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The category of a map and of a cohomology class

by

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The purpose of this paper is to prove several results concerning the n-dimensional category of a topological space X in the sense of Fox [7] and the category of a cohomology class $u \in H^q(X; G)$ in the sense of Fary [6]. The category of a map, a concept which goes back to Fox ([7], p. 368), will play a unifying role in the present setting: among other things, we prove that, provided X is a reasonale space, both the n-dimensional category of X and the category of u coincide with the categories of certain maps of X into standard spaces of homotopy theory.

1. The category of a map. Let $f\colon X \to Y$ be a (continuous) map of arbitrary topological spaces.

Definition 1.1. $\operatorname{cat} f$ *is the least integer* $k \geqslant 1$ *with the property that* X *may be covered by* k *open subsets* U_m *such that the maps* $f\,|\,U_m\colon U_m \to Y$ *defined by* f *are nullhomotopic; if no such integer exists, we put* $\operatorname{cat} f = \infty$.

We shall denote by $\operatorname{cat} X$ the Lusternik-Schnirelmann category of X, i.e., the least integer $k \geqslant 1$ with the property that X may be covered by k open subsets which are contractible in X; if no such integer exists, $\operatorname{cat} X = \infty$.

The following results are easy to check:

1.2. $\operatorname{cat} f \leqslant \min\{\operatorname{cat} X, \operatorname{cat} Y\}$.

1.3. $\operatorname{cat} \theta = \operatorname{cat} X$ *if* θ *is the identity map of* X.

1.4. $\operatorname{cat} g \circ f \leqslant \min\{\operatorname{cat} f, \operatorname{cat} g\}$ *for any map* $g\colon Y \to Z$.

1.5. $\operatorname{cat} h_0 = \operatorname{cat} h_1$ *if* $h_t\colon X \to Y$ *is a homotopy.*

Next, since a CW-pair has the homotopy extension property and since a CW-complex is locally contractible, we have

1.6. *If a CW-complex* X *is the union of* k *subcomplexes which are contractible in* X, *then* $\operatorname{cat} X \leqslant k$.

We now prove

Proposition 1.7. *If* X *is a CW-complex and* $f\colon X \to Y$ *is an arbitrary map, then the following statements are equivalent:*

(i) *X may be covered by k open subsets U_m such that $f \circ g_m \simeq 0$ for any CW-complex L_m and any map $g_m\colon L_m \to U_m$;*

(ii) *there exists a CW-complex Z and maps $\varphi\colon X \to Z$, $\psi\colon Z \to Y$ such that Z is the union of k self-contractible subcomplexes and $f \simeq \psi \circ \varphi$;*

(iii) $\operatorname{cat} f \leqslant k$.

Proof. Select, as we may, a simplicial CW-complex L and a homotopy equivalence $g\colon L \to X$. In a suitable subdivision of L there are subcomplexes L_m such that $L = \bigcup L_m$ and $L_m \subset g^{-1}(U_m)$. Let Z denote the CW-complex which results by attaching to L a cone CL_m over each L_m; thus

$$Z = CL_1 \cup \ldots \cup CL_k \quad \text{and} \quad CL_m \cap CL_n = L_m \cap L_n .$$

Let $\omega\colon L \to Z$ denote the inclusion map, and let $g_m\colon L_m \to U_m$ be the map defined by g. We have $f \circ g_m \simeq 0$ so that $f \circ g$ extends to every CL_m yielding a map $\psi\colon Z \to Y$ such that $\psi \circ \omega = f \circ g$. Let h be a homotopy inverse of g, and let $\varphi = \omega \circ h$. Then, $f \simeq f \circ g \circ h = \psi \circ \varphi$ and we have proved that (i) implies (ii). Next, it follows from 1.6 that $\operatorname{cat} Z \leqslant k$, and by 1.5, 1.4, 1.2, we obtain $\operatorname{cat} f \leqslant k$. Finally, it is clear that (iii) implies (i).

Next we compare 1.1 with the extension to maps of the G. W. Whitehead [17] definition of a category. Let Y^k be the k-fold Cartesian power of an arbitrary space Y, and for any $y_0 \in Y$ let $T(Y, y_0; k)$ denote the subspace consisting of all points $(y_1, \ldots, y_k)$ such that $y_m = y_0$ for some m with $1 \leqslant m \leqslant k$; let $j\colon T(Y, y_0; k) \to Y^k$ be the inclusion map and let the diagonal map $\Delta_Y\colon Y \to Y^k$ be given by $\Delta_Y(y) = (y, \ldots, y)$. For any map $f\colon X \to Y$ one has $\Delta_Y \circ f = f^k \circ \Delta_X$, where f^k is the k-fold Cartesian power of f.

Proposition 1.8. *Suppose X is normal, Y is 0-connected, $y_0 \in Y$ has a neighborhood N which is contractible in Y, and let $f\colon X \to Y$ be a map. Then, $\operatorname{cat} f \leqslant k$ if and only if there is a map $g\colon X \to T(Y, y_0; k)$ such that $j \circ g \simeq \Delta_Y \circ f$.*

Proof. Let $h_t\colon X \to Y^k$ be a homotopy such that $h_0 = \Delta_Y \circ f$ and $h_1 = j \circ g$. For every m with $1 \leqslant m \leqslant k$ let $U_m = h_1^{-1}\big(p_m^{-1}(N)\big)$, where $p_m\colon Y^k \to Y$ sends $(y_1, \ldots, y_k)$ into y_m. Clearly, every U_m is open and $X = \bigcup U_m$. We have $p_m \circ h_0 = f$ and $p_m \circ h_1(U_m) \subset N$; therefore, and since N is contractible in Y, $f \mid U_m \simeq 0$ so that $\operatorname{cat} f \leqslant k$. Conversely, if $\operatorname{cat} f \leqslant k$, there are open subsets V_m of X and homotopies $h_m\colon V_m \times I \to Y$ such that

$$X = \bigcup V_m \quad \text{and} \quad h_m(x, 0) = f(x)\,, \quad h_m(x, 1) = y_m$$

for every $x \in V_m$, $1 \leqslant m \leqslant k$. Since Y is 0-connected, we may assume that $y_m = y_0$ for all m. Since X is normal, there are closed subsets A_m of X, open subsets W_m of X, and continuous functions $r_m\colon X \to I$ such that

$$X = \bigcup A_m\,, \quad A_m \subset W_m \subset \overline{W}_m \subset V_m\,,$$
$$r_m(A_m) = 1\,, \quad r_m(X - W_m) = 0\,.$$

For every m define a homotopy $d_m\colon X \times I \to Y$ by

$$d_m(x, t) = \begin{cases} f(x) & \text{if} \quad x \in X - \overline{W}_m\,, \\ h_m(x, t r_m(x)) & \text{if} \quad x \in V_m\,. \end{cases}$$

Let $d(x, t) = \big(d_1(x, t), \ldots, d_k(x, t)\big) \in Y^k$; then, clearly, $d(x, 0) = \Delta_Y \circ f(x)$, and, since every $x \in X$ belongs to some A_m, $d(x, 1) \in T(Y, y_0; k)$.

