{\mathbf{L},m,z,q,x}$ which expresses the right side of [17] (7) in the applicable case and subcase.

(a) *For each* r, *and each* $i=0,\ldots,r+9$, *there is a formula* $\mathbf{S}_i$ *(without free symbols) such that, for each odd* m $(=\langle 0,n_1,\ldots,n_r\rangle$ *with* $n_r>0$ *if* $r>0)$, *each* z,q,x, *each* $\alpha_1^1,\ldots,\alpha_{n_1}^1,\ldots,\alpha_1^r,\ldots,\alpha_{n_r}^r$ *(briefly,* $\mathfrak{a}$*), and each formula* $\mathbf{L}$ *such that* $\mathbf{L}(-1)$ *red* I: $\mathbf{S}_i(\mathbf{L},\mathbf{m},\mathbf{z},\mathbf{q},\mathbf{x},\alpha_1^1,\ldots,\alpha_{n_r}^r)$ $\mathfrak{a}$*-red* $\mathbf{R}_{\mathbf{L},m,z,q,x}$ *when the* i*-case of* [17] (7) *applies to that* m,z,q,x.

Proof. We begin with the easier cases. $\mathbf{S}_{r+9}$ (for Case 10) is simply I. $\mathbf{S}_4$ (Case 5a) is $\lambda lmzqx \cdot l\big(m, (z)_2, q, \prod_{i<x} p_i^{(x)_i+1}\big)$, and $\mathbf{S}_6, \mathbf{S}_7$ (Cases 6a, 6b) are constructed similarly. $\mathbf{S}_{r+8}$ (Case 9) is $\lambda lmzqx \cdot z\big(I, l, m, (x)_0, q, \prod_{i<x} p_i^{(x)_i+1}\big)$. $\mathbf{S}_3$ (Case 4) is $\lambda lmzqx \cdot \mathbf{G}\Big((m)_1 + \ldots + (m)_r, \lambda d \cdot l\big(m, (z)_2, q, 2^d \cdot \prod_{i<x} p_{i+i}^{(x)_i}\big), l\big(m, (z)_3, q, x\big)\Big)$ for the $\mathbf{G}$ of (25), and $\mathbf{S}_5$ (Case 5b) is constructed similarly.

To construct $\mathbf{S}_0$ (Case 1), we first apply (28) with $\lambda mbc \cdot m\big(I, b(c)\big)$ as the $\mathbf{A}_0$ and H_r from (23) as the $\mathbf{H}$ to obtain $\mathbf{G}_1$ such that $\mathbf{G}_1(\mathbf{n}, \mathbf{m})$ red $\lambda bca_1^{(r)} \ldots a_n^{(r)} \cdot b(H_r, a_1^{(r)}, \ldots, H_r, a_n^{(r)}, c)$. Next applying (28) with $\lambda m \cdot \mathbf{G}_1\big((m)_r, m\big)$ as the $\mathbf{A}_0$ and H_{r-1} from (23) as the $\mathbf{H}$, we obtain $\mathbf{G}_2$ such that $\mathbf{G}_2(\mathbf{n}, \mathbf{m})$ red $\lambda bca_1^{(r-1)} \ldots a_n^{(r-1)} a_1^{(r)} \ldots a_{n_r}^{(r)} \cdot b(H_{r-1}, a_1^{(r-1)}, \ldots, H_{r-1}, a_n^{(r-1)}, H_r, a_1^{(r)}, \ldots, H_r, a_{n_r}^{(r)}, c)$ where $n_r = (m)_r$. Continuing thus, we eventually obtain $\mathbf{G}_r$ such that $\mathbf{G}_r(\mathbf{n}, \mathbf{m})$ red $\lambda bca_1^{(1)} \ldots a_n^{(1)} a_1^{(2)} \ldots a_{n_2}^{(2)} \ldots a_1^{(r)} \ldots a_{n_r}^{(r)} \cdot b(H_1, a_1^{(1)}, \ldots, H_1, a_n^{(1)}, H_2, a_1^{(2)}, \ldots, H_2, a_{n_2}^{(2)}, \ldots, H_r, a_1^{(r)}, \ldots, H_r, a_{n_r}^{(r)}, c)$. Let $\mathbf{S}_0 \to \lambda lmzq \cdot \mathbf{G}_r\Big((m)_1, m, I, l\big(-1, z, I, q, I, (x)_0 + 1\big)\Big)$. $\mathbf{S}_1$ and $\mathbf{S}_2$ (Cases 2 and 3) are constructed similarly.

