E_3: l'ensemble

$$\left\{\left(\frac{1}{n}, y\right):\ n = 1, 2, \dots, 0 \leqslant y \leqslant \frac{1}{n}\right\} \cup \{(x, 0):\ 0 \leqslant x \leqslant 1\}$$

du plan euclidien;

E_4: l'ensemble

$$\bigcup_{n=1}^{\infty} \{(x, y):\ 0 \leqslant x \leqslant 1/n^2,\ y = nx\}$$

du plan euclidien.

L'espace E_i ($i = 1, \dots, 4$) satisfait aux conditions (4.1)-(4.4), excepté (4.i).

Remarquons encore que les représentants des graphes finis et connexes, appelés *courbes ordinaires* par K. Menger, ont été caractérisés topologiquement par cet auteur de la façon suivante (voir [5], p. 304 et [6], p. 266; cf. aussi [2], p. 325, théorème V):

Pour que l'espace G soit homéomorphe à un représentant d'un graphe fini et connexe, il faut et il suffit que G soit un continu métrisable, ne contenant que des points d'ordre fini et un nombre fini de points de ramification.

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Some remarks on Borsuk generalized cohomotopy groups *

by

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1. In [3] K. Borsuk introduced the concept of generalized cohomotopy groups. We recall some of the basic definitions.

Let X be a topological space. A closed subset A_k of X is called a *k-skeleton* of X, if $\dim A_k \leqslant k$ and if every closed subset of X of dimension $\leqslant k$ can be continuously deformed in the space X into A_k.

If X is a polyhedron and K is a triangulation of X (or, more generally, if X is a CW-complex given in a cellular decomposition K), then the k-skeleton K^m of the complex K is also an m-skeleton in the sense of Borsuk of the space X. Borsuk also showed that if X is a compact ANR-space satisfying the so-called condition (Δ), then there exists a k-skeleton of X for every $k = 0, 1, \dots$ (see [2]).

Let S be an ANR-space and let A be a closed subset of the space X. Consider the set S^X of all continuous mappings of X into S and denote by $S^{A \subset X}$ the subset of S^A consisting of all mappings $f\colon A \to S$ which are extendable over X.

If $f \in S^X$ then we denote by $[f]$ the homotopy class of the mapping f, by $[S^X]$ the set of homotopy classes of the mappings $f \in S^X$, and by $[S^{A \subset X}]$ the set of homotopy classes of mappings $f \in S^{A \subset X}$. By the homotopy extension theorem, $[S^{A \subset X}]$ is a subset of $[S^A]$.

If S is the n-sphere S^n and $\dim A < 2n-1$, then a group operation in $[S^A]$ can be defined and, under this operation, $[S^A]$ becomes an Abelian group which is called the n-th *cohomotopy group* $\pi^n(A)$ of A (see [5]).

Let $\pi^n(A \subset X)$ denote the subgroup of $\pi^n(A)$ generated by the elements of $[S^{A \subset X}]$. If $A = X$ and $\dim X < 2n-1$, then $\pi^n(A \subset X) = \pi^n(X)$.

Borsuk showed that if A and B are two k-skeletons of X and $k < 2n-1$, then the groups $\pi^n(A \subset X)$ and $\pi^n(B \subset X)$ are isomorphic. Hence if $k < 2n-1$ and if the space X possesses a k-skeleton, then an abstract group $\pi_k^n(X)$, isomorphic to $\pi^n(A \subset X)$, can be defined. If $\dim X = k$, then $\pi_k^n(X) = \pi^n(X)$. In particular, if X is a polyhedron (or, more generally,

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a CW-complex, or a compact ANR-space with property (Δ)), then the group $\pi^n_k(X)$ is defined for every n and $k < 2n-1$.

In this paper we prove some elementary properties of the groups $\pi^n_k(X)$ and express them in the most simple cases in terms of known homological invariants.