Proposition 1.9. *If X is a CW-complex and Y has a continuous multiplication, then $\operatorname{cat} f_1 f_2 \leqslant \operatorname{cat} f_1 + \operatorname{cat} f_2 - 1$ for any two maps $f_m\colon X \to Y$.*

Proof. The product map $f_1 f_2$ is equal to the composition

$$X \xrightarrow{\Delta_X} X \times X \xrightarrow{f_1 \times f_2} Y \times Y \xrightarrow{\mu} Y$$

in which μ is the multiplication. According to 1.7 and 1.6 there are CW-complexes Z_m and maps $\varphi_m\colon X \to Z_m$, $\psi_m\colon Z_m \to Y$ such that $\operatorname{cat} Z_m \leqslant \operatorname{cat} f_m$ and $f_m \simeq \psi_m \circ \varphi_m$. Then, $f_1 \times f_2$ is homotopic to the composition

$$X \times X \xrightarrow{\varphi_1 \times \varphi_2} Z_1 \times Z_2 \xrightarrow{\psi_1 \times \psi_2} Y \times Y\,,$$

and the classical theorem on the category of Cartesian products [7] may easily be extended to products of CW-complexes so as to yield $\operatorname{cat} Z_1 \times Z_2 \leqslant \operatorname{cat} Z_1 + \operatorname{cat} Z_2 - 1$. The result now follows from 1.5, 1.4, and 1.2.

We close this section by generalizing to maps some of the standard estimations of category. We use singular homology and cohomology groups with arbitrary coefficients G, but omit the symbol G whenever we work over the integers; we shall use cup products of cohomology classes with coefficients in an arbitrary commutative ring R.

Proposition 1.10. *Let $f\colon X \to Y$ be a map with $\operatorname{cat} f \leqslant k$. Then, for any cohomology classes $v_m \in H^{q_m}(Y; R)$ with $q_m \geqslant 1$ and $q = q_1 + \ldots + q_k$ one has $f^*(v_1 \cup \ldots \cup v_k) = 0 \in H^q(X; R)$.*

Proof. As in [9], 2.1, let U_m be open subsets of X such that $X = \bigcup U_m$ and $f \mid U_m \simeq 0$, $1 \leqslant m \leqslant k$. Since $q_m \geqslant 1$, in the diagram

$$\begin{array}{ccccc} H^{q_m}(X, U_m; R) & \xrightarrow{j_m^*} & H^{q_m}(X; R) & \xrightarrow{i_m^*} & H^{q_m}(U_m; R) \\ & & \uparrow{\scriptstyle f^*} & & \\ & & H^{q_m}(Y; R) & & \end{array}$$

we have $i_m^* \circ f^* = 0$ so that, by exactness, $f^*(v_m) = j_m^*(w_m)$ for some $w_m \in H^{q_m}(X, U_m; R)$. One has $w_1 \cup \ldots \cup w_k \in H^q(X, \bigcup U_m; R) = 0$ and the naturality of the cup product finally yields

$$f^*(v_1 \cup \ldots \cup v_k) = j_1^*(w_1) \cup \ldots \cup j_k^*(w_k) = j^*(w_1 \cup \ldots \cup w_k),$$

where j^* is induced by the inclusion map $j: X \to (X, \bigcup U_m)$.

PROPOSITION 1.11. *If X is a CW-complex such that $H_r(X)$ is free and $H_q(X) = 0$ for $q > r$, and if Y is a $(p-1)$-connected space $(p \geqslant 2)$, then any map $f: X \to Y$ satisfies* $\mathrm{cat} f \leqslant r/p + 1$.

Proof. If $r < p$, a standard obstruction argument yields $f \simeq 0$ and $\mathrm{cat} f = 1$. Suppose $r \geqslant p$, and let Z and $\varphi: X \to Z$ denote the CW-complex and the identification map which result by pinching to a point the $(p-1)$-skeleton X^{p-1} of X. Since Y is $(p-1)$-connected, $f \mid X^{p-1} \simeq 0$ and therefore there is a map $\psi: Z \to Y$ such that $f \simeq \psi \circ \varphi$. By 1.5, 1.4, and 1.2 we have $\mathrm{cat} f \leqslant \mathrm{cat} Z$. As is easily seen, Z is $(p-1)$-connected and

$$H_p(Z) \approx H_p(X) \oplus F, \quad H_q(Z) \approx H_q(X) \quad \text{for} \quad q > p,$$

where $\oplus$ denotes direct summation and F is a subgroup of the free group $H_{p-1}(X^{p-1})$. The extension of the Grossman [10] theorem given in [8], 1.1, now yields $\mathrm{cat} Z \leqslant r/p + 1$.

Remark 1.12. Another upper bound for $\mathrm{cat} f$, still in case X is a CW-complex and Y is a $(p-1)$-connected space $(p \geqslant 2)$, may be obtained by a similar argument replacing the Grossmann theorem by the following generalization ([8], 1.2) of a result by Eckmann and Hilton [4]:

Let Z be a 1-connected CW-complex, $E = \{q > 0 \mid H_q(Z) \neq 0\}$, $p = \min E$, $r = \max E$. If E is contained in the union of k closed linear intervals, each of length $p-2$, then $\mathrm{cat} Z \leqslant k+1$; *if $H_r(Z)$ is free, the interval containing r may have length $p-1$.*

2. The n-dimensional category.

We shall need the generalization to arbitrary maps of an invariant first defined by Fox ([7], p. 368) for inclusion maps and then by Švarc [13] for fibre maps.