To construct $\mathbf{S}_8$ (Case 7), let $\mathbf{A}' \to \lambda mqbca_s^{(1)} \cdot \mathbf{G}_r\big((m)_1 \dot{-} (q)_{1,0}, m, b, a_s^{(1)}(c)\big)$ for the $\mathbf{G}_r$ for $\mathbf{S}_0$, so (using the Case 7 hypothesis) $\mathbf{A}'(\mathbf{m}, \mathbf{q})$ red $\lambda bca_s^{(1)} \ldots a_{n_1}^{(1)} a_1^{(2)} \ldots a_{n_2}^{(2)} \ldots a_1^{(r)} \ldots a_{n_r}^{(r)} \cdot b\big(H_1, a_{s+1}^{(1)}, \ldots, H_1, a_{n_1}^{(1)}, H_2, a_1^{(2)}, \ldots, H_2, a_{n_2}^{(2)}, \ldots, H_r, a_1^{(r)}, \ldots, H_r, a_{n_r}^{(r)}, a_s^{(1)}(c)\big)$ where $s = (q)_{1,0}$. Applying (28) with $\lambda t \cdot \mathbf{A}'\big((t)_0, (t)_1\big)$ as the $\mathbf{A}_0$ and H_1 as the $\mathbf{H}$, we obtain $\mathbf{G}'$ such that $\mathbf{G}'\big((\mathbf{q})_{1,0} \dot{-} 1, \langle \mathbf{m}, \mathbf{q} \rangle\big)$ red $\lambda bca_1^{(1)} \ldots a_{n_r}^{(r)} \cdot b\big(H_1, a_1^{(1)}, \ldots, H_1, a_{s-1}^{(1)}, H_1, a_{s+1}^{(1)}, \ldots, H_r, a_{n_r}^{(r)}, a_s^{(1)}(c)\big)$. Let $\mathbf{S}_8 \to \lambda lmzqx \cdot \mathbf{G}'\Big((q)_{1,0} \dot{-} 1, \langle m, q \rangle, I, l\big(-1, z, I, (x)_0\big)\Big)$.

To construct $\mathbf{S}_9$ (Case 8.2), we first construct similarly to $\mathbf{G}'$ for $\mathbf{S}_8$ a formula $\mathbf{G}''$ such that $\mathbf{G}''\big((\mathbf{m})_1, \langle \mathbf{m}, \mathbf{q} \rangle\big)$ red $\lambda bca_1^{(1)} \ldots a_{n_r}^{(r)} \cdot b\big(H_1, a_1^{(1)}, \ldots, H_2, a_{s-1}^{(2)}, H_2, a_{s+1}^{(2)}, \ldots, H_r, a_{n_r}^{(r)}, a_s^{(2)}(c)\big)$ where $s = (q)_{2,0}$. By (26), there is a formula $\mathbf{E}$ such that $\mathbf{E}\big((\mathbf{m})_1 + \ldots + (\mathbf{m})_r\big)$ red $\lambda ba_1^{(1)} \ldots a_{n_r}^{(r)} t \cdot b(t, a_1^{(1)}, \ldots, a_{n_r}^{(r)})$. Using the $\mathbf{G}$ of (25), let $\mathbf{S}_9 \to \lambda lmzqx \cdot \mathbf{G}\Big((m)_1 + \ldots + (m)_r, \mathbf{G}''\big((m)_1, \langle m, q \rangle, I\big), \mathbf{E}\Big((m)_1 + \ldots + (m)_r, \lambda t \cdot l\big(m, (z)_3, q, 2^t \cdot \prod_{i<x} p_{i+1}^{(x)_i}\big)\Big)\Big)$.