2. If X, Y, S are three spaces, and $f\colon Y\to X$ is a mapping, then the assignment $\alpha\to\alpha f$, for $\alpha\in S^X$, defines a function

$$f^{\#}\colon [S^X]\to[S^Y]$$

which has the following properties:

(i) *If $f\colon X\to X$ is the identity mapping, then $f^{\#}$ is the identity.*

(ii) *If Z is another space and $g\colon Z\to Y$ is a mapping, then*

$$(fg)^{\#} = g^{\#}f^{\#}.$$

(iii) *If $g\colon Y\to X$ and $g\simeq f$, then $g^{\#} = f^{\#}$.*

If S is the n-sphere S^n and $\dim X$, $\dim Y < 2n-1$, then $f^{\#}$ is a homomorphism of the cohomotopy groups, $f^{\#}\colon \pi^n(X)\to\pi^n(Y)$ (see [5], p. 214).

THEOREM 1. *Let A be a closed subset of X, B a closed subset of Y and let S be an* ANR-*space. Let $f\colon Y\to X$ be a mapping such that $f(B)\subset A$ and let $f_0\colon B\to A$ be the mapping defined by f. Then f_0 maps $[S^{A\subset X}]$ into $[S^{B\subset Y}]$ and therefore defines a function*

$$\bar{f}\colon [S^{A\subset X}]\to[S^{B\subset Y}]$$

which has the following properties:

(i) *If $f\colon X\to X$ is the identity, then $\bar{f}\colon [S^{A\subset X}]\to[S^{B\subset Y}]$ is the identity.*

(ii) *If C is a closed subset of Z and $g\colon Z\to Y$ is a mapping such that $g(C)\subset B$, then $\overline{fg} = \bar{g}\bar{f}$.*

(iii) *If $g\colon Y\to X$ is a mapping such that $g(B)\subset A$ and $g\simeq f$, then $\bar{g} = \bar{f}$.*

Proof. If $[\alpha]\in[S^{A\subset X}]$, $\alpha\colon A\to S$ and $\alpha'\colon X\to S$ is an extension of α, then $\alpha' f\colon Y\to S$ is an extension of the mapping $\alpha f_0\colon B\to S$; but $[\alpha f_0] = f_0^{\#}[\alpha]$.

Properties (i) and (ii) follow from the corresponding properties above. To prove property (iii) we notice that the mapping αf_0 can be written as $\alpha' f i_B$, where $i_B\colon B\to Y$ is the inclusion mapping. Since $f\simeq g\colon Y\to X$, then $fi_B\simeq gi_B\colon B\to X$ and hence $\alpha' fi_B\simeq\alpha' gi_B\colon B\to S$.

LEMMA 1. *Let X be a compact* ANR-*space with property* (Δ), *A a k-skeleton of X, B a compact subset of Y of dimension $\leqslant k$ and f a mapping of Y into X. Then f is homotopic to a mapping $f'\colon Y\to X$ such that $f'(B)\subset A$.*

Proof. By [1], p. 79, the mapping f is homotopic to a mapping $f_1\colon Y\to X$ such that $\dim f_1(B)\leqslant k$. Since A is a k-skeleton of X, there exists a mapping $\mu\colon f_1(B)\to A$ such that the inclusion $i_1\colon f_1(B)\to X$ and the mapping $i_A\mu\colon f_1(B)\to X$, where $i_A\colon A\to X$ is the inclusion, are homotopic. It follows that the partial mapping $f_1 i_B\colon B\to X$, where $i_B\colon B\to Y$ is the inclusion, is homotopic to a mapping $f_2\colon B\to X$ such that $f_2(B)\subset A$. Since X is an ANR-space, the mapping f_2 can be extended to a mapping $f'\colon Y\to X$ such that $f'\simeq f$. Then f' has the desired properties.