DEFINITION 2.1. *The genus of a map $f: X \to Y$ is the least integer $k \geqslant 1$ for which there are open subsets V_m of Y and maps $g_m: V_m \to X$ such that $Y = \bigcup V_m$ and $f \circ g_m \simeq j_m$, where $j_m: V_m \to Y$ is the inclusion map $(1 \leqslant m \leqslant k)$; if no such integer exists, genus $f = \infty$.*

The following results are easy to check:

2.2. $\mathrm{genus} f \leqslant \mathrm{cat}\, Y$ *if Y is 0-connected.*

2.3. $\mathrm{genus}\, \theta = 1$ *if θ is the identity map of X.*

2.4. $\mathrm{genus}\, h_0 = \mathrm{genus}\, h_1$ *if $h_t: X \to Y$ is a homotopy.*

2.5. $\mathrm{genus} f \circ g = \mathrm{genus} f = \mathrm{genus}\, h \circ f$ *if $g: W \to X$ and $h: Y \to Z$ are homotopy equivalences.*

Recall that a sequence $F \xrightarrow{i} E \xrightarrow{p} B$ of spaces and maps is a fibration if i defines a homeomorphism of F onto $p^{-1}(b)$ for some $b \in B$, and if for any space A, any homotopy $h_t: A \to B$ and any map $k: A \to E$ such that $p \circ k = h_0$, there is a homotopy $k_t: A \to E$ such that $k_0 = k$ and $p \circ k_t = h_t$.

PROPOSITION 2.6. *Let $\mathcal{F}: F \xrightarrow{i} E \xrightarrow{p} B$ be a sequence of spaces and maps. If $p \circ i \simeq 0$, then $\mathrm{cat}\, p \leqslant \mathrm{genus}\, i$; if $\mathcal{F}$ is a fibration and B is 0-connected, then $\mathrm{cat}\, p = \mathrm{genus}\, i$.*

Proof. Let V be an open subset of E with inclusion map j. If $g: V \to F$ satisfies $i \circ g \simeq j$, then $p \mid V = p \circ j \simeq p \circ i \circ g \simeq 0$. Next, suppose that $\mathcal{F}$ is a fibration and let $h_t: V \to B$ be a homotopy such that $h_0 = p \circ j$ and $h_1(V) = b$. Since B is 0-connected, we may assume that $b = p \circ i(F)$. Also, there is a homotopy $k_t: V \to E$ such that $k_0 = j$ and $p \circ k_t = h_t$; therefore, $k_1(V) \subset p^{-1}(b)$ and there is a map $g: V \to F$ such that $i \circ g \simeq j$.

PROPOSITION 2.7. *If Y is a CW-complex and $f: X \to Y$ is an arbitrary map, then the following statements are equivalent:*

(i) *Y may be covered by k open subsets V_m with the property that for any CW-complexes L_m and any maps $h_m: L_m \to V_m$ there are maps $g_m: L_m \to X$ such that $f \circ g_m \simeq j_m \circ h_m$, where $j_m: V_m \to Y$ are the inclusion maps;*

(ii) $\mathrm{genus} f \leqslant k$.

Proof. Select, as we may, a simplicial CW-complex L and a homotopy equivalence $h: L \to Y$. In a suitable subdivision of L there are subcomplexes L_m such that $L = \bigcup L_m$ and $L_m \subset h^{-1}(V_m)$. Let $h_m: L_m \to V_m$ be the map defined by h and let $g_m: L_m \to X$ satisfy $f \circ g_m \simeq j_m \circ h_m$. There are open subsets U_m of L and maps $r_m: U_m \to L_m$ such that $U_m \supset L_m$ and $e_m \circ r_m \simeq i_m$, where $e_m: L_m \to L$ and $i_m: U_m \to L$ are the inclusion maps. Let d be a homotopy inverse of h. Then,

$$d \circ f \circ g_m \circ r_m \simeq d \circ j_m \circ h_m \circ r_m = d \circ h \circ e_m \circ r_m \simeq d \circ h \circ i_m \simeq i_m$$

so that $\mathrm{genus}\, d \circ f \leqslant k$ and, by 2.5, $\mathrm{genus} f \leqslant k$. The converse is obvious and the proof is complete.

Following J. H. C. Whitehead ([18], p. 214) we re-define the n-dimensional category of a space using maps of arbitrary n-dimensional CW-complexes instead of maps of finite n-dimensional polyhedra.

DEFINITION 2.8. *For any $n \geqslant 0$, $\mathrm{cat}_n X$ is the least integer $k \geqslant 1$ with the property that X may be covered by k open n-categorical subsets, i.e. open subsets U_m such that, for any map $h: L \to U_m$ of any CW-complex L of dimension $\leqslant n$, the map $L \to X$ defined by h is nullhomotopic; if no such integer exists, $\mathrm{cat}_n X = \infty$.*

Evidently, $\mathrm{cat}_n X \leqslant \mathrm{cat}\, X$.

We shall be concerned with spaces obtained by killing off the homotopy groups in dimensions $\leqslant n$ or $> n$ of a given 0-connected space.

Theorem 2.9. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z$ *be maps of connected* CW-*complexes. Let* $n \geqslant 1$. *If*

$$\pi_q(X) = 0 \quad \text{for} \quad q \leqslant n \quad \text{and} \quad f_q\colon \pi_q(X) \approx \pi_q(Y) \quad \text{for} \quad q > n,$$
$$\pi_q(Z) = 0 \quad \text{for} \quad q > n \quad \text{and} \quad g_q\colon \pi_q(Y) \approx \pi_q(Z) \quad \text{for} \quad q \leqslant n,$$

then $\operatorname{genus} f = \operatorname{cat}_n Y = \operatorname{cat} g$.

Proof. Let V be an open subset of Y with inclusion map $j\colon V \to Y$.

Suppose that V is n-categorical, let L be an arbitrary CW-complex with n-skeleton L^n, and let $h\colon L \to V$ be an arbitrary map. Use of the mapping cylinder of f will enable us to assume that f is an inclusion map. Then, we have $\pi_q(Y, X) = 0$ for $q > n$, and the assumption on V implies that $j \circ h \mid L^n \simeq 0$. Standard deformation arguments now yield a map $d\colon L \to X$ such that $f \circ d \simeq j \circ h$, and by 2.7 we obtain $\operatorname{genus} f \leqslant \operatorname{cat}_n Y$.

Next, suppose $g \circ j \simeq 0$, let L be an arbitrary CW-complex of dimension $\leqslant n$, and let $h\colon L \to V$ be an arbitrary map. Without altering the homotopy type of Y we may assume that g is a fibre map with fibre F and inclusion map $i\colon F \to Y$. Since $g \circ j \circ h \simeq 0$ and Z is connected, there is a map $d\colon L \to F$ such that $i \circ d \simeq j \circ h$. We have $\pi_q(F) = 0$ for $q \leqslant n$ so that, since $\dim L \leqslant n$, $d \simeq 0$; therefore, $j \circ h \simeq 0$ and we have proved that $\operatorname{cat}_n Y \leqslant \operatorname{cat} g$.