To construct $\mathbf{S}_{j+7}$ (Case $8 \cdot j$) for $3 \leqslant j \leqslant r$, we first obtain $\mathbf{G}^{(j)}$ such that $\mathbf{G}^{(j)}\big((\mathbf{m})_1, \langle \mathbf{m}, \mathbf{q} \rangle\big)$ red $\lambda bca_1^{(1)} \ldots a_{n_r}^{(r)} \cdot b\big(H_1, a_1^{(1)}, \ldots, H_j, a_{s-1}^{(j)}, H_j, a_{s+1}^{(j)}, \ldots, H_r, a_{n_r}^{(r)}, a_s^{(j)}(c)\big)$ where $s = (q)_{j,0}$. By (26), there is an $\mathbf{E}$ such that $\mathbf{E}\big((m)_{j-1} + \ldots + (m)_r\big)$ red $\lambda ba_1^{(j-1)} \ldots a_{n_r}^{(r)} t \cdot b\big(t, a_1^{(j-1)}, \ldots, a_{n_r}^{(r)}\big)$ Taking the $\mathbf{A}_0$ for (27) to be $\lambda m \cdot \mathbf{E}\big((m)_{j-1} + \ldots + (m)_r\big)$, we obtain a formula $\mathbf{E}_j$ such that $\mathbf{E}_j\big((\mathbf{m})_1 + \ldots + (\mathbf{m})_{j-2}, \mathbf{m}\big)$ red $\lambda ba_1^{(1)} \ldots a_{n_r}^{(r)} t \cdot b\big(a_1^{(1)}, \ldots, a_{n_{j-2}}^{(j-2)}, t, a_1^{(j-1)}, \ldots, a_{n_r}^{(r)}\big)$. Using the $\mathbf{G}$ of (25), let $\mathbf{S}_{j+7} \to \lambda lmzqx \cdot \mathbf{G}\Big((m)_1 + \ldots + (m)_r, \mathbf{G}^{(j)}\big((m)_1, \langle m, q \rangle, I\big), \mathbf{E}_j\Big((m)_1 + \ldots + (m)_{j-2}, m, l\big(\mathbf{p}_{j-2} m, (z)_3, \mathbf{Q}_{8,j,m,q}, x\big)\Big)\Big)$ where $\mathbf{Q}_{8,j,m,q}$ is the result of translating $\bar{q}_8$ with m, q as variables.

(b) *For each r, there is a formula $\mathbf{L}$ such that $\mathbf{L}(-1)$ red I, and, for each odd m ($= \langle 0, n_1, \ldots, n_r \rangle$ with $n_r > 0$ if $r > 0$), each z, q, x, and each $\alpha_1^1, \ldots, \alpha_{n_1}^1, \ldots, \alpha_1^r, \ldots, \alpha_{n_r}^r$ (briefly, $\mathfrak{a}$): $\mathbf{L}(\mathbf{m}, \mathbf{z}, \mathbf{q}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ $\mathfrak{a}$-red $\mathbf{R}_{\mathbf{L},m,z,q,x}$.*

Proof. The case hypotheses are mutually exclusive primitive recursive predicates of m, z, q, x, so there is a primitive recursive function $\varkappa$ such that in the i-case $\varkappa(m, z, q, x) = i$. By (22), there is a formula $\mathbf{K}$ which λ-defines $\varkappa$. By (24) for $k = r + 10$, there is a formula $\mathbf{S}$ such that $\mathbf{S}(\mathbf{i})$ red $\mathbf{S}_i$ ($i = 0, \ldots, r+9$). Let $\mathbf{F} \to \lambda mlzqx \cdot \mathbf{S}\big(\mathbf{K}(m, z, q, x), l, m, z, q, x\big)$, and choose $\mathbf{L}$ by (19).

6.3. *For each r, each odd m ($= \langle 0, n_0, \ldots, n_r \rangle$ with $n_r > 0$ if $r > 0$), each z, q, x, and each $\alpha_1^1, \ldots, \alpha_{n_r}^r$ (briefly, $\mathfrak{a}$): If $\varphi_m(z, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ is defined, then $\mathbf{L}(\mathbf{m}, \mathbf{z}, \mathbf{q}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ $\mathfrak{a}$-red $\mathbf{y}$ where $y = \varphi_m(z, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$.*

Proof, by induction over the computation of $\varphi_m(z, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ by the recursion [17] (7). Since $\varphi_m(z, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ is defined, one of Cases 1-9 of [17] (7) must apply, etc.