THEOREM 2. *Let X and Y be two spaces such that X is a compact* ANR-*space with property* (Δ) *and let S be an* ANR-*space. Let A be a k-skeleton of X and let B a compact subset of Y such that $\dim B\leqslant k$. Let f be a mapping of Y into X. Then there exists a unique function*

$$\bar{f}_{A,A}\colon [S^{A\subset X}]\to[S^{B\subset Y}]$$

which has the following properties:

(i) *If $f\colon X\to X$ is the identity, then $\bar{f}_{A,A}$ is the identity.*

(ii) *If Y is a compact* ANR-*space, B a k-skeleton of Y, Z is another space and C is a compact subset of Z with $\dim C\leqslant k$ and $g\colon Z\to Y$ is a mapping, then*

$$\overline{(fg)}_{A,C} = \bar{g}_{B,C}\bar{f}_{A,B}.$$

(iii) *If $f, g\colon Y\to X$ and $f\simeq g$, then $\bar{f}_{A,B} = \bar{g}_{A,B}$.*

(iv) *If $f\colon Y\to X$ and $f(A)\subset B$, then $\bar{f}_{A,B} = \bar{f}$.*

Proof. Using Lemma 1 we replace f by a mapping $f'\colon Y\to X$ homotopic to f and such that $f'(B)\subset A$ and we define $\bar{f}_{A,B} = \bar{f}'$. If $f''\colon Y\to X$ is another mapping homotopic to f such that $f''(B)\subset A$, then, by property (iii) in Theorem 1, $\overline{f''} = \overline{f'}$. Hence the function $\bar{f}_{A,B}$ is uniquely defined and, obviously, it has properties (iii) and (iv) of Theorem 2. This implies that the function $\bar{f}_{A,B}$ is unique. The properties (i) and (ii) follow from the corresponding properties of Theorem 1.

THEOREM 3 *Let A be a closed subset of X, B a closed subset of of Y and S be the n-sphere S^n. Let $f\colon Y\to X$ be a mapping such that $f(B)\subset A$, and $f_0\colon B\to A$ be the mapping defined by f. Then the homomorphism $f_0^{\#}\colon \pi^n(A)\to\pi^n(B)$ of the cohomotopy groups induced by f_0 maps $\pi^n(A\subset X)$ into $\pi^n(B\subset Y)$ and hence induces a homomorphism*

$$\tilde{f}\colon \pi^n(A\subset X)\to\pi^n(B\subset Y)$$

which has the following properties:

(i) *If $f\colon X\to X$ is the identity, then $\tilde{f}$ is the identity.*

(ii) *If C is a closed subset of Z and $g\colon Z\to Y$ is a mapping such that $g(C)\subset B$, then $(\widetilde{fg}) = \tilde{g}\tilde{f}$.*

(iii) *If f, $g\colon Y\to X$ are two mappings such that $f(B)\subset A$, $g(B)\subset A$ and $f\simeq g$, then $\tilde{f} = \tilde{g}$.*

Theorem 3 will be a consequence of the following algebraic lemma the proof of which is evident:

LEMMA 2. *Let G, H be Abelian groups, $\mathfrak{A}$ a subset of G, $\mathfrak{B}$ a subset of H and $\varphi\colon G\to H$ a homomorphism such that $\varphi(\mathfrak{A})\subset\mathfrak{B}$. Let G_0 be the subgroup of G generated by $\mathfrak{A}$ and H_0 the subgroup of H generated by $\mathfrak{B}$. Then: (1) φ maps G_0 into H_0 and therefore defines a homomorphism $\varphi_0\colon G_0\to H_0$ such that $\varphi_0(a)=\varphi(a)$, for every $a\in G_0$; (2) If $\varphi,\psi\colon G\to H$ are two homomorphisms such that $\varphi|\mathfrak{A}=\psi|\mathfrak{A}$, then $\varphi|G_0=\psi|H_0$; (3) If φ maps $\mathfrak{A}$ onto $\mathfrak{B}$, then φ_0 is an epimorphism.*

Proof of Theorem 3. By Theorem 1, we can apply the lemma to the case, when

$$G=\pi^n(A)\,,\quad H=\pi^n(B)\,,\quad \mathfrak{A}=[S^{A\subset X}]\,,\quad \mathfrak{B}=[S^{B\subset Y}]\,,$$
$$G_0=\pi^n(A\subset X)\,,\quad H_0=\pi^n(B\subset Y)$$

and φ is the homomorphism of cohomotopy groups induced by f_0. The properties (i), (ii), (iii) follow from the corresponding properties in Theorem 1.