Finally, since $\pi_q(X) = 0$ for $q \leqslant n$, the n-skeleton X^n of X is contractible in X and $g \circ f$ extends to $X \cup CX^n$, where CX^n is the cone over X^n; since $\pi_q(Z) = 0$ for $q > n$, we may further extend $g \circ f$ to the cone over the whole of X so that $g \circ f \simeq 0$, and, by 2.6, we obtain $\operatorname{cat} g \leqslant \operatorname{genus} f$.

Under slightly different assumptions, similar arguments yield results related to 2.9; thus, one has

Proposition 2.10. *Let* $g\colon Y \to Z$ *be a map of connected* CW-*complexes, and let* $n \geqslant 1$. *If* $\pi_q(Z) = 0$ *for* $q > n$ *and if the homomorphism* $g_n\colon \pi_n(Y) \to \pi_n(Z)$ *is trivial, then* $\operatorname{cat} g \leqslant \operatorname{cat}_{n-1} Y$.

Proof. Let V be an open $(n-1)$-categorical subset of Y. Let L be an arbitrary CW-complex with m-skeleton L^m, and let $h\colon L \to Y$ be an arbitrary map such that $h(L) \subset V$. Since $h \mid L^{n-1} \simeq 0$, h extends to a map $k\colon L \cup CL^{n-1} \to Y$. Since the homomorphism

$$g_n \circ k_n\colon \pi_n(L \cup CL^{n-1}) \to \pi_n(Z)$$

is trivial, $g \circ k$ extends to a map $f\colon L \cup CL^n \to Z$. Finally, since $\pi_q(Z) = 0$ for $q > n$, f extends to a map $d\colon CL \to Z$. Therefore, $g \circ h = d \circ i \simeq 0$, where $i\colon L \to CL$ is the inclusion map, and, by 1.7, we have $\operatorname{cat} g \leqslant \operatorname{cat}_{n-1} Y$.

3. The category of a cohomology class.

Let X be an arbitrary space, G an Abelian group, and $n \geqslant 1$ an integer.

The following definition is essentially due to Fary [6]:

Definition 3.1. *For any* $u \in H^n(X; G)$, $\operatorname{cat} u$ *is the least integer* $k \geqslant 1$ *with the property that* X *may be covered by* k *open subset* U_m *such that* $j_m^*(u) = 0$, *where* $j_m^*\colon H^n(X; G) \to H^n(U_m; G)$ *is induced by the inclusion map* $j_m\colon U_m \to X$; *if no such integer exists*, $\operatorname{cat} u = \infty$.

Let $K(G, n)$ denote any Eilenberg-MacLane CW-complex L such that $\pi_n(L) \approx G$ and $\pi_q(L) = 0$ for $q \neq n$; the homotopy type of L is uniquely determined. The group $H^n(K(G, n); G)$ may be identified with the group $\operatorname{Hom}(G, G)$, and the identity map of G then corresponds to the fundamental class $\iota \in H^n(K(G, n); G)$. If X is a CW-complex, the set $\pi(X, K(G, n))$ of homotopy classes of maps $X \to K(G, n)$ and the group $H^n(X; G)$ are in a 1:1 correspondence which is given by

$$f \to f^*(\iota) \quad \text{where} \quad f^*\colon H^n(K(G, n); G) \to H^n(X; G)$$

is induced by $f\colon X \to K(G, n)$. For any $u \in H^n(X; G)$ we shall denote by f_u any of the (homotopically equivalent) maps $f\colon X \to K(G, n)$ such that $f^*(\iota) = u$.

Theorem 3.2. *If* X *is a* CW-*complex and if* $u \in H^n(X; G)$, *then* $\operatorname{cat} u = \operatorname{cat} f_u$.

Proof. Let V be an open subset of X with inclusion map $j\colon V \to X$. If $f_u \circ j \simeq 0$, then $j^*(u) = j^* \circ f_u^*(\iota) = 0$ so that $\operatorname{cat} u \leqslant \operatorname{cat} f_u$. Conversely, if $j^*(u) = 0$, then, for any CW-complex L and any map $h\colon L \to V$, one has $h^* \circ j^* \circ f_u^*(\iota) = 0$ so that $f_u \circ j \circ h \simeq 0$ and, by 1.7, we have $\operatorname{cat} f_u \leqslant \operatorname{cat} u$.

From 3.2 and 1.11 we obtain

Theorem 3.3. *If* X *is a* CW-*complex of dimension* $\leqslant r$ *and if* $u \in H^n(X; G)$, *then* $\operatorname{cat} u \leqslant r/n + 1$.

Proposition 3.4. *If* X *is a* CW-*complex and if* $u, v \in H^n(X; G)$, *then* $\operatorname{cat}(-u) = \operatorname{cat} u$ *and* $\operatorname{cat}(u+v) \leqslant \operatorname{cat} u + \operatorname{cat} v - 1$.

Proof. The first statement is obvious by 3.1. In order to derive the second, recall that $K(G, n)$ has an H-space structure which converts the set $\pi(X, K(G, n))$ into a group which is isomorphic to $H^n(X; G)$ under the 1:1 correspondence displayed above. Therefore, $f_{u+v} = f_u f_v$ and the result now follows from 3.2 and 1.9.

Proposition 3.5. *If* X *is any space and if* $u_m \in H^{q_m}(X; R)$, *then* $\operatorname{cat}(u_1 \cup u_2) \leqslant \min\{\operatorname{cat} u_1, \operatorname{cat} u_2\}$.

Proof. If U with the inclusion map $i\colon U \to X$ is an open subset of X such that $i^*(u_1) = 0$, then $i^*(u_1 \cup u_2) = i^*(u_1) \cup i^*(u_2) = 0$ so that $\operatorname{cat}(u_1 \cup u_2) \leqslant \operatorname{cat} u_1$; similarly, $\operatorname{cat}(u_1 \cup u_2) \leqslant \operatorname{cat} u_2$ and 3.5 is proved.

Proposition 3.6. *If* X *is an arbitrary space and* $v \in H^n(X; G')$ *is the image of* $u \in H^m(X; G)$ *under a cohomology operation* T, *then* $\operatorname{cat} v \leqslant \operatorname{cat} u$.