Case $8 \cdot j \cdot s$, for any $j = 3, \ldots, r$ and $s = 1, \ldots, n_j$. In this case by 6.2, (i) $\mathbf{L}(\mathbf{m}, \mathbf{z}, \mathbf{q}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ $\mathfrak{a}$-red $\alpha_s^j\big(\lambda t \cdot \mathbf{L}(\mathbf{p}_{j-2}\mathbf{m}, (\mathbf{z})_3, \mathbf{Q}_{8,j,\mathbf{m},\mathbf{q}}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_{j-2}}^{j-2}, t, \alpha_1^{j-1}, \ldots, \alpha_{n_r}^r)\big)$. Furthermore, (ii) $\big\{\lambda t \cdot \mathbf{L}(\mathbf{p}_{j-2}\mathbf{m}, (\mathbf{z})_3, \mathbf{Q}_{8,j,\mathbf{m},\mathbf{q}}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_{j-2}}^{j-2}, t, \alpha_1^{j-1}, \ldots, \alpha_{n_r}^r)\big\}(\sigma^{j-2})$ red $\mathbf{L}(\mathbf{m}\dagger, \mathbf{z}\dagger, \mathbf{q}\dagger, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_{j-2}}^{j-2}, \sigma^{j-2}, \alpha_1^{j-1}, \ldots, \alpha_{n_r}^r)$ where $m\dagger = p_{j-2}m$, $z\dagger = (z)_3$, $q\dagger = \bar{q}_8$. Since $\varphi_m(z, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ is defined, by the case hypothesis $\varphi_{m\dagger}(z\dagger, q\dagger, x, \alpha_1^1, \ldots, \alpha_{n_r}^r, \sigma^{j-2})$ is defined for each σ^{j-2}, $= \beta^{j-1}(\sigma^{j-2})$ say. So by the hypothesis of the induction, for each σ^{j-2} $\mathbf{L}(\mathbf{m}\dagger, \mathbf{z}\dagger, \mathbf{q}\dagger, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_{j-2}}^{j-2}, \sigma^{j-2}, \alpha_1^{j-1}, \ldots, \alpha_{n_r}^r)$ $\mathfrak{a}, \sigma^{j-2}$-red $\mathbf{v}$ where $v = \beta^{j-1}(\sigma^{j-2})$. Using also (ii), $\lambda t \cdot \mathbf{L}(\mathbf{p}_{j-2}\mathbf{m}, (\mathbf{z})_3, \mathbf{Q}_{8,j,\mathbf{m},\mathbf{q}}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_{j-2}}^{j-2}, t, \alpha_1^{j-1}, \ldots, \alpha_{n_r}^r)$ λ-defines β^{j-1} from $\mathfrak{a}$. So by an α-reduction, $\alpha_s^j\big(\lambda t \cdot \mathbf{L}(\mathbf{p}_{j-2}\mathbf{m}, (\mathbf{z})_3, \mathbf{Q}_{8,j,\mathbf{m},\mathbf{q}}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_{j-2}}^{j-2}, t, \alpha_1^{j-1}, \ldots, \alpha_{n_r}^r)\big)$ $\mathfrak{a}$-red $\mathbf{y}$ where $y = \alpha_s^j(\beta^{j-1})$, which is the value of $\varphi_m(z, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ in this case. This with (i) gives the desired conclusion.

6.4. *For each r, etc.: If $\varphi_m(z, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ is undefined, then $\mathbf{L}(\mathbf{m}, \mathbf{z}, \mathbf{q}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ has no $\mathfrak{a}$-normal form (and hence is $\mathfrak{a}$-convertible to no numeral $\mathbf{y}$).*

Proof. Then by the analog of RF 9.1 LIII for computation trees by [17] (7) (cf. [17] 3.13), there is in the computation tree for $\varphi_m(z, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ by (7) an (uppermost) infinite branch, each position on which [below which] is occupied by a tuple $(\bar{m}, \bar{z}, \ldots)$ for which $\varphi_{\bar{m}}(\bar{z}, \ldots)$ is undefined [defined]. Say the tuple at the k-position on a *given* such infinite branch is $(m_k, z_k, \ldots)$ ($k = 0, 1, 2, \ldots$). We shall infer that, in a certain (indefinite) $\mathfrak{a}$-reduction tree with $\mathbf{L}(\mathbf{m}, \mathbf{z}, \ldots)$ at the 0-position, there is a branch having a formula containing as part $\mathbf{L}(\mathbf{m}_k, \mathbf{z}_k, \ldots)$ at its s_k-position, where $0 = s_0 < s_1 < s_2 < \ldots$. Thus that $\mathfrak{a}$-reduction tree has an infinite branch, and it will follow by Theorem 5 that $\mathbf{L}(\mathbf{m}, \mathbf{z}, \ldots)$ has no $\mathfrak{a}$-normal form.