THEOREM 4. *Let X, Y be two spaces such that X is a compact ANR-space with property (Δ) and let S be the n-sphere S^n. Let A be a k-skeleton of X and B a compact subset of Y with $\dim B\leqslant k$, where $k<2n-1$. Let $f\colon Y\to X$. Then f defines a unique homomorphism*

$$f^{\#}_{A,B}\colon \pi^n(A\subset X)\to\pi^n(B\subset Y)$$

which has the following properties:

(i) *If $f\colon X\to X$ is the identity, then $f^{\#}_{A,A}$ is the identity.*

(ii) *If Y is a compact ANR-space, B is a k-skeleton of Y, C is a compact subset of a space Z with $\dim C\leqslant k$ and $g\colon Z\to Y$, then*

$$(fg)^{\#}_{A,C}=g^{\#}_{B,C}f^{\#}_{A,B}\,.$$

(iii) *If $f,g\colon Y\to X$ and $f\simeq g$, then $f^{\#}_{A,B}=g^{\#}_{A,B}$.*

(iv) *If $f\colon Y\to X$ and $f(A)\subset B$, then $f^{\#}_{A,B}=\bar{f}$.*

(v) *$f^{\#}_{A,B}|[S^{A\subset X}]=\bar{f}_{A,B}$.*

Proof. The existence of $f_{A,B}$ and property (v) follows from Theorem 2 and from the first part of Lemma 2. Properties (i), (ii) follow from the corresponding properties in Theorem 2. Properties (iii) and (iv) follow from Theorem 2 and 3 and from the second part Lemma 2.

COROLLARY 1. *Let A and B be two k-skeletons of a compact ANR-space X with property (Δ) and let S be an ANR-space. Then the function $\bar{e}_{A,B}$, where $e\colon X\to X$ is the identity mapping, establishes a* unique *one-to-one correspondence between the sets $[S^{A\subset X}]$ and $[S^{B\subset X}]$.*

COROLLARY 2. *Let S be the n-sphere S^n and let A and B be two k-skeletons of a compact ANR-space X with property (Δ) such that $k<2n-1$. Then the identity mapping $e\colon X\to X$ induces a* unique *isomorphism $e^{\#}_{A,B}\colon \pi^n(A\subset X)\approx\pi^n(B\subset X)$.*

The fact that $e^{\#}_{A,B}$ is an isomorphism follows from properties (i) and (ii) in Theorem 4.

Thus we have obtained the theorem proved by Borsuk on isomorphism between $\pi^n(A\subset X)$ and $\pi^n(B\subset X)$. We stress the fact that this isomorphism is unique; for this allows us to state the following

COROLLARY 3. *Let X and Y be two compact ANR-spaces with property (Δ) and let $k<2n-1$. Let $f\colon Y\to X$ be a mapping. Then f induces a unique homomorphism*

$$f^{\#}\colon \pi^n_k(X)\to\pi^n_k(Y) \tag{1}$$

which has the following properties:

(i) *If $f\colon X\to X$ is the identity, then $f^{\#}$ is the identity.*

(ii) *If Z is a compact ANR-space with property (Δ), and $g\colon Z\to Y$, then $(fg)^{\#}=g^{\#}f^{\#}$.*

(iii) *If $f,g\colon Y\to X$ and $f\simeq g$, then $f^{\#}=g^{\#}$.*

(iv) *If $k=\dim X$, then $f^{\#}$ coincides with the homomorphism of cohomotopy groups induced by f.*

Property (iv) justifies notation (1).