Proof. Let U with the inclusion map $j\colon U\to X$ be an open subset of X such that $j^*(u)=0$. Then, commutativity in the diagram

$$\begin{array}{ccc} H^m(X;G) & \xrightarrow{j^*} & H^m(U;G) \\ \downarrow T_X & & \downarrow T_U \\ H^n(X;G') & \xrightarrow{j'^*} & H^n(U;G') \end{array}$$

yields $j'^*(v)=j'^*\circ T_X(u)=T_U\circ j^*(u)=0$, and 3.6 is proved.

PROPOSITION 3.7. *Let $g\colon E\to X$ be a map of arbitrary spaces and let $u\in H^n(X;G)$. Then*

(i) $\operatorname{cat} g^*(u)\leqslant \operatorname{cat} u$;

(ii) *if X is a connected* CW-*complex*, $\operatorname{cat} u\leqslant k$ *if and only if there is a connected* CW-*complex Z, a cohomology class $w\in H^n(Z;G)$, and a map $\varphi\colon X\to Z$ such that* $\operatorname{cat} Z\leqslant k$ *and* $u=\varphi^*(w)$.

Proof. If U with inclusion map $j\colon U\to X$ is an open subset of X such that $j^*(u)=0$, then $i^*\circ g^*(u)=d^*\circ j^*(u)=0$, where $N=g^{-1}(U)$, $i\colon N\to E$ is the inclusion map, and $d\colon N\to U$ is defined by g. This proves (i). The sufficiency part of (ii) is an immediate consequence of (i) and of the obvious fact that $\operatorname{cat} w\leqslant \operatorname{cat} Z$. Finally, if $\operatorname{cat} u\leqslant k$, then, by 3.2, $\operatorname{cat} f_u\leqslant k$ so that, by 1.7, there is a connected CW-complex Z and maps $\varphi\colon X\to Z$, $\psi\colon Z\to K(G,n)$ such that $\operatorname{cat} Z\leqslant k$ and $f_u\simeq\psi\circ\varphi$; with $w=\psi^*(\iota)$ one has $u=f_u^*(\iota)=\varphi^*(w)$, and the proof is complete.

Next, we shall establish relations between the n-dimensional category of X and the category of certain cohomology classes. We shall need a slight generalization of cohomology suspension, which is based on consideration of the spaces and maps

$$E(X^k;T_k,\Delta_k)\quad\text{and}\quad p_k\colon E(X^k;T_k,\Delta_k)\to X$$

associated with any space X with base-point $x_0\in X$ and any $k\geqslant 2$. With the notation of 1.8, these are defined as follows: $E(X^k;T_k,\Delta_k)$ is the compact-open topologized space of all paths $\lambda=(\lambda_1,\dots,\lambda_k)$ in the k-fold Cartesian power X^k which emanate from the subset $T_k=T(X,x_0;k)$ and end in the diagonal subset Δ_k of X^k; p_k is given by $p_k(\lambda)=\Delta_X^{-1}(\lambda(1))$, where $\Delta_X\colon X\to X^k$ is the diagonal map.

THEOREM 3.8. *Let X be a connected* CW-*complex with base-point a 0-cell $x_0\in X$. Let $n\geqslant 1$ and $u\in H^n(X;G)$. Then*

(i) $\operatorname{cat} u\leqslant \operatorname{cat}_n X$;

(ii) $\operatorname{cat} u\leqslant(\operatorname{cat}_n X+k-1)/k$ *if* $p_k^*(u)=0$;

(iii) $\operatorname{cat} u\leqslant(\operatorname{cat}_n X+1)/2$ *if u is in the kernel of the cohomology suspension*;

(iv) $\operatorname{cat} u\leqslant \operatorname{cat}_{n-1} X$ *if u is a spherical annihilator*;

(v) $\operatorname{cat}(u_1\cup\dots\cup u_k)\leqslant(\operatorname{cat}_n X+k-1)/k$ *if* $u_m\in H^{q_m}(X;R)$ *and* $1\leqslant q_m\leqslant n$ *for* $m=1,\dots,k$.

Proof. The first statement follows from 3.2 and 2.10 with n replaced by $n+1$.

In order to prove (ii) it obviously suffices to show that the inequality $\operatorname{cat}_n X\leqslant rk$ implies $\operatorname{cat} u\leqslant r$. Let then V_{sm} $(1\leqslant s\leqslant r,\ 1\leqslant m\leqslant k)$ be open n-categorical subsets which cover X and let $V_s=\bigcup V_{sm}$. Let L, with n-skeleton $L^{(n)}$, be an arbitrary CW-complex and let $h\colon L\to X$ be an arbitrary map such that $h(L)\subset V_s$. Without altering the homotopy type of L we may assume that it is a simplicial CW-complex, and then in a suitable subdivision of $L^{(n)}$ there are subcomplexes L_m such that $L^{(n)}=\bigcup L_m$ and $h(L_m)\subset V_{sm}$. Therefore, with $d=h\,|\,L^{(n)}$, we have $d\,|\,L_m\simeq 0$. Since the CW-pair $(L^{(n)},L_m)$ has the homotopy extension property and since X is 0-connected, we may invoke the last argument in the proof of 1.8 to obtain a map $g\colon L^{(n)}\to T(X,x_0;k)$ and a homotopy $H_t\colon L^{(n)}\to X^k$ such that $H_0=j\circ g$ and $H_1=\Delta_X\circ d$, where j embeds $T(X,x_0;k)$ in X^k. Define a map $e\colon L^{(n)}\to E(X^k;T_k,\Delta_k)$ by setting $e(a)(t)=H_t(a)$ for every $a\in L^{(n)}$ and $t\in I$. Then, $p_k\circ e=d$ and, therefore,

$$d^*\circ f_u^*(\iota)=e^*\circ p_k^*\circ f_u^*(\iota)=e^*\circ p_k^*(u)=0\in H^n(L^{(n)};G)\,.$$

As a consequence, the map $f_u\circ d\colon L^{(n)}\to K(G,n)$ is nullhomotopic so that the map $f_u\circ h$ extends to a map $L\cup CL^{(n)}\to K(G,n)$ which, since $\pi_q(K(G,n))=0$ for $q>n$, may further be extended to a map $CL\to K(G,n)$. As a result, $f_u\circ h\simeq 0$ and, by 1.7, $\operatorname{cat} f_u\leqslant r$ so that, by 3.2, $\operatorname{cat} u\leqslant r$, and (ii) is proved.