We construct the $\mathfrak{a}$-reduction tree by stages. Suppose we have reached the s_k-position on the branch in question, where the interpretation is $\mathfrak{a}_k$. We show how to reach the s_{k+1}-position. To match the notation of 6.2 and 6.3, we now drop the subscript k in writing $\varphi_{m_k}(z_k, \ldots)$, $(m_k, z_k, \ldots)$, $\mathbf{L}(\mathbf{m}_k, \mathbf{z}_k, \ldots)$, $\mathfrak{a}_k$. Since $\varphi_m(z, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ is undefined, Cases 1-3 and 7 of [17] (7) cannot apply. We take first:

Case $8 \cdot j \cdot s$ ($j = 3, \ldots, r$; $s = 1, \ldots, n_j$). Pick that σ^{j-2} for which the $k+1$-position of the given branch is occupied by $(p_{j-2}m, (z)_3, \bar{q}_s, x, \alpha_1^1, \ldots, \alpha_{n_r}^r, \sigma^{j-2})$. First we perform the reductions (i) of 6.3 on the part $\mathbf{L}(\mathbf{m}, \mathbf{z}, \ldots)$ in question (simultaneously adjoining at the lower next positions to the nodes the requisite $\mathfrak{a}$- or $\mathfrak{a}, \alpha^{j-2}$-reduction trees, available by Theorem 6). Then we perform the reductions (ii) starting from that lower next position to the node thus reached (whose upper next position will be unfilled) which is determined by σ^{j-2} as the interpretation of the new constant σ^{j-2}. In this manner we reach the s_{k+1}-position.

Case 4: Using 6.2, a non-empty sequence of λ-reductions takes $\mathbf{L}(\mathbf{m}, \mathbf{z}, \mathbf{q}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ into $\mathbf{L}\big(\mathbf{m}, \mathbf{z}\dagger, \mathbf{q}, \big(2 \exp \mathbf{L}(\mathbf{m}, \mathbf{z}\ddagger, \mathbf{q}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_r}^r)\big) \cdot \prod_{i<\mathbf{x}} p_{i+1}^{(\mathbf{x})_i}, \alpha_1^1, \ldots, \alpha_{n_r}^r\big)$ where $z\dagger = (z)_2$ and $z\ddagger = (z)_3$. We perform these reductions on the part $\mathbf{L}(\mathbf{m}, \mathbf{z}, \ldots)$ in question of the formula at the s_k-position.

Subcase 1: the $k+1$-position in the given branch is the upper one. Then $\varphi_m\big((z)_3, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r\big)$ is defined, say with value v, and that $k+1$-position is occupied by $(m, z\dagger, q, x\dagger, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ where $x\dagger = 2^v \cdot \prod_{i<x} p_{i+1}^{(x)_i}$. By 6.3, $\mathbf{L}(\mathbf{m}, \mathbf{z}\ddagger, \mathbf{q}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ $\mathfrak{a}$-red $\mathbf{v}$. Using this, the part in question further reduces to $\mathbf{L}(\mathbf{m}, \mathbf{z}\dagger, \mathbf{q}, \mathbf{x}\dagger, \alpha_1^1, \ldots, \alpha_{n_r}^r)$, which is our $\mathbf{L}(\mathbf{m}_{k+1}, \mathbf{z}_{k+1}, \ldots)$.

Subcase 2: the $k+1$-position is the lower one, occupied by $(m, z\ddagger, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$. The part $\mathbf{L}(\mathbf{m}, \mathbf{z}\ddagger, \mathbf{q}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ within the result of the above reduction of the part $\mathbf{L}(\mathbf{m}, \mathbf{z}, \ldots)$ is our $\mathbf{L}(\mathbf{m}_{k+1}, \mathbf{z}_{k+1}, \ldots)$.

6.5. Suppose that $\varphi(a_1, \ldots, a_{n_0}, \alpha_1^1, \ldots, \alpha_{n_1}^1, \ldots, \alpha_1^r, \ldots, \alpha_{n_r}^r)$ is partial recursive with index z, where $n_r > 0$ if $r > 0$ Let $\mathbf{F} \to \lambda a_1 \ldots a_{n_0} \cdot \mathbf{L}(\mathbf{m}, \mathbf{z}, \mathbf{q}, \langle a_1, \ldots, a_{n_0} \rangle)$ for the m, q of [17] (6). For given $a_1, \ldots, a_{n_0}$, let $x = \langle a_1, \ldots, a_{n_0} \rangle$; then $\mathbf{F}(\mathbf{a}_1, \ldots, \mathbf{a}_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$ red $\mathbf{L}(\mathbf{m}, \mathbf{z}, \mathbf{q}, \mathbf{x}, \alpha_1^1, \ldots, \alpha_{n_r}^r)$, and by [17] (6) $\varphi(a_1, \ldots, a_{n_0}, \alpha_1^1, \ldots, \alpha_{n_r}^r) \simeq \varphi_m(z, q, x, \alpha_1^1, \ldots, \alpha_{n_r}^r)$. The conclusion of Theorem 8 follows by 6.3 and 6.4.

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