3. We assume throughout this section that X is a compact ANR-set with property (Δ). Then, as it has been shown by Borsuk, for every $k=0,1,\ldots$ there exists in X a k-skeleton A_k. Let $e\colon X\to X$ be the identity mapping and let $k\leqslant l\leqslant\dim X$. Consider the function

$$i_{l,k}=\bar{e}_{A_l,A_k}\colon [S^{A_l\subset X}]\to[S^{A_k\subset X}]$$

defined by e (see Theorem 2). We observe that in the case, when $A_k\subset A_l$, the function $i_{k,k}$ is onto; hence by the uniqueness property stressed in Theorem 2 and in Corollary 1 it follows that the function $i_{l,k}$ is always onto. This can also be shown directly, for the function $i_{l,k}$ can be defined as follows:

Let $a=[\alpha]\in[S^{A_l\subset X}]$, where $\alpha\colon A_l\to S$. Then there exists an extension $\alpha'\colon X\to S$. We claim that the restriction $\alpha' i_{A_k}\colon A_k\to S$, where $i_{A_k}\colon A_k\to X$ is the inclusion, *represents* $i_{l,k}(a)$. For, by Lemma 1, there exists a mapping $f\colon X\to X$ homotopic to the identity e such that $f(A_k)\subset A_l$. Then the function $\bar{f}\colon [S^{A_l\subset X}]\to[S^{A_k\subset X}]$ (see Theorem 1) defined by f coincides, by property (iii) and (iv) in Theorem 2, with $i_{l,k}=\bar{e}_{A_l,A_k}$. But $\bar{f}[\alpha]=[\alpha' f i_{A_k}]$, and, since $f\simeq e\colon X\to X$, then $\alpha' f i_{A_k}\simeq\alpha' e i_{A_k}=\alpha' i_{A_k}$.

If S is the n-sphere and $l < 2n-1$, then we have a homomorphism

$$h_{l,k} = e^{\#}_{A_l,A_k}\colon \pi^n(A_l \subset X) \to \pi^n(A_k \subset X)$$

(see Theorem 4). By the third part of Lemma 2 we infer that $h_{l,k}$ is an *epimorphism*.

Denote $i_{k,k-1} = i_k$, $h_{k,k-1} = h_k$. Therefore, if $k < 2n-1$, then $\pi^n(A_{k-1} \subset X)$ is the image of $\pi^n(A_k \subset X)$ under the homomorphism h_k. We have therefore a sequence

$$\ldots [S^{A_{2n}\subset X}] \xrightarrow{i_{2n}} [S^{A_{2n-1}\subset X}] \xrightarrow{i_{2n-1}} [S^{A_{2n-2}\subset X}] \subset \pi^n(A_{2n-2} \subset X)$$
$$\xrightarrow{h_{2n-2}} \pi^n(A_{2n-3} \subset X) \xrightarrow{h_{2n-3}} \ldots \xrightarrow{h_{n+2}} \pi^n(A_{n+1} \subset X) \xrightarrow{h_{n+1}} \pi^n(A_n \subset X) \to 0$$

such that each i_k is onto, and each h_k, for $2n-2 \leqslant k \leqslant n$, is an epimorphism.

It is clearly seen that the kernel of $h_{l,k}$: $\pi^n(A_l \subset X) \to \pi^n(A_k \subset X)$ is contained in $[S^{A_l\subset X}]$; in fact, an element $a \in \pi^n(A_l \subset X)$ belongs to the kernel of $h_{k,l}$ if and only if it is represented by a mapping α: $A_l \to S$ extendable over X and inessential on A^k. We also observe that, by the definition of k-skeleton, a mapping of X into S is inessential on a k-skeleton of X if and only if it is inessential on every closed subset of X of dimension $\leqslant k$.