The proof of (iii) will be based on (ii) and on several arguments essentially due to G. W. Whitehead [16] Introduce the subsets

$$E_0=\{\lambda\mid\lambda(0)=x_0\}\,,\qquad E_1=\{\lambda\mid\lambda(1)=x_0\}\,,\qquad \Omega=E_0\cap E_1$$

of the space X^I of all paths λ in X, and define a homeomorphism

$$\Phi\colon E(X^2;T_2,\Delta_2)\to E_0\cup E_1$$

by

$$\Phi(\lambda_1,\lambda_2)(s)=\begin{cases}\lambda_1(2s) & \text{if}\quad 0\leqslant s\leqslant\tfrac12\,,\\ \lambda_2(2-2s) & \text{if}\quad \tfrac12\leqslant s\leqslant 1\,.\end{cases}$$

With ε denoting the constant path at x_0 in X, we have the diagram

$$\begin{array}{ccccccc} H^n(E_0\cup E_1,\varepsilon;G) & \xleftarrow{j^*} & H^n(E_0\cup E_1,E_1;G) & \xrightarrow{k^*} & H^n(E_0,\Omega;G) & \xleftarrow{\delta} & H^{n-1}(\Omega,\varepsilon;G) \\ \downarrow\Phi^* & & & \nwarrow r^* & \uparrow p^* & & \\ H^n\big(E(X^2;T_2,\Delta_2),(\varepsilon,\varepsilon);G\big) & & \xleftarrow{\quad p_2^*\quad} & & H^n(X,x_0;G) & & \end{array}$$

in which j^* and k^* are induced by inclusion maps, while p^* and r^* are induced by the maps p and r given by $\lambda\to\lambda(1)$. Define a homotopy $h_t\colon\big(E(X^2;T_2,\Delta_2),(\varepsilon,\varepsilon)\big)\to(X,x_0)$ by $h_t(\lambda_1,\lambda_2)=\lambda_2(t)$ so that

$$h_0(\lambda_1,\lambda_2)=\Phi(\lambda_1,\lambda_2)(1)=r\circ j\circ\Phi(\lambda_1,\lambda_2)\,,\qquad h_1(\lambda_1,\lambda_2)=p_2(\lambda_1,\lambda_2)$$

and $p_2^* = \Phi^* \circ j^* \circ r^*$. It is also clear that $p^* = k^* \circ r^*$. Now, since $(X^I; E_0, E_1)$ has the homotopy type of a CW-triad [11], the excision k^* is isomorphic. Since E_0 and E_1 are contractible, consideration of the cohomology sequences of the triples $(E_0 \cup E_1, E_1, \varepsilon)$ and $(E_0, \Omega, \varepsilon)$ reveals that j^* and δ are isomorphic. Therefore, $\Phi^* \circ j^* \circ (k^*)^{-1} \circ \delta$ is an isomorphism and

$$p_2^* = \Phi^* \circ j^* \circ (k^*)^{-1} \circ \delta \circ \sigma^* \quad \text{where} \quad \sigma^* = \delta^{-1} \circ p^*$$

is the standard cohomology suspension. As a result, $\sigma^*(u) = 0$ implies $p_2^*(u) = 0$ and, by (ii), we obtain $\operatorname{cat} u \leqslant (\operatorname{cat}_n X + 1)/2$.

The fourth statement is an immediate consequence of 3.2, 2.10, and of the fact that u is a spherical annihilator if and only if the homomorphism $(f_u)_n\colon \pi_n(X) \to \pi_n(K(G, n))$ is trivial.

In order to prove (v), again let V_{sm} $(1 \leqslant s \leqslant r, 1 \leqslant m \leqslant k)$ be open n-categorical subsets which cover X and let $V_s = \bigcup V_{sm}$. Let

$$V_{sm} \xrightarrow{i_{sm}} X \xrightarrow{j_{sm}} (X, V_{sm}), \qquad V_s \xrightarrow{i_s} X \xrightarrow{j_s} (X, V_s)$$

be inclusion maps. Since V_{sm} certainly is q_m-categorical, by 3.2 and the proof of 2.10 we have $i_{sm}^*(u_m) = 0$ so that, by exactness, there is a $v_{sm} \in H^{q_m}(X, V_{sm}; R)$ such that $u_m = j_{sm}^*(v_{sm})$. Therefore,

$$u_1 \cup \ldots \cup u_k = j_s^*(v_{s1} \cup \ldots \cup v_{sk})$$

and, by exactness, $i_s^*(u_1 \cup \ldots \cup u_k) = 0$. Evidently, $X = \bigcup V_s$ and we have proved that the inequality $\operatorname{cat}_n X \leqslant rk$ implies $\operatorname{cat}(u_1 \cup \ldots \cup u_k) \leqslant r$.

Remark 3.9. It is well known that a cohomology class which is annihilated by cohomology suspension is a spherical annihilator; therefore, in (iii) we also have $\operatorname{cat} u \leqslant \operatorname{cat}_{n-1} X$. Next, it follows from (v) that $\operatorname{cat}_n X \geqslant k+1$ if X has a non-trivial k-fold cup product of cohomology classes of positive dimensions $\leqslant n$; this is the "n-dimensional" version of the well known result according to which $\operatorname{cat} X \geqslant k+1$ if X has a non-trivial k-fold cup product of positive dimensional cohomology classes. Finally, in connection with (i) we point out

PROPOSITION 3.10. *If X is an $(n-1)$-connected n-simple $(n \geqslant 1)$ CW-complex and if $u \in H^n(X; \pi_n(X))$ is the primary obstruction to contracting X into a point, then $\operatorname{cat} u = \operatorname{cat}_n X$.*

Proof. One has $H^n(X; \pi_n(X)) \approx \operatorname{Hom}(H_n(X), \pi_n(X))$ and u corresponds to the inverse Hurewicz isomorphism $\varrho^{-1}\colon H_n(X) \to \pi_n(X)$. The map $f_u\colon X \to K(\pi_n(X), n)$ induces an isomorphism of homotopy groups in dimension n and the result now follows from 3.2 and 2.9.

A lower bound for the category of a cohomology class is provided by

THEOREM 3.11. *If X is a CW-complex and if the k-fold cup product $u \cup \ldots \cup u$ of a positive dimensional cohomology class $u \in H^n(X; R)$ is non-trivial, then $\operatorname{cat} u \geqslant k+1$.*

Proof. One has $f_u^*(\iota \cup \ldots \cup \iota) = u \cup \ldots \cup u \neq 0$ so that, by 3.2 and 1.10, $\operatorname{cat} u = \operatorname{cat} f_u \geqslant k+1$.