Passing to the abstract groups, we obtain a sequence of epimorphisms

$$\pi^n_{2n-2}(X) \xrightarrow{h_{2n-2}} \pi^n_{2n-3}(X) \xrightarrow{h_{2n-3}} \ldots \xrightarrow{h_{n+2}} \pi^n_{n+1}(X) \xrightarrow{h_{n+1}} \pi^n_n(X) \to 0 .$$

4. A theorem proved by Borsuk (see [2]) implies that

(∗) *If A_k is a k-skeleton of a compact* ANR-*space X with property* (Δ), *then the homomorphism j^*: $H^n(X) \to H^n(A_k)$ of the integral cohomology groups induced by the inclusion j: $A_k \to X$ is a monomorphism.*

Let S be the n-sphere S^n. In the case when $\dim X \leqslant n+2$, the classification theorems of Hopf, Pontriagin and Steenrod can be used to determine the groups $\pi^n_k(X)$. Thus, for example, the Hopf classification and extension theorem can be formulated as follows:

THEOREM 5. *If* $\dim X \leqslant n+1$ *then* $\pi^n_k(X) \approx H^n(X)$.

Proof. Let s^n be the generator of $H^n(S^n)$. If α is a mapping of an n-skeleton A_n of X into S^n then, by the Hopf classification theorem, the assignment $\alpha \to \alpha^* s^n$ defines an isomorphism η: $H^n(A_n) \approx [S^{A_n}]$. If $u \in H^n(X)$, then, by the Hopf extension theorem, the class $\vartheta(u) = \eta j^*(u)$ is extendable over X, i.e. $\vartheta(u) \in [S^{A_n\subset X}]$. If $a \in [S^{A_n\subset X}]$ is an arbitrary element represented by α: $A_n \to S$ and α': $X \to S$ is an extension of α, then $\vartheta\alpha'^*(s^n) = a$, hence ϑ maps $H^n(X)$ onto $[S^{A_n\subset X}]$. Since η is an isomorphism and, by (∗), j^* is a monomorphism, it follows that ϑ is an isomorphism.

In particular it follows that in the case when $\dim X \leqslant n+1$, $[S^{A_n\subset X}]$ is a subgroup of $\pi^n(A_n)$, i.e. $\pi^n(A_n \subset X) = [S^{A_n\subset X}]$.

Let now $n = 2$ and $\dim X \leqslant 3$. Consider the function $[S^X] \xrightarrow{i_3} \pi^2_2(X) \approx H^2(X)$ (see No. 3); it maps $[S^X]$ onto $\pi^2_2(X)$. If $a \in \pi^2_2(X)$ and $u \in H^2(X)$ corresponds to a, then, by the Pontriagin classification theorem (see [4]), the set $i_3^{-1}(a)$ is in one-to-one correspondence with the subgroup $H^1(X) \smile 2u$ of $H^3(X)$ (the cup product of $H^1(X)$ by $2u$).

5. Let now $n > 2$ and $\dim X \leqslant n+2$. Then the Steenrod extension theorem can be formulated as follows:

THEOREM 6. *If* $\dim X \leqslant n+2$, *then* $\pi^n_n(X)$ *is isomorphic to the kernel of the Steenrod square homomorphism* Sq^2: $H^n(X) \to H^{n+2}(X, Z_2)$ (*the coefficient group* Z_2 *here is the group of integers modulo* 2).

Proof. If A_n is an n-skeleton of X, then an isomorphism ϑ: $\mathrm{Ker}(\mathrm{Sq}^2) \approx [S^{A_n\subset X}]$ is defined here in the same way as the isomorphism ϑ in the proof of Theorem 5: If $u \in \mathrm{Ker}(\mathrm{Sq}^2) \subset H^n(X)$, j^*: $H^n(X) \to H^n(A_n)$ is the homomorphism induced by the inclusion and η: $H^n(A_n) \approx [S^{A_n}]$ is the Hopf isomorphism, then we define $\vartheta(u) = \eta j^*(u)$. Then by the Steenrod extension theorem it follows that $\vartheta(u) \in [S^{A_n\subset X}]$ and ϑ maps $\mathrm{Ker}(\mathrm{Sq}^2)$ onto $[S^{A_n\subset X}]$; by proposition (∗), ϑ is an isomorphism.