Remark 3.12. Let M_n be the complex projective space of $2n-2$ real dimensions. One has $\pi_1(M_n) = 0$ so that, by [10], $\operatorname{cat} M_n \leqslant n$. Also, $H^{2q}(M_n) \approx Z$ for $1 \leqslant q \leqslant n-1$, where Z is the ring of integers. If $u \in H^2(M_n)$ is a generator, then $\operatorname{cat} u \leqslant \operatorname{cat} M_n \leqslant n$ and, since the $(n-1)$-fold cup product $u \cup \ldots \cup u$ generates $H^{2n-2}(M_n)$, by 3.11 one has $\operatorname{cat} u \geqslant n$. Thus, the category of a cohomology class may assume any given value.

4. The category of a rational cohomology class. The purpose of this section is to provide a recipe for the computation of the category of a cohomology class with rational coefficients. We shall use reduced homology groups, and consistently denote by Q the additive group or the field of rational numbers.

LEMMA 4.1. *Let W be a 1-connected space and let $n \geqslant 3$ be an odd integer. Then, the following statements are equivalent:*

(i) *$H_n(W) \approx Q$ and $H_q(W) = 0$ if $q \neq n$;*

(ii) *$\pi_n(W) \approx Q$ and $\pi_q(W) = 0$ if $q \neq n$.*

Proof. The result could be derived within the framework of the Cartan [3] computation of the groups $H(\pi, n)$. A more geometric proof runs as follows. For every integer $r \geqslant 1$ select two copies X_r and Y_r of the n-sphere S^n and let $f_r\colon X_r \to Y_r$ be a cellular map of degree r. Let C_r denote the mapping cylinder of f_r in which X_r and Y_r are embedded in the standard way. Let K_s denote the CW-complex which results from the disjoint union $C_1 \cup \ldots \cup C_s$ upon identifying Y_r with X_{r+1} for all $r < s$. We have $K_s \subset K_{s+1}$ and we define a CW-complex K as the union $\bigcup K_s$ with the weak topology. The homology and homotopy groups of K are the direct limits of the systems consisting of the homology or homotopy groups of the K_s with the homomorphisms induced by the inclusion maps $j_s\colon K_s \to K_{s+1}$, $s = 1, 2, \ldots$ As an immediate consequence we have

$$\pi_q(K) = H_q(K) = 0 \quad \text{if} \quad q < n, \qquad H_q(K) = 0 \quad \text{if} \quad q > n.$$

Notice next that the inclusion map $K_s \to K_{s+t}$ is homotopically equivalent to the map $f_{s+t} \circ \ldots \circ f_{s+1}\colon S^n \to S^n$. Therefore, $(j_s)_*(a_s) = (s+1)a_{s+1}$, where a_m is a generator of the free cyclic group $H_n(K_m)$, so that

$$H_n(K) \approx Q, \quad \text{whence} \quad \pi_n(K) \approx Q.$$

Now, select an integer $q > n$ and recall that, since n is odd, $\pi_q(S^n)$ is a finite group ([14], Chap. V, 3); therefore, it follows from [15], Chap. V, 1, that, provided t is large enough, the map $f_{s+t} \circ \ldots \circ f_{s+1}$, whose degree is $(s+1)\ldots(s+t)$, induces the trivial homomorphism of $\pi_q(S^n)$ into itself. Obviously, this implies that

$$\pi_q(K) = 0 \quad \text{if} \quad q > n .$$

Thus we have constructed a CW-complex K which satisfies both (i) and (ii), and the assumption that $n \geqslant 3$ has not yet been used. Now, if W is an arbitrary 1-connected space satisfying either (i) or (ii), then, since $n \geqslant 3$, its singular polytope will have the homotopy type of K, and 4.1 is proved.

COROLLARY 4.2. *If $n \geqslant 3$ is odd, then* $\operatorname{cat} K(Q, n) = 2$.

Proof. According to [12], Appendice, there exists (and in the proof of 4.1 we have actually constructed) a 1-connected CW-complex X such that $H_{n-1}(X) \approx Q$ and $H_q(X) = 0$ if $q \neq n-1$. The suspension ΣX is 1-connected, $H_n(\Sigma X) \approx Q$, $H_q(\Sigma X) = 0$ if $q \neq n$, and, evidently, $\operatorname{cat} \Sigma X = 2$. The result now follows upon noticing that, by 4.1, $K(Q, n)$ has the homotopy type of ΣX.

THEOREM 4.3. *Let X be a CW-complex and let $u \in H^n(X; Q)$, $u \neq 0$, $n > 1$. Then*

(i) *if n is odd,* $\operatorname{cat} u = 2$;

(ii) *if n is even,* $\operatorname{cat} u$ *is equal to the least (finite or infinite) integer $k > 1$ for which the k-fold cup product $u \cup \ldots \cup u$ vanishes.*

Proof. The first statement is an immediate consequence of 3.2, 1.2, and 4.2. In order to prove (ii), notice first, that as an immediate consequence of 3.11, we have $\operatorname{cat} u \geqslant k$. To prove the converse inequality let $Y = K(Q, n)$ and let $f_u\colon X \to Y$ satisfy $f_u^*(\iota) = u$, where $\iota \in H^n(Y; Q)$ is the fundamental class. We may assume that Y is a countable CW-complex whose $(n-1)$-skeleton is a 0-cell $y_0 \in Y$. As a result, with the notation of 1.8 we have (¹)

$$(1) \qquad (Y^k)^{(kn-1)} \subset T_k$$

where $T_k = T(Y, y_0; k)$. Let M denote the space of all paths in Y^k which emanate from T_k and end in the point $(y_0, \ldots, y_0) \in Y^k$; as is easily seen,

$$(2) \qquad \pi_q(M) \approx \pi_{q+1}(Y^k, T_k) \quad \text{for all} \quad q \geqslant 0 .$$

Since $n > 1$ and $k > 1$, it follows from (1) that T_k certainly contains the 3-skeleton of Y^k so that $\pi_k(Y^k, T_k) = 0$ for $1 \leqslant q \leqslant 2$, and, by (2),

$$(3) \qquad M \text{ is 1-connected} .$$

(¹) The p-skeleton of any CW-complex L will be denoted by $L^{(p)}$; L^p will denote the p-fold Cartesian power of L, which is known to be a CW-complex if L is countable.