If $\dim X \leqslant n+1$ then the kernel of the epimorphism

$$\pi^n_{n+1}(X) = \pi^n(X) \xrightarrow{h_{n+1}} \pi^n_n(X) \approx H^n(X)$$

is also given by the Steenrod classification theorem. It is isomorphic to the finite quotient group $H^{n+1}(X, Z_2)/\mathrm{Sq}^2 H^{n-1}(X)$ (we also use here the proposition (∗)).

As an example, let us consider the case when X is a polyhedral compact $(n+1)$-dimensional pseudomanifold without boundary, with $n > 2$, given in a triangulation. Then $H^{n+1}(X, Z_2) \approx Z_2$ and it follows from the above remark that either h^{n+1}: $\pi^n(X) \to \pi^n_n(X)$ is an isomorphism, i.e. $\pi^n(X) \approx H^n(X)$, or its kernel is isomorphic to Z_2. This can also be shown directly, without using the Steenrod squares. For if a mapping α: $X \to S^n$ represents an element in the kernel of h_{n+1}, then we may assume that α maps a (polyhedral) n-skeleton A_n of X into a single point $y_0 \in S^n$. Hence on each $(n+1)$-cell of X, α represents an element of the homotopy group $\pi_{n+1}(S^n) \approx Z_2$. It is clearly seen that if σ is a fixed $(n+1)$-cell of X, then α is homotopic to a mapping α': $X \to S^n$ such that $\alpha'(\overline{X-\sigma}) = y_0$. Then the homotopy class of α can be represented by one of the two elements of $\pi_{n+1}(S^n)$ which can correspond to $\alpha'|\sigma$.

If X is a pseudomanifold with boundary then, evidently, $\pi^n(X) \approx \pi^n_n(X) \approx H^n(X)$.

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The category of a map and of a cohomology class

by

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The purpose of this paper is to prove several results concerning the n-dimensional category of a topological space X in the sense of Fox [7] and the category of a cohomology class $u \in H^q(X; G)$ in the sense of Fary [6]. The category of a map, a concept which goes back to Fox ([7], p. 368), will play a unifying role in the present setting: among other things, we prove that, provided X is a reasonale space, both the n-dimensional category of X and the category of u coincide with the categories of certain maps of X into standard spaces of homotopy theory.

1. The category of a map. Let f: $X \to Y$ be a (continuous) map of arbitrary topological spaces.

DEFINITION 1.1. $\operatorname{cat} f$ *is the least integer* $k \geqslant 1$ *with the property that* X *may be covered by* k *open subsets* U_m *such that the maps* $f \mid U_m$: $U_m \to Y$ *defined by* f *are nullhomotopic; if no such integer exists, we put* $\operatorname{cat} f = \infty$.

We shall denote by $\operatorname{cat} X$ the Lusternik-Schnirelmann category of X, i.e., the least integer $k \geqslant 1$ with the property that X may be covered by k open subsets which are contractible in X; if no such integer exists, $\operatorname{cat} X = \infty$.

The following results are easy to check:

1.2. $\operatorname{cat} f \leqslant \min\{\operatorname{cat} X, \operatorname{cat} Y\}$.

1.3. $\operatorname{cat} \theta = \operatorname{cat} X$ *if* θ *is the identity map of* X.

1.4. $\operatorname{cat} g \circ f \leqslant \min\{\operatorname{cat} f, \operatorname{cat} g\}$ *for any map* g: $Y \to Z$.

1.5. $\operatorname{cat} h_0 = \operatorname{cat} h_1$ *if* h_t: $X \to Y$ *is a homotopy.*

Next, since a CW-pair has the homotopy extension property and since a CW-complex is locally contractible, we have

1.6. *If a* CW-*complex* X *is the union of* k *subcomplexes which are contractible in* X, *then* $\operatorname{cat} X \leqslant k$.

We now prove

PROPOSITION 1.7. *If* X *is a* CW-*complex and* f: $X \to Y$ *is an arbitrary map, then the following statements are equivalent:*