Let EY and EY^k denote the spaces of all paths in Y, respectively Y^k, which end in the point y_0, respectively $(y_0, \ldots, y_0)$. As noticed in [1] p. 446, the pair (EY^k, M) is homeomorphic to the k-fold Cartesian power $(EY, \Omega Y)^k$ of the pair $(EY, \Omega Y)$, where ΩY is the loop space of Y. Since Y is a CW-complex, it follows from [11] that $(EY, \Omega Y)$ has the homotopy type of a CW-pair and we may use the relative Künneth theorem to calculate the homology of $(EY, \Omega Y)^k$. Since ΩY is a space of type $(Q, n-1)$ and n is even, by 4.1 we have

$$(4) \qquad H_{n-1}(\Omega Y) \approx Q \quad \text{and} \quad H_q(\Omega Y) = 0 \quad \text{if} \quad q \neq n-1 .$$

Therefore, the homology of ΩY is torsion-free and, since EY and EY^k are contractible, for $q > 0$ we obtain

$$(5) \qquad H_q(M) \approx \sum H_{q_1}(\Omega Y) \otimes \ldots \otimes H_{q_k}(\Omega Y)$$

with (direct) summation extended over all sequences $(q_1, \ldots, q_k)$ such that $q_1 + \ldots + q_k = q - k + 1$ and $q_i > 0$ for all i. From (4) and (5), and since $Q \otimes \ldots \otimes Q = Q$, we obtain

$$H_{kn-1}(M) \approx Q \quad \text{and} \quad H_q(M) = 0 \quad \text{if} \quad q \neq kn-1 .$$

Since $kn-1$ is odd, (3) and 4.1 now imply that M is a space of type $(Q, kn-1)$ so that, by (2), we finally obtain

$$(6) \qquad \pi_q(Y^k, T_k) = 0 \quad \text{for} \quad q \neq kn .$$

According to 1.8, in order to prove that $\operatorname{cat} u \leqslant k$ it suffices to show that the map $\Delta_Y \circ f_u\colon X \to Y^k$ may be compressed into T_k. Now, by (1) and the cellular approximation theorem, we may already assume that

$$\Delta_Y \circ f_u(X^{(kn-1)}) \subset T_k .$$

Next, the primary obstruction to compressing $\Delta_Y \circ f_u$ into T_k is given by the element

$$f_u^* \circ \Delta_Y^* \circ j^*(\theta) \in H^{kn}(X; \pi_{kn}(Y^k, T_k)) ,$$

where

$$j^*\colon H^{kn}(Y^k, T_k; \pi_{kn}(Y^k, T_k)) \to H^{kn}(Y^k; \pi_{kn}(Y^k, T_k))$$

is induced by inclusion, and

$$\theta \in H^{kn}(Y^k, T_k; \pi_{kn}(Y^k, T_k))$$

is the fundamental class of the $(kn-1)$-connected pair (Y^k, T_k). Upon identifying $\pi_{kn}(Y^k, T_k)$ with

$$H_{kn}(Y^k, T_k) = H_n(Y, y_0) \otimes \ldots \otimes H_n(Y, y_0) = Q \otimes \ldots \otimes Q ,$$

θ becomes the identity map in the group

$$\operatorname{Hom}\big(H_{kn}(Y^k, T_k), Q\otimes\ldots\otimes Q\big)$$

and, therefore, $j^*(\theta)$ is equal to the k-fold cross product $\iota\times\ldots\times\iota$. Since $Q\otimes\ldots\otimes Q = Q$, we obatin

$$f_u^*\circ\Delta_Y^*\circ j^*(\theta) = f_u^*\circ\Delta_Y^*(\iota\times\ldots\times\iota) = f_u^*(\iota\cup\ldots\cup\iota) = u\cup\ldots\cup u\,.$$

By assumption, we have $u\cup\ldots\cup u = 0$ so that, as in the proof of the Pontrjagin-Postnikov theorem ([2]; § 10), the map $\Delta_Y\circ f_u \mid X^{(kn)}$ may be compressed into T_k. Use of (6) with $q > kn$ implies that the compression of $\Delta_Y\circ f_u\mid X^{(kn)}$ may be further extended to a compression of $\Delta_Y\circ f_u$ over the whole of X, and the proof is complete.

We now examine the case $n = 1$.

Proposition 4.4 *Let X be a CW-complex and let $u\in H^1(X; Q)$, $u\neq 0$. Then*

(i) $\operatorname{cat} u\leqslant 3$ *and the case* $\operatorname{cat} u = 3$ *is actually possible*;

(ii) *if X is finite*, $\operatorname{cat} u = 2$.

Proof. According to 3.2 and 1.2 we have $\operatorname{cat} u\leqslant\operatorname{cat} K(Q, 1)$; also, by 1.3, we have $\operatorname{cat}\iota = \operatorname{cat} K(Q,1)$, where $\iota\in H^1\big(K(Q,1); Q\big)$ is the fundamental class. Therefore, in order to prove (i) it suffices to show that $\operatorname{cat} K(Q,1) = 3$. This is a consequence of the following two remarks: as in the proof of 4.1, $K(Q,1)$ may be constructed so as to have dimension 2 and, therefore, $\operatorname{cat} K(Q,1)\leqslant 3$; secondly, since $H^2\big(K(Q,1)\big)\neq 0$, it follows from [5] that $\operatorname{cat} K(Q,1)\geqslant 3$. In order to prove (ii), we shall again take for $K(Q,1)$ the CW-complex $K = \bigcup K_s$ constructed in the proof of 4.1. Then, for some $s\geqslant 1$, the compact subset $f_u(X)$ of K is contained in the subcomplex K_s and, since the latter is homotopically equivalent to a 1-sphere, we have $\operatorname{cat} f_u = 2$.

Remark. 4.5. The preceding results do not hold for finitely generated coefficient groups. For, the Cartan [3] computation of the groups $H(\pi, n)$ shows that if G is a finitely generated Abelian group and if $n\geqslant 2$, then, for some prime $p\geqslant 2$, the cohomology ring of $K(G, n)$ with coefficients in Z_p contains non-trivial cup products of arbitrarily many positive dimensional elements. Thus, we have

Proposition 4.6. *If G is a finitely generated Abelian group and if $n\geqslant 2$, then* $\operatorname{cat} K(G, n) = \infty$.

This is also the reason for which most of the estimations given in § 3 involve only invariants of X disregarding the coefficient group.